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Second-Order Sliding-Mode Differentiators: An Experimental Comparative Analysis Using Van der Pol oscillator

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Abstract
This article provides a comparative study of four different second-order sliding-mode (SOSM) differentiators proposed in the literature, namely, standard higher-order sliding-modes (HOSM) differentiator, non-homogeneous HOSM differentiator, uniform robust exact differentiator and hybrid fixed-time differentiator. Based on sliding mode principles, these differentiators can provide robust exact differentiation with finite/fixed-time convergence. First, a comprehensive summary of the different methods is provided. Then, the differentiators are applied experimentally to estimate the states of a Van der Pol oscillator. Through experiments, it is shown that the different differentiators outperformed a Kalman-like observer, high-gain differentiator and extended Kalman filter. Finally, some suggestions are provided on the selection of SOSM differentiators for various applications.

KEYWORDS
Sliding mode differentiation, Van der Pol oscillator, Real-time estimation

1. Introduction

Numerical differentiation of noisy signals is very useful in the context of state estimation, parameter identification, filtering, fault detection and control of dynamical systems. Numerical differentiation can be found in a wide varieties of applications. For example - calculation of velocity and acceleration from position measurement [De Loza et al. (2012); Guerra et al. (2016); Salgado et al. (2017); Davila et al. (2005)], fault detection [Ríos et al. (2016); H. Ahmed et al. (2016)], control [Salgado, Camacho, et al. (2014); Levant (2003)], parameter identification [Davila et al. (2006); Shtessel & Poznyak (2005); Iqbal et al. (2011); Q. Ahmed & Bhatti (2011); Imine et al. (2015)], synchronization of nonlinear systems [H. Ahmed et al. (2017)], secure communication

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[Perruquetti et al. (2008)], environmental monitoring [H. Ahmed et al. (2015)], frequency estimation of sinusoidal signals [Wu et al. (2004)] *etc.* to name a few. Due to its various practical applications, this area has attracted a lot attention from the scientific community in recent times.

Two popular numerical differentiation schemes that are used frequently to calculate time derivatives are difference method (also known as Euler method) and Kalman filter [Kalman (1960)]. The difference method can estimate the derivatives of signals but its result is sensitive to the noise which is inevitable in practical applications. To solve this problem, engineers often use low-pass filtering, this however, introduces undesirable delay in real-time applications. In the presence of non-Gaussian noise, Kalman-filer is known as the best linear estimator. However, Kalman-filter depends on the model/knowledge of the plant. For various industrial applications, models are often not available. As a result, model-free robust derivative estimation in the presence of measurement noise is a challenging problem from practical point of view. To tackle this problem, various robust techniques have been proposed in the literature. They are based on methods like-sliding-mode (SM) [Levant (1998, 2003)], homogeneity based approach [Perruquetti et al. (2008); Polyakov et al. (2014)], algebraic approach [Ibrir (2003)], high-gain approach [Dabroom & Khalil (1999)], to name a few.

Out of all the previously mentioned methods, in this work we focus on sliding-mode based differentiators. Sliding-mode differentiators have some very nice properties like - robustness to noise, finite/fixed-time convergence, easy implementation, computational simplicity *etc.* All these properties have made sliding-mode differentiator a very suitable choice for solving practical problems. The sliding-mode differentiator has already been successfully applied to numerous practical applications [Amamra et al. (2017); Reichhartinger & Horn (2009); Salgado et al. (2017); Rodriguez et al. (2009); Salgado, Chairez, et al. (2014); H. Ahmed et al. (2016)].

The sliding-mode differentiator based on 2-sliding algorithm was initially proposed by Levant in 1998 [Levant (1998)]. It allows to estimate the first order derivative of a bounded noisy signal. In 2003, a generalization of the original differentiator to an arbitrary order case was proposed in [Levant (2003)]. Since then, various types of sliding-mode differentiators have been proposed based on high order sliding-mode principles [D. V. Efimov & Fridman (2011); Cruz-Zavala et al. (2011); Sidhom et al. (2015); Vázquez et al. (2016); Basin et al. (2017); Ghanes et al. (2017); Levant (2014); Barbot et al. (2016); Ríos & Teel (2018)]. These variants consider different objectives. The result proposed in [D. V. Efimov & Fridman (2011)] provides global derivative estimation independently on amplitude of the differentiated signal and measurement noise. Currently, in the literature, there exist only a few works related to the fixed-time property: in the differentiation framework, *i.e.* [Cruz-Zavala et al. (2011)] for the design of a first-order fixed-time robust exact differentiator, and [Angulo et al. (2013)] for arbitrary order differentiation under some type of disturbances; and in the observer design framework [Andrieu et al. (2008)], [Lopez-Ramirez et al. (2018, 2017)] and [Ríos & Teel (2018)], but only for the ideal linear case. Finite- and fixed-settling time differentiators utilizing a non-recursive higher-order sliding mode is proposed in [Basin et al. (2017)]. The differentiators have also been applied to hypersonic missile. In [Ghanes et al. (2017)], SOSM differentiator with a variable exponent is proposed. However, finite-time convergence of the differentiation error is obtained only for a particular value of the exponent. In [Levant (2014)], a finite-time convergent exact differentiator with variable gains is presented. This differentiator works through the addition of linear terms to the recursive structure of [Levant (2003)]. Variable gain approach is also used in [Edwards & Shtessel (2016)] where equivalent control based dual-layer is used. However, the number of parameter to tune is relatively higher than non-adaptive version. Unlike the deterministic settings used in the previously mentioned references, [A. S. Poznyak (2018); A. Poznyak (2017)] proposes the application of stochastic super-twisting for SM based observer and controller design. The advantage of this method is that it can explicitly take into account the noise effect. However, real-time implementation of this algorithm is complicated. For example, the gain adaptation mechanism in [A. Poznyak (2017)] requires the calculation of partial derivative. The results of [Vázquez et al. (2016)] and [Sidhom et al. (2015)] provide robust derivative estimation in the context of hydraulic system application. In the same line of research, several adaptive differentiators have already been proposed for particular purposes. In [Alwi & Edwards (2013)], an adaptive differentiator has been proposed to estimate an actuator oscillatory fault. Fault reconstruction of uncertain nonlinear systems using adaptive sliding-mode differentiator is discussed in [X.-G. Yan & Edwards (2008)]. Sensorless control of electric motor using adaptive sliding-mode differentiator can be found in [Furuhashi et al. (1992)]. For a detail bibliographical survey on sliding-mode observer/differentiator, readers may consult [Spurgeon (2008)].

There are some existing works in the literature, where different differentiation techniques have been compared by simulation. In [Hongwei & Heping (2015)], the authors compared various tracking differentiators (including sliding-mode differentiator) by simulation. Comparative analysis of differentiators for pneumatic application was performed in [X. Yan et al. (2014)]. Comparative analysis of differentiators for aerospace application can be found in [Cieslak et al. (2016)]. The same analysis for environmental monitoring application can be found in [H. Ahmed et al. (2015)]. To the best of the author's knowledge, up to now, no experimental study has been carried out to compare the performances of the various HOSM differentiators proposed in the literature. The goal of this article is to fill this void. To achieve this goal, four techniques [Levant (1998); D. V. Efimov & Fridman (2011); Cruz-Zavala et al. (2011); Ríos & Teel (2018)] based on HOSM have been selected. They have been chosen because of their comparative advantages as summarized below:

(1) HOSM Differentiator (HOSMD) [Levant (2003)]: This is the first HOSM differentiator proposed in the literature. Exact finite-time differentiation along with robustness to measurement noises are its main features. Various variants of this differentiator have been proposed in the literature. This differentiator has been applied to numerous practical problems (e.g., Chawda et al. (2011); Imine et al. (2015); Madani & Benallegue (2007) etc.). It is considered as a benchmark HOSM differentiator.
(2) Non-homogeneous HOSM differentiator (NHOSMD) [D. V. Efimov & Fridman (2011)]: Using novel Lyapunov function, this differentiator provide simple estimates on the time of convergence and accuracy of derivative calculation. An interesting property of this differentiator is that the solutions of the differentiator stay bounded even for wrongly chosen parameters. This differentiator is also applied to solve various practical problems (e.g., H. Ahmed et al. (2016); D. Efimov et al. (2013); de Loza et al. (2015)).
(3) Uniform robust exact differentiator (URED) [Cruz-Zavala et al. (2011)]: This differentiator is a variant of the Levant's differentiator and includes high-degree terms providing finite-time, and exact convergence to the derivative of the input signal, with a convergence time that is bounded by some constant independent of the initial conditions of the differentiation error. This is the first SOSM differentiator that provided fixed-time convergence. The fixed-time convergence

property is proved using strong Lyapunov functions. This differentiator also also found to be very effective in practical applications (e.g., Amamra et al. (2017); Espinoza-Moreno et al. (2014); Hussain et al. (2013) etc.).

(4) Hybrid fixed-time differentiator (HFTD) [Ríos & Teel (2018)]: This differentiator is a combination between the Levant's differentiator and the general homogeneous differentiator given by [Levant (2005)] and [Angulo et al. (2013)]. A hybrid hysteresis mechanism is proposed to combine the main properties of these differentiators, *i.e.* finite-time convergence and exactness, and uniform convergence with respect to initial errors, providing the fixed-time convergence property. This differentiator is one of the few arbitrary-order differentiators with fixed-time convergence.

The above mentioned techniques will be compared experimentally for an interesting practical application. In this work only the estimation of first derivative will be considered for the sake of simplicity. It is to be noted here that there are also other sliding-mode differentiators available in the literature. However, for the sake of simplicity and tractability, we decided not to include them here.

The rest of the article is organized as follows: Section 2 provides some preliminaries about different types of stabilities of nonlinear systems, Section 3 provides the formal problem statement, Section 4 provides a summary of the selected techniques, Results and discussions can be found in Section 5 and finally Section 6 concludes this article.

2. Preliminaries

2.1. *Notation*

Throughout the paper the following notations are used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geq 0\}$, where $\mathbb{R}$ is the set of all real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define the norm $||d||_{[t_0,t_1)} = \operatorname{ess\,sup}_{t\in[t_0,t_1)} ||d\,(t)\,||$, then $||d||_\infty = ||d||_{[0,+\infty)}$ and the set of $d(t)$ with the property $||d||_\infty < +\infty$ is denoted as $\mathcal{L}_\infty$ (the set of essentially bounded measurable functions); and $\mathcal{L}_D = \{d \in \mathcal{L}_\infty : ||d||_\infty \leq D\}$ for any $D > 0$.

2.2. *Stability of nonlinear systems*

This section gives the notion of different types of stability of nonlinear systems. These notions will be helpful to understand the types of convergence of the selected differentiators described in Section 4.

Consider the following nonlinear systems

$$\dot{x}\,(t) = f\,(x\,(t)\,, d\,(t))\,, t \geq 0, \tag{1}$$

where $x\,(t) \in \mathbb{R}^n$ is the state, $d\,(t) \in \mathbb{R}^m$ is the input, $d\,(t) \in \mathcal{L}_\infty$, $f \colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ ensures forward existence of the system solutions at least locally, $f\,(0,0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty$, define the corresponding solution by $\varphi(t, x_0, d)$ for any $t \geq 0$ for which the solution exists. Let Ω be an open neighborhood of the origin in $\mathbb{R}^n$ and $D > 0$.

Definition 1. At the steady state $x = 0$ the system (1) for any $d \in \mathcal{L}_D$ is said to be

(1) *uniformly Lyapunov stable* in Ω if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for any $x_0 \in \Omega$ and $d \in \mathcal{L}_D$, if $||x_0|| < \delta(\varepsilon)$ then $||\varphi(t, x_0, d)|| < \varepsilon$ for all $t \geq 0$;
(2) *uniformly asymptotically stable* in Ω if it is uniformly Lyapunov stable in Ω and for any $\kappa > 0$ and $\varepsilon > 0$ there exists $T(\kappa, \varepsilon) \geq 0$ such that for any $x_0 \in \Omega$ and $d \in \mathcal{L}_D$, if $||x_0|| < \kappa$ then $||\varphi(t, x_0, d)|| < \varepsilon$ for all $t \geq T(\kappa, \varepsilon)$;
(3) *uniformly finite-time stable* in Ω if it is uniformly Lyapunov stable in Ω and uniformly finite-time converging in Ω, *i.e.* for any $x_0 \in \Omega$ and all $d \in \mathcal{L}_D$ there exists $0 \leq T \leq +\infty$ such that $\varphi(t, x_0, d) = 0$ for all $t \geq T$. The function $T_0(x_0) = \inf\{T \geq 0\colon \varphi(t, x_0, d) = 0 \, \forall t \geq T, \forall d \in \mathcal{L}_D\}$ is called the uniform settling time of the system (1);
(4) *uniformly fixed-time stable* in Ω if it is uniformly finite-time stable in Ω and $\sup_{x_0 \in \Omega} T_0(x_0) < +\infty$. The set Ω is called the domain of stability/attraction.

If $\Omega = \mathbb{R}^n$, then the corresponding properties are called global uniform Lyapunov/asymptotic/finite-time/fixed-time stability of (1) for $d \in \mathcal{L}_D$ at $x = 0$. For details, please consult [D. Efimov, Levant, et al. (2017)] and the references therein.

3. Problem Statement

Let us consider the Van der Pol oscillator which is a popular second-order benchmark nonlinear system. The Van der Pol oscillator is an oscillator with nonlinear damping governed by the second-order differential equation given below,

$$\ddot{x}(t) - \epsilon\left[1 - x^2(t)\right]\dot{x}(t) + \omega^2 x(t) = 0, \gamma, \omega > 0. \tag{2}$$

In model (2), ϵ and ω are model parameters. By considering $\omega^2 = 1$, $\dot{x} = x_1$ and $\ddot{x} = x_2$, model (2) can be written in state-space form as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \epsilon x_2\left(1 - x_1^2\right), \\ y &= x_1, \end{aligned} \tag{3}$$

where $x_1, x_2 \in \mathbb{R}$ are state variables, $y \in \mathbb{R}$ is the output and ϵ is the model parameter. Model (3) has been successfully applied in various fields like biomedical engineering [Ryzhii & Ryzhii (2014); Kaplan et al. (2008)], control [Puvsenjak et al. (2014); Landau et al. (2008)], electrical networks [Sinha et al. (2016)], etc. Because of its practical importance, state estimation of model (3) is a very interesting problem.

Model (3) can be re-written as:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[\underbrace{-x_1 + \epsilon x_2\left(1 - x_1^2\right)}_{d(t,x)}\right] \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \tag{4}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $d\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is an unknown bounded function, $|d(t,x)| \leq d_0, \forall (t,x) \in \mathbb{R}^3$ with d_0 being a known positive number. For model (4), estimating the state x_2 from measurement x_1 is essentially a problem of derivative estimation. Estimating the state x_2 can be done using the observer framework as well. For this purpose, in this work various SOSM observer/differentiator will be used. As comparison tools, several other popular methods namely high-gain observer, extended Kalman filter and Kalman-like observer will also be used. A short summary of all these methods can be found in the following section.

4. Summary of the selected differentiators

In this section, a short summary of the selected SOSM differentiators [Levant (1998); D. V. Efimov & Fridman (2011); Cruz-Zavala et al. (2011); Ríos & Teel (2018)] will be given for further analysis. Kalman-like observer [Besançon et al. (2010)], high-gain observer [Khalil (2017)] and extended Kalman filter [Simon (2006)] have been selected for comparison purpose. Short summary of these methods can also be found at the end of this section.

4.1. *Summary of the SOSM differentiators*

4.1.1. HOSM Differentiator (HOSMD) [Levant (2003)]

Consider an input signal $f(t)$ be a function defined on $[0,\infty)$ consisting of a bounded Lebesgue-measurable noise with unknown features and an unknown base signal $f_0(t)$ having a derivative with known Lipschitz's constant $L > 0$, *i.e.* $|\ddot{f}_0(t)| \leq L$. To calculate the derivative of $f(t)$, consider an auxiliary equation $\dot{z} = u$ where $z(t)$ denotes the estimate of the original signal $f(t)$. The control law u is designed to drive the estimation error, *i.e.* $e(t) = f(t) - z(t)$, to zero. A. Levant proposed in Levant (2003) a control based on super-twisting principle that guarantees the finite-time convergence to zero of the derivative estimation error $e(t)$. The differentiator is given by:

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 \lceil x_1 - f(t) \rfloor^{\frac{1}{2}} + x_2, \\ \dot{x}_2 &= -\lambda_2 \lceil x_1 - f(t) \rfloor^{0}, \end{aligned} \tag{5}$$

where $\lceil x \rfloor^\gamma := |x|^\gamma \text{sign}(x), \gamma \geq 0$, $x_1, x_2 \in \mathbb{R}$ are the state variables of the system, λ_1 and λ_2 are tuning parameters with $\lambda_1 > 0$ and $\lambda_2 > 0$. The variable $x_1(t)$ serves as an estimate of the function $f(t)$ and $x_2(t)$ converges to $\dot{f}(t)$, *i.e.* it provides the derivative estimate. Tuning of parameters λ_1 and λ_2 is provided in Levant (2003). They can be selected as: $\lambda_1 = 1.5L^{\frac{1}{2}}$ and $\lambda_2 = 1.1L$. The gains of the super-twisting differentiator (5) can also be tuned based on a strict Lyapunov function as proposed in [Moreno & Osorio (2012)].The system (5) is discontinuous. The classical theory of differential equations is now not applicable since Lipschitz assumptions are employed to guarantee the existence of unique solutions. The solutions of (5) are to be understood in Filippov sense [Filippov (1960)]. The solution concept proposed by Filippov for differential equation with discontinuous right hand sides constructs a solution as the "average" of the solutions obtained from approaching the point of discontinuity from different directions. In [Levant (2003)] and [Moreno & Osorio (2012)], it was shown that differentiator (5) is finite-time convergent. To prove the finite-time convergence property of (5), [Levant (1998)] used the homogeneity based approach as such no estimate of the convergence time is available. In the presence of measurement noise, the accuracy of this differentiator is proportional to $\varphi^{\frac{1}{2}}$, where φ is the maximal measurement noise magnitude.

4.1.2. Non-homogeneous HOSM differentiator (NHOSMD) [D. V. Efimov & Fridman (2011)]

Recently, D. Efimov and L. Fridman proposed a variant of the super-twisting differentiator [Levant (1998)]. It can provide simple estimates on the time of convergence and accuracy of the derivative calculation. The differentiator is given by:

$$\begin{aligned} \dot{x}_1 &= -\alpha \lceil x_1 - f(t) \rfloor^{\frac{1}{2}} + x_2, \\ \dot{x}_2 &= -\beta \lceil x_1 - f(t) \rfloor^{0} - \chi \lceil x_2 \rfloor^{0} - x_2, \end{aligned} \tag{6}$$

where, x_1, x_2 and $f(t)$ are as defined in Section 4.1.1, α, β and χ are tuning parameters with $\alpha, \beta, \chi > 0$. Compared to (5), differentiator (6) has two additional feedback loops involving x_2. Appearance of x_2 causes the loss of homogeneity property but does not affect the excellent performance of super-twisting differentiator. The main advantage of (6) is that solutions of the differentiator stay bounded even for wrongly chosen parameters. The system (6) is discontinuous, its solutions are understood in the Filippov sense [Filippov (1960)]. Using Lyapunov function argument, in [D. V. Efimov & Fridman (2011)], it was shown that the differentiator (6) is finite-time convergent. The gains α, β and χ of this differentiator can be chosen according to the following formula (given in Corollary 1 of [D. V. Efimov & Fridman (2011)]):

$$\begin{aligned} \chi &= 0.25\sqrt[4]{2}L_1 + \nu, \beta > L_1 + L_2 + 3\chi, \\ \alpha &= 4\frac{\sqrt{2(\beta + L_1 + L_2 + 2\chi)}\chi + \sqrt{\beta + L_1 + L_2 + 3\chi}\,(L_1 + L_2 + 2\chi)}{2\beta - L_1 - L_2 - 2\chi}, \end{aligned} \tag{7}$$

where $|\dot{f}_0(t)| \leq L_1$, $|\ddot{f}_0(t)| \leq L_2$ and $\nu \geq 0$. According to Corollary 1 of [D. V. Efimov & Fridman (2011)], if the gains are selected as in (7) and the initial conditions are selected as $x_1(0) = f(0)$ and $x_2(0) = 0$, then for $\nu \geq 0$, the convergence time T_0 is bounded by $T_0 \leq L_1 / \left(0.25\sqrt[4]{2}L_1 + \nu\right)$. This simple estimate of convergence time was one of the big advantage of [D. V. Efimov & Fridman (2011)] over [Levant (1998)]. As in [Levant (1998)] the requirement on the existence of the second derivative can be replaced with Lipschitz continuity of the first derivative, then L_2 is the corresponding Lipschitz constant. An interesting property of this differentiator is that the solutions of the differentiator stay bounded even for wrongly chosen parameters [(D. V. Efimov & Fridman, 2011, Lemma 1)]. In the presence of measurement noise, the accuracy of this differentiator is proportional to $\varphi^{\frac{1}{4}}$, where φ is the maximal measurement- noise magnitude. It is also to be pointed out here that many new Lyapunov functions are now available for the Super Twisting algorithm that allows to estimate convergence rates and attraction regions in the presence of noise (cf. [Moreno & Osorio (2012)]).

4.1.3. *Uniform robust exact differentiator (URED) [Cruz-Zavala et al. (2011)]*

Super-twisting differentiators like [Levant (1998)] and [D. V. Efimov & Fridman (2011)] are well known for their exact and finite-time convergence properties. However, the convergence time grows unboundedly when the initial conditions of the differentiation error grow. To overcome this problem, E. Cruz-Zavala *et al.* proposed a variant of [Levant (2003)] by considering high-degree terms. The main advantage of this method is that it can provide finite-time, and exact convergence to the derivative of the input signal, with a convergence time that is bounded by some constant independent of the initial conditions of the differentiation error. This differentiator is given by:

$$\begin{aligned}
\dot{x}_1 &= -k_1 \left[\lceil x_1 - f(t) \rfloor^{\frac{1}{2}} + \mu \lceil x_1 - f(t) \rfloor^{\frac{3}{2}} \right] + x_2, \\
\dot{x}_2 &= -k_2 \left[\frac{1}{2} \lceil x_1 - f(t) \rfloor^{0} + 2\mu \left(x_1 - f(t) \right) + \frac{3}{2} \mu^2 \lceil x_1 - f(t) \rfloor^{2} \right],
\end{aligned} \tag{8}$$

where, $k_1, k_2 > 0$ are differentiators gains and $\mu > 0$ is a scalar. When $\mu = 0$, the standard robust exact differentiator is recovered [Levant (1998)]. As before, $x_1(t)$ and $x_2(t)$ represent the estimate of signal $f(t)$ and its first derivative. The system (8) is discontinuous, its solutions are understood in the Filippov sense [Filippov (1960)]. Using Lyapunov function argument, in [Cruz-Zavala et al. (2011)], it was shown that the differentiator (8) is fixed-time convergent if the gains k_1 and k_2 are in the set

$$\mathcal{K} = \left\{ (k_1, k_2) \in \mathbb{R}^2 | 0 < k_1 \le 2\sqrt{L}, k_2 > \frac{k_1^2}{4} + \frac{4L^2}{k_1^2} \right\} \cup \left\{ (k_1, k_2) \in \mathbb{R}^2 | k_1 > 2\sqrt{L}, k_2 > 2L \right\}. \tag{9}$$

The convergence time estimate of differentiator (8) is quite complex and requires to solve linear matrix inequality. So, the estimate is avoided here for the purpose of brevity. Interested reader may consult eq. (12) of [Cruz-Zavala et al. (2011)]. Finally, in the presence of measurement noise, the accuracy of this differentiator is the same to that of HOSMD.

4.1.4. *Hybrid fixed-time differentiator (HFTD) [Ríos & Teel (2018)]*

In many real-time control applications, fixed-time convergence of the derivative is a highly desirable property. This property implies the existence of a bound for the convergence time, and such a bound is independent of the initial estimation error. To achieve this goal, H. Rï¿œos and A. R. Teel recently proposed a fixed-time observer/differentiator in [Ríos & Teel (2018)] using the hybrid systems framework given in [Goebel et al. (2012)]. Based on homogeneity properties, the robustness of the hybrid differentiator is analyzed in [Ríos & Teel (2018)]. The main idea of [Ríos & Teel (2018)] is summarized below:

Consider the system of the following form:

$$\begin{aligned}
&\dot{x} = Ax + D\ddot{f}_0(t), \; x(0) = x_0, \\
&y = Cx,
\end{aligned} \tag{10}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \begin{bmatrix} f_0 & \dot{f}_0 \end{bmatrix}^T \in \mathbb{R}^2$, $y \in \mathbb{R}$ and $f_0 \in \mathbb{R}$ are the states, the output and the signal to be differentiated, respectively. The matrices are defined as

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

For this system, estimating the state x_2 from the measurement x_1 is essentially the problem of the estimating the first derivative. The following differentiator has been proposed in Ríos & Teel (2018),

$$\begin{aligned} \hat{x} &= z + \Phi^{-1} v, \\ \frac{d}{dt}\begin{bmatrix} z \\ v \\ \bar{v} \\ q \end{bmatrix} &= \begin{bmatrix} Az + H\left(y - Cz\right) \\ \bar{A}v + \chi_q\left(\varepsilon_1, v_1\right) \\ \bar{A}\bar{v} + \chi_2\left(\varepsilon_1, \bar{v}_1\right) \\ 0 \end{bmatrix}, (\hat{x}, z, v, \bar{v}, q) \in \mathcal{C}, \\ \begin{bmatrix} z^+ \\ v^+ \\ \bar{v}^+ \\ q^+ \end{bmatrix} &= \begin{bmatrix} z \\ v \\ \bar{v} \\ 3 - q \end{bmatrix}, (\hat{x}, z, v, \bar{v}, q) \in \mathcal{D}, \end{aligned} \tag{11}$$

where, $\hat{x} = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix}^T \in \mathbb{R}^2$ is the state estimation of x through the hybrid observer and $z \in \mathbb{R}^2$ is the state estimation of x through Luenberger observer. The matrix $\bar{A} := \Phi A_h \Phi^{-1} := \Phi\left(A - HC\right)\Phi^{-1}$ with the linear correction term $H = \begin{bmatrix} h_1 & h_2 \end{bmatrix}^T \in \mathbb{R}^2$ and the transformation matrix $\Phi \in \mathbb{R}^{2x2}$ given as $\begin{bmatrix} A_h\zeta & \zeta \end{bmatrix} / |\zeta|^2$; where $\zeta := \mathcal{O}\bar{h}$, with $\bar{h} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\mathcal{O}$ is the observability matrix of the pair (A_h, C). The vectors $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T \in \mathbb{R}^2$ and $\bar{v} = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 \end{bmatrix}^T \in \mathbb{R}^2$ provide estimations of the error between the system state x (*i.e.* $x - \hat{x}$) and the Luenberger state estimation z, *i.e.* $x - z$, respectively. The nonlinear injection $\chi_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2, i = 1, 2$ take the following structure:

$$\begin{aligned} \chi_1\left(\varepsilon_1 - v_1\right) &= \begin{bmatrix} \alpha_1 \lceil \varepsilon_1 - v_1 \rfloor^{\frac{1}{2}} \\ \alpha_2 \lceil \varepsilon_1 - v_1 \rfloor^{0} \end{bmatrix}, \\ \chi_2\left(\varepsilon_1 - v_1\right) &= \begin{bmatrix} \beta_1 \lceil \varepsilon_1 - v_1 \rfloor^{\frac{2+\gamma}{2}} \\ \beta_2 \lceil \varepsilon_1 - v_1 \rfloor^{1+\gamma} \end{bmatrix}. \end{aligned}$$

The parameters α_i, β_i and γ are positive constants to be designed, and the variable $\varepsilon_1 = y - Cz$. The corresponding initial conditions are given as $z(0,0) = z_0$, $v(0,0) = v_0$, $\bar{v}(0,0) = \bar{v}_0$, $q(0,0) = q_0$, and the flow and jump sets are defined as

$$\mathcal{C} := \{(\hat{x}, z, v, \bar{v}, q) \in \mathbb{R}^8 \times \{1, 2\} : v - \bar{v} \in \mathcal{C}_q\}, \tag{12}$$

$$\mathcal{D} := \{(\hat{x}, z, v, \bar{v}, q) \in \mathbb{R}^8 \times \{1, 2\} : v - \bar{v} \in \mathcal{D}_q\}, \tag{13}$$

where

$$\begin{aligned} &\mathcal{C}_1 := \left\{\tilde{v} \in \mathbb{R}^2 : |\tilde{v}| \leq \epsilon + \rho\right\}, \; \mathcal{C}_2 := \overline{\mathbb{R}^2 \setminus \mathcal{D}_2}, \\ &\mathcal{D}_1 := \overline{\mathbb{R}^2 \setminus \mathcal{C}_1}, \; \mathcal{D}_2 := \left\{\tilde{v} \in \mathbb{R}^2 : |\tilde{v}| \leq \epsilon\right\}, \end{aligned}$$

with $\epsilon > 0$ and $\rho > 0$. The idea of this hybrid observer is to switch between the two injections χ_1 and χ_2 in order to provide fixed-time attractiveness of the set corresponding to zero estimation error by means of using χ_1 near the estimation error origin and χ_2 far from it. Thus, based on a hysteresis mechanism [Goebel et al. (2012)], a hybrid injection involving the logic variable $q \in \{1, 2\}$ and the decision variable $\tilde{v} := v - \bar{v}$ is proposed. The mechanism sets χ_1 if the variable q equals to 1 and $\tilde{v} \in \mathcal{C}_1$, and sets χ_2 if the variable q equals to 2 and $\tilde{v} \in \mathcal{C}_2$, otherwise the variable q switches. The auxiliary variable $\bar{v}$, which never switches, help us to define the flow and jump sets only in terms of known variables as follows: the subset $\mathcal{C}_1$ should be taken as a compact neighborhood of the origin of $\tilde{v}$ that is contained in the attraction domain when using χ_2, while $\mathcal{D}_2$ should be taken as another compact neighborhood of the origin of $\tilde{v}$ such that the solutions using χ_1 that start in $\mathcal{D}_2$ do not reach the boundary of $\mathcal{C}_1$.

Using homogeneity properties, in [Ríos & Teel (2018)], it was shown that the differentiator (11) is fixed-time convergent if the observer parameters are design as follows:

(1) H such that matrix A_h is Hurwitz, and $\Phi^{-1} := \begin{bmatrix} A_l\zeta, & \zeta \end{bmatrix} / |\zeta|^2$.
(2) $\alpha_1 = 1.5L^{\frac{1}{2}}$ and $\alpha_2 = 1.1L$.
(3) $\gamma > 0$ is selected sufficiently small and β_1 and β_2 are chosen such that the following matrix is Hurwitz

$$\begin{bmatrix} -\beta_1 & 1 \\ -\beta_2 & 0 \end{bmatrix}.$$

(4) The flow and jump sets as in (12)-(13), with $\epsilon > 0$ a small positive constant and $\rho > 0$ sufficiently large such that $|\tilde{v}_{\chi_1}(0,0)| \leq \epsilon$ then $|\tilde{v}_{\chi_1}| \leq \epsilon + \rho$, where $\tilde{v}_{\chi_1} := v_{\chi_1} - \bar{v}$, with v_{χ_1} the trajectories of v for $q = 1$.

Since $\tilde{v}$ is measurable one can calculate ρ by means of simulations. Particularly, fixing $|\tilde{v}(0,0)| = \epsilon$, *i.e.* $\tilde{v}(0,0) \in \mathcal{D}_2$, it is possible to do some simulations in order to estimate the region ρ and then ensure that any trajectory starting in $\mathcal{D}_2$ does not reach the boundary of $\mathcal{C}_1$ while χ_1 is used. In other words, the value of ρ should be greater or equal to the overshot given by the Levant's differentiator. Since the fixed-time convergence property of differentiator (11) is proved using homogeneity properties,ass such no convergence time estimate is available. Differentiator (11) uses the HOSMD in its hybrid structure. As a result, it can be substantiated that in the presence of measurement noise, the accuracy of this differentiator is the same to that of HOSMD.

4.2. *Summary of the non-SM differentiators*

4.2.1. Kalman-like observer (KO) [Besançon et al. (2010)]

In state-space form, model (2) can be written as,

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \left(x_3 - x_3 x_1^2\right) x_2 - x_4 x_1, \\ \dot{x}_i &= 0, i = 3, 4, \\ y &= x_1, \end{aligned} \tag{14}$$

where, $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \mu$ and $x_4 = \omega^2$. Next, consider a global transformation defined as $z_i = x_i, i = 1, 3, 4$ and

$$z_2 = x_2 + x_3 \left(\frac{x_1^3}{3} - x_1 \right),$$

Then, system (14) can be written as:

$$\begin{aligned} \dot{z} &= A\left(y(t)\right) z, a(y) = y - \frac{y^3}{3}, \\ y &= z. \end{aligned} \tag{15}$$

where

$$A\left(y\left(t\right)\right) = \begin{bmatrix} 0 & 1 & a(y) & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

For system (15), the following observer was proposed in [Besançon et al. (2010)]:

$$\begin{aligned} \dot{\hat{z}}(t) &= A\left(y\left(t\right)\right)\hat{z}(t) - S^{-1}(t)C^T\left[C\hat{z}(t) - y(t)\right]; S\left(0\right) > 0, \\ \dot{S}\left(t\right) &= -\rho S\left(t\right) - A^T\left(y\left(t\right)\right)S\left(t\right) - S\left(t\right)A\left(y\left(t\right)\right) + C^T C. \end{aligned} \tag{16}$$

KO (16) provides exponential convergence for the extended Van der Pol oscillator model (15). Specialty of the KO (16) lies in the fact that it provides simultaneous state and parameter estimation. For the detail convergence analysis of observer (16), please consult [Besançon et al. (2010)].

4.2.2. High-Gain Differentiator (HG) [Khalil (2017)]

This differentiator is given by:

$$\begin{aligned} \dot{x}_1 &= x_2 + \frac{\eta_1}{\kappa}\left\{f\left(t\right) - x_1\right\}, \\ \dot{x}_2 &= \frac{\eta_2}{\kappa^2}\left\{f\left(t\right) - x_1\right\}, \end{aligned} \tag{17}$$

where, x_1 and x_2 are as defined in Section 4.1.1, η_1, η_2 and κ are positive tuning parameters. The parametersη_1 and η_2 are designed in a way such that the roots of the equation $s^2 + \eta_1 s + \eta_2 = 0$ become negative and $\kappa \ll 1$. Convergence time of HG (17) can be controlled by properly selecting the gain κ. However, for very small values of κ, HG (17) shows impulsive-like transient behavior known as peaking phenomenon.

4.2.3. Extended Kalman Filter (EKF) [Simon (2006)]

Consider the following nonlinear system affected by process and measurement noise

$$\begin{aligned} \dot{x}\left(t\right) &= f\left(x\right) + w\left(t\right), \\ y &= Cx + v\left(t\right), \end{aligned} \tag{18}$$

Table 1. Summary of the properties of the selected sliding-mode differentiators

Differentiator/Observer	Type of convergence	Knowledge required	Asymptotic Accuracy
HOSMD	Finite-time	Lipschitz constant of the first derivative	$\varphi^{\frac{1}{2}}$
NHOSMD	Finite-time	Lipschitz constants of the signal and the first derivative	$\varphi^{\frac{1}{4}}$
URED	Fixed-time	Lipschitz constant of the first derivative	$\varphi^{\frac{1}{2}}$
HFTD	Fixed-time	Lipschitz constant of the first derivative	$\varphi^{\frac{1}{2}}$

where $x(t) \in \mathbb{R}^2$ are the states, $w(t)$ is the process noise, $v(t)$ is the measurement noise and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. It is assumed here that both $w(t)$ and $v(t)$ are zero mean Gaussian noise with their covariance matrices are denoted by Q and R. The Jacobian matrix of systems (18) can be denoted as $F(x) = \frac{\partial f(x)}{\partial x}$. EKF obtain the state estimation through predictor-corrector structure whose equations are given below:
The prediction step

$$\begin{aligned} x_{k|k-1} &= x_{k-1|k-1} + f\left(x_{k-1|k-1}\right) T_s, \\ P_{k|k-1} &= P_{k-1|k-1} + \left[F_{k-1} P_{k-1|k-1} + P_{k-1|k-1} F_{k-1|k-1}^T\right] T_s + Q. \end{aligned} \tag{19}$$

The correction step

$$\begin{aligned} K_k &= P_{k|k-1} C^T \left(C P_{k|k-1} C^T + R\right)^{-1}, \\ x_{k|k} &= x_{k|k-1} + K_k \left(y_k - C x_{k|k-1}\right), \\ P_{k|k} &= P_{k|k-1} - K_k C P_{k|k-1}, \end{aligned} \tag{20}$$

where

T_s	sampling interval;
k	sampling instant;
Q	process noise covariance matrix;
R	measurement noise covariance matrix;
$x_{k\|k-1}$	predicted state estimate;
$x_{k\|k}$	optimal state estimate;
$P_{k\|k-1}$	predicted error covariance estimate;
$P_{k\|k}$	optimal error covariance estimate;
K_k	Kalman filter gain.

In the literature, very few results are available for the systematic tuning of EKF parameters Q and R. Trial and error method is frequently used for the tuning of Q and R.

A short summary of the properties of the selected sliding-mode differentiators/observer can be found in Table 1.

5. Results and Discussions

In this section, the performance of the selected differentiators/observers will be compared through experimental study of a Van der Pol oscillator. For model (4), estimating the state x_2 from measurement x_1 is essentially a problem of derivative estimation. As a result, the differentiators described in Section 4 will be very useful for this purpose. Electronic circuit diagram of the model (4) can be seen in Fig. 1. The circuit parameters are: $R_i = 1M\Omega, i = \overline{1,6}$, $R_7 = 130\Omega$, $R_8 = 1.2K\Omega$, $R_9 = 100\Omega$, $R_{10} = 1.5K\Omega$, $C_1 = C_2 = 1\mu F$, LM741 is a general purpose operational amplifier and AD633 is a

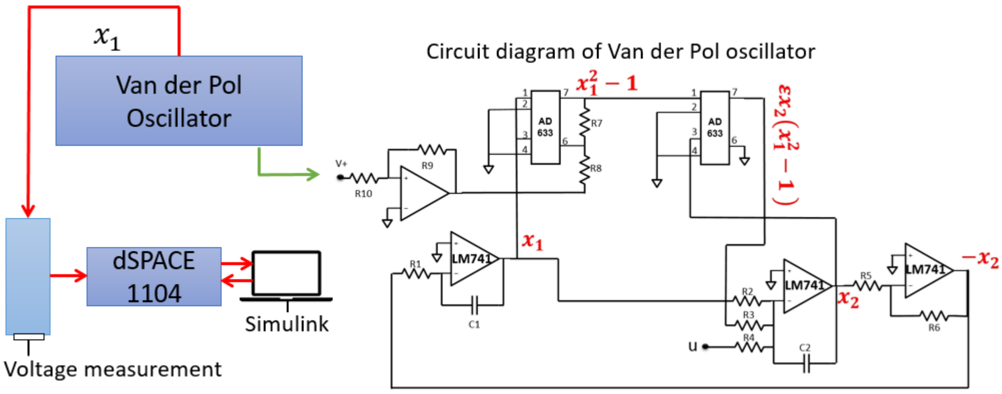

Figure 1. Left - overview of the experimental setup, Right - electronic circuit diagram of (2) [H. Ahmed et al. (2018)].

4-quadrant multiplier operational amplifier. The parameters of the circuits are set in a way so that $\epsilon \approx 0.1$. Then we can consider $d_0 = 3$ as the bound of $d(t, x)$. We have used dSPACE 1104 board as rapid prototyping solution. The differentiators were implemented using Simulink. The solver was the fixed-step Euler and the time step was 0.001 seconds.

Discretization plays an important role in the real-time implementation of continuous HOSM differentiators. Recent literature on this topic [Acary & Brogliato (2010); D. Efimov, Polyakov, et al. (2017); Levant et al. (2016)] have considered the implicit Euler method of discretization. Implicit Euler method has some very nice property like global asymptotic stability even in the case of very slow sampling. However, it has higher computational complexity than the explicit one. That is why, explicit Euler scheme is mostly used in real-time implementation where computational complexity is a big issue. Moreover in [D. Efimov, Polyakov, et al. (2017)], it was shown that for fast sampling, explicit Euler scheme has better convergence speed than implicit one. In our real-time implementation, we have considered sampling frequency of 1KHz which is sufficiently fast. As such for the sake of computational complexity, explicit Euler method was used in our work. However, through study on the discretization scheme will be considered in a future work.

Next, the four differentiators were applied to estimate x_1 and x_2 from measurement of x_1. States x_1 and x_2 are the output voltages of LM741 op-amps as given in Fig. 1. The values of the differentiators parameters can be seen in Table 2. The gains were selected according to the tuning rules presented in Sec. 4.1.1, 4.1.2, 4.1.3 and 4.1.4 respectively.

There are also some existing results on the state estimation of the Van der pol oscillator. We have selected KO, HG and EKF (described in Sec. 4.2.1, 4.2.2 and 4.2.3) to compare the performances of the second-order sliding-mode differentiators. The values of the different parameters of these three techniques can also be found in Table 2. The initial conditions are selected as: HOSMD, NHOSMD, URED, HFTD = $(0, 1)$, KO, HG = $(0, 0)$ and EKF = $(-0.1, 0)$. Slightly different initial conditions are selected to avoid numerical stability issues.

Remark 2. To find the parameters of the EKF, some information about the type of process and measurement noises are needed. The main source of measurement noise in our case is the analog to digital conversion process. The noise introduced by this process is very often of Gaussian nature [Ruscak & Singer (1995)]. To separate the

Table 2. Parameters values of the differentiators

Differentiator/Observer	Parameter Values
HOSMD	$\lambda_1 = 1.5L,\ \lambda_2 = 1.1L, L = 5$
NHOSMD	$\alpha = 9.18, \beta = 5.92, \chi = 0.3, \nu = 0.1$
URED	$k_1 = 2\sqrt{5}, k_2 = 18, \mu = 1$
HFTD	$H = \begin{bmatrix} 22 & 120 \end{bmatrix}^T, \Phi = I_2, \alpha_1 = 2.5981,$ $\alpha_2 = 3.3000, \gamma = 0.75, \beta_1 = 10.2560, \beta_2 = 40.0743$ $\mathcal{C}_1 = \{\tilde{v} \in \mathbb{R}^n : \lvert\tilde{v}\rvert \leq 28.5270\}, \mathcal{C}_2 = \overline{\mathbb{R}^n \setminus \mathcal{D}_2},$ $\mathcal{D}_1 = \overline{\mathbb{R}^n \setminus \mathcal{C}_1}, \mathcal{D}_2 = \{\tilde{v} \in \mathbb{R}^n : \lvert\tilde{v}\rvert \leq 18.5270\}$
KO	$\rho = 10$
HG	$\eta_1 = 2,\ \eta_2 = 1,\ \kappa = 0.05$
EKF	$Q = \mathrm{diag}\,[0.2\ 0.1]\,, R = 0.0356$

noise, measured signal x_1 was passed through a high pass filter. Then the covariance of the noise signal can be easily calculated. To select the process noise covariance matrix Q, trial and error method was used. In the literature, very few results can be found regarding the systematic tuning of Q and R.

The performances of the selected methods can be seen in Fig. 2 and 3. From Fig. 3, it can be seen that the selected differentiators perform very well to estimate the state x_2 with good convergence time. Except HOSMD, the convergence time is around 0.2 sec. which is quite good. Differentiation/observation errors in the case of state x_2 can be seen in Fig. 4 and 5. In noise free case, the selected differentiators provide finite/fixed time convergence of estimation error. In the presence of measurement noise, the differentiation errors are supposed to converge to a close vicinity of zero. In this application, the measurement x_1 was collected after analog-to-digital conversion (ADC). The initial analog measurement had noise. ADC process also introduced errors[1]. As a result, the differentiation errors obtained by the selected differentiators converge to a close vicinity of zero (Fig. 4 and 5) in accordance with the prior theoretical findings. Out of three comparison techniques, HG's performance is close to the worst performing (in terms of maximum steady-state error) sliding-mode differentiator *i.e.* NHOSMD. This observation is similar to the findings reported in [Vasiljevic & Khalil (2008)] where the performance of HG was compared with super-twisting differentiator. However, the maximum error magnitude in steady-state is relatively high for the methods KO[2] and EKF. For example, the maximum error given by HFTD is $0.09V$, while it is almost four times higher in the case of KO and EKF. Moreover, KO and EKF are computationally complex. KO requires the inversion of a 4×4 matrix while several equations have to be solved for EKF. One point to be noted here is that KO showed very good convergence time comparable to SOSM differentiators. The selected HOSM differentiators (except HFTD) are very easy to implement both in analog and digital form. HOSM differentiators also provide finite/fixed time convergence which is very useful for observer based control as separation principle (separate design of observer and controller) can be guaranteed. These demonstrate the suitability of HOSM differentiators for real-time state estimation of nonlinear systems in practice. In this particular case, the maximum steady-state error performance of HOSM differentiators are better than the other selected methods. URED, HOSMD and HFTD provide al-

[1]For details about the ADC performance of dSPACE 1104 board, please consult DS1104 (2006) (page 159)

[2]By choosing sufficiently small ρ, the maximum error magnitude may be reduced. However, this will increase significantly the convergence time.

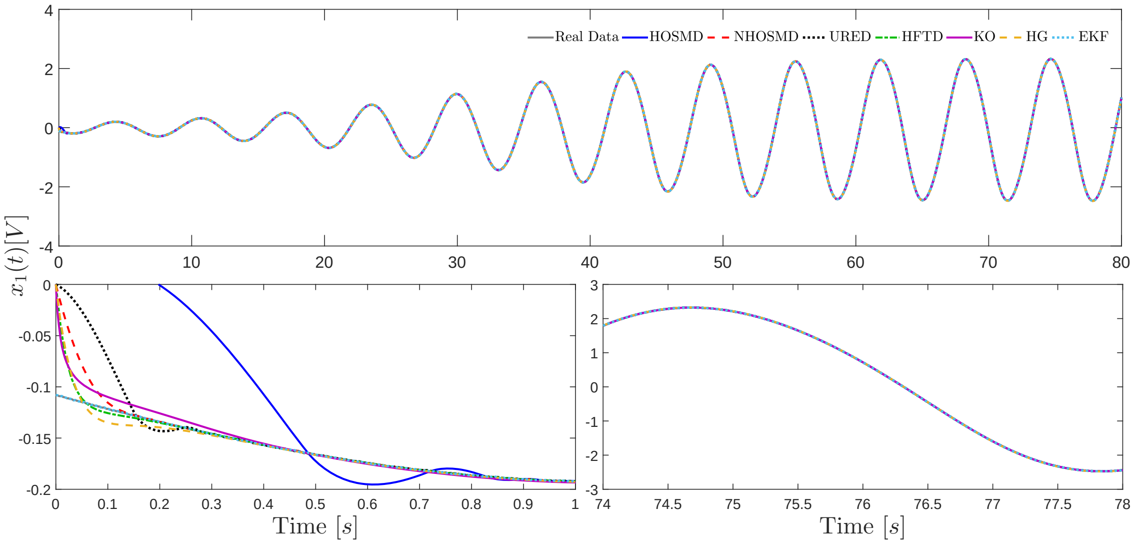

Figure 2. x_1 and its estimates by different methods

most similar performances. The maximum differentiation error of these methods varies between $0.09 - 0.0135V$. Although the estimation error by NHOSMD in steady-state is relatively higher than the others but still much better than the result of HG, KO and EKF. A comparative summary of the different methods can be seen in Table 3. Graphical representation of Table 3 can be seen in Fig 8. One point to be noted in Table 3, is that we have considered only the maximum steady stare error along with convergence time as performance criteria. Steady state error is also used as performance indicator for differentiator performance comparison in [Vasiljevic & Khalil (2008)]. However, there are other criteria as well, for example - maximum overshoot of the estimation error, robustness to high or low frequency noise, *etc*. For the sake of simplicity, these criteria are not considered in this work.

Figure 2, 3, 4 and 5 show that SOSM differentiators are suitable choices for real-time online differentiation purposes. These observers provide robust estimation of the unmeasured state x_2 from the noisy digital measurement x_1. Moreover, the estimation errors converge to a close vicinity of zero in a finite/fixed time. Their estimation error performances are better than HG and significantly better than KO and EKF as given in Fig. 4 and 5. The overall performances of the differentiators are similar since all these differentiators are based on the same principle *i.e.* sliding mode. However, depending on the application, the choice of particular SOSM differentiator can be different. If the focus is number of tunable parameters, then HOSMD is a good choice. For convergence time sensitive applications, URED and HFTD are good choice. From computational complexity view point, HOSMD, NHOSMD and URED can be good options as they are easy to implement. From asymptotic accuracy view point, HOSMD, HFTD and URED can be chosen. In this work, only first derivative was considered. Some practical applications require higher order derivatives, for example, acceleration feedback from position measurements only. In this case, some of the selected differentiators (or their extensions) can provide higher order derivative estimation also. Except NHOSMD, the selected differentiators can be used for the purpose of estimating higher order derivatives. In this work, robustness with respect to high or low frequency noise was not considered. This will be considered in a future work.

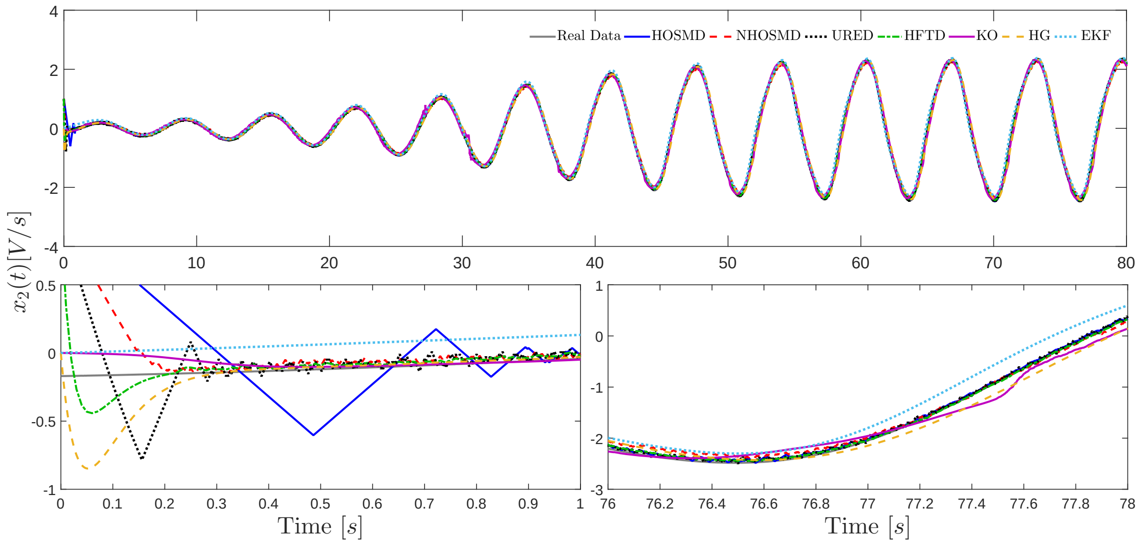

Figure 3. x_2 and its estimates by different methods.

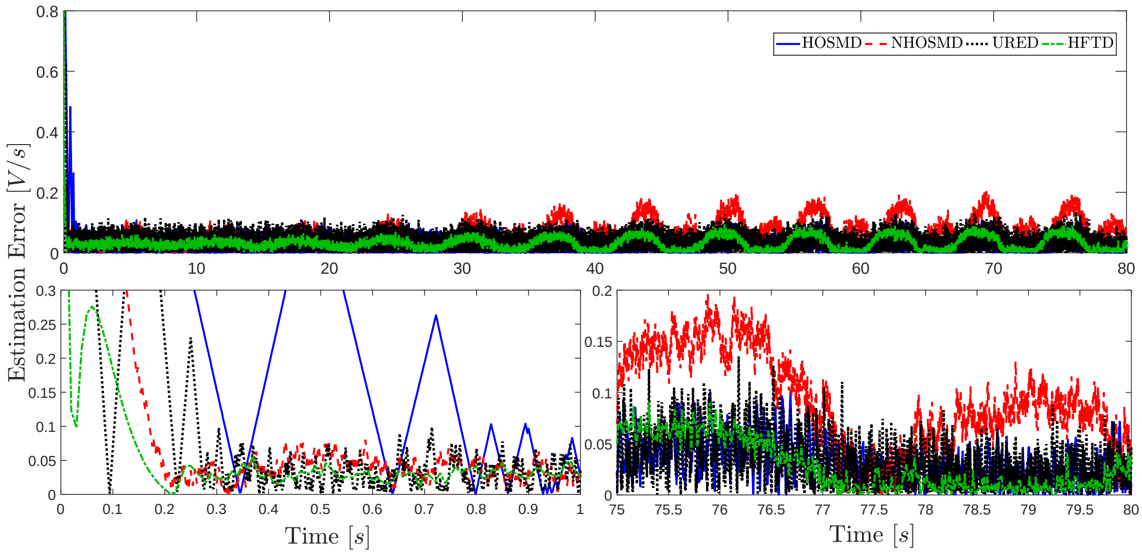

Figure 4. Estimation error for state x_2 by different SOSM methods.

Table 3. Estimation error $x_2 - \hat{x}_2$ statistics of different methods when the oscillator reached the limit cycle.

Method	Maximum error (V)	$\frac{\max(x_2-\hat{x}_2)}{\max(x_2)}$ (in %)	Convergence Time (in sec)[3]
HFTD	0.09	3.9	0.13
HOSMD	0.12	5.1	0.77
URED	0.135	5.8	0.26
NHOSMD	0.2	8.6	0.1
HG	0.25	11.1	0.2
KO	0.32	13.7	0.26
EKF	0.37	16	3.35

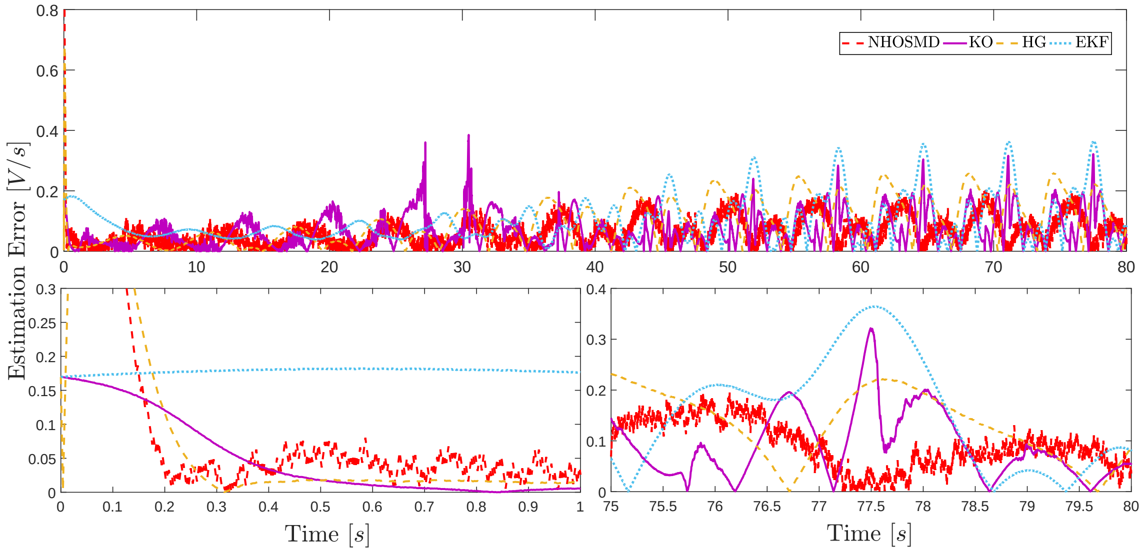

Figure 5. Estimation error for state x_2 by worst performing SOSM method and the comparison techniques.

6. Conclusion

In this paper, a comparison study of four different SOSM differentiators was performed. The performances of the differentiators were compared experimentally by estimating the states of Van der Pol oscillator. Experimental results showed that the performances of the different differentiators are similar for this particular application. The differentiators were also compared with other methods from the literature. Comparative analysis showed the superiority of SOSM differentiators over other selected methods. Finally, some conclusions were drawn based on theoretical and experimental findings regarding the selection of SOSM differentiator in practice for various applications. They are summarized below:

- In terms of parameter tuning, HOSMD is the simplest since only the Lipschitz constant of the second derivative has to be known. URED and NHOSMD have one more parameter to tune than HOSMD.
- URED and HFTD ensure fixed-time convergence while NHOSMD and HOSMD provide finite-time convergence.
- HOSMD, NHOSMD and URED are computationally simple and easy to implement.
- HOSMD, UFTD and URED have better asymptotic accuracy than NHOSMD.
- HOSMD, HFTD and extension of URED can be used to obtain higher order derivatives.

References

Acary, V., & Brogliato, B. (2010). Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems. *Systems & Control Letters*, *59*(5), 284–293.

[3]When the first time, the estimation error entered the region $|x_2 - \hat{x}_2| \leq 0.1$ and stayed there for a while, we considered that time as the convergence time. For the purpose of convergence time estimation, Fig. 6 was used. For some differentiators/observers, the estimation error left this region later on. This is quite natural because the oscillator took some time to reach the stable limit cycle.

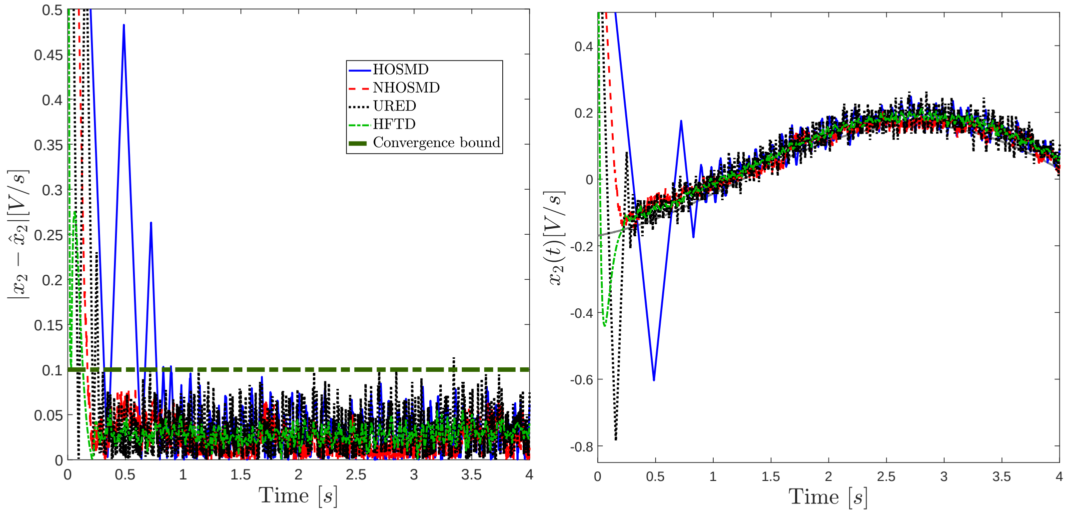

Figure 6. Convergence time estimation plot for different SOSM methods.

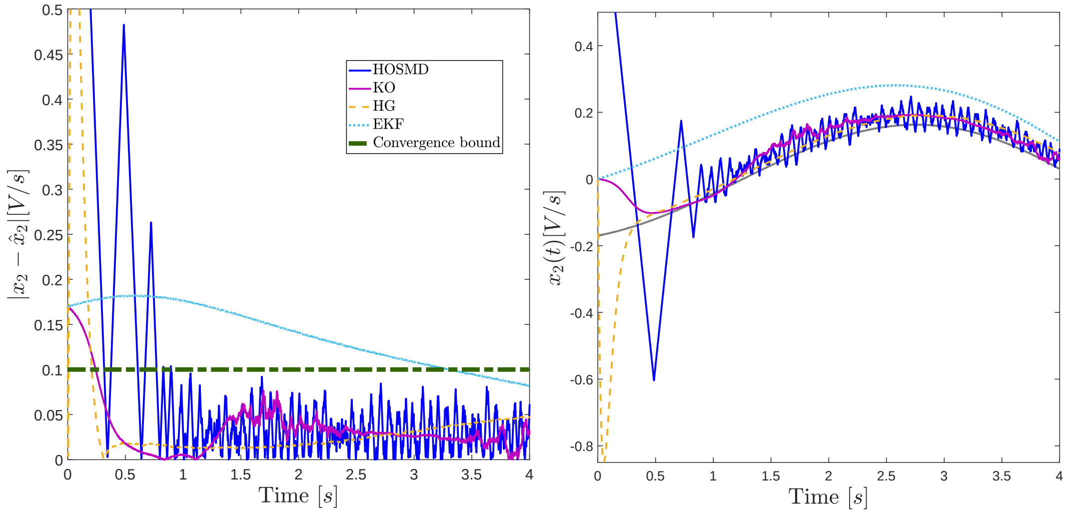

Figure 7. Convergence time estimation plot for worst performing SOSM method and the comparison techniques.

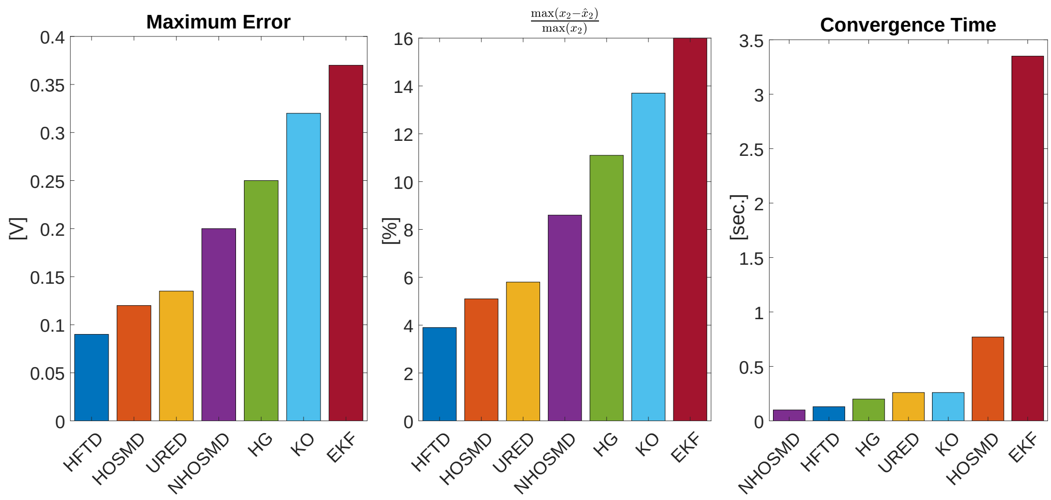

Figure 8. Graphical representation of Table 3.

Ahmed, H., Salgado, I., & Ríos, H. (2018). Robust synchronization of master-slave chaotic systems using approximate model: An experimental study. *ISA Transactions*, doi: 10.1016/j.isatra.2018.01.009.

Ahmed, H., Ushirobira, R., Efimov, D., Fridman, L., & Wang, Y. (2017). Oscillatory global output synchronization of nonidentical nonlinear systems. *IFAC-PapersOnLine*, *50*(1), 2708–2713.

Ahmed, H., Ushirobira, R., Efimov, D., Tran, D., & Massabuau, J.-C. (2015). Velocity estimation of valve movement in oysters for water quality surveillance. *IFAC-PapersOnLine*, *48*(11), 333–338.

Ahmed, H., Ushirobira, R., Efimov, D., Tran, D., Sow, M., Payton, L., & Massabuau, J.-C. (2016). A fault detection method for automatic detection of spawning in oysters. *IEEE Transactions on Control Systems Technology*, *24*(3), 1140–1147.

Ahmed, Q., & Bhatti, A. I. (2011). Estimating SI engine efficiencies and parameters in second-order sliding modes. *IEEE Transactions on Industrial Electronics*, *58*(10), 4837–4846.

Alwi, H., & Edwards, C. (2013). An adaptive sliding mode differentiator for actuator oscillatory failure case reconstruction. *Automatica*, *49*(2), 642–651.

Amamra, S.-A., Ahmed, H., & Elsehiemy, R. (2017). Firefly algorithm optimized robust protection scheme for DC microgrid. *Electric Power Components and Systems*, *45*(10), 1141-1151.

Andrieu, V., Praly, L., & Astolfi, A. (2008). Homogeneous approximation, recursive observer design, and output feedback. *SIAM J. Control Optimization*, *47*(4), 1814–1850.

Angulo, M. T., Moreno, J. A., & Fridman, L. (2013). Robust exact uniformly convergent arbitrary order differentiator. *Automatica*, *49*(8), 2489–2495.

Barbot, J. P., Levant, A., Livne, M., & Lunz, D. (2016). Discrete sliding-mode-based differentiators. In *2016 14th international workshop on Variable Structure Systems (VSS)* (p. 166-171.).

Basin, M., Yu, P., & Shtessel, Y. (2017). Finite- and fixed-time differentiators utilising HOSM techniques. *IET Control Theory Applications*, *11*(8), 1144-1152. doi:

Besançon, G., Voda, A., & Jouffroy, G. (2010). A note on state and parameter estimation in a Van der Pol oscillator. *Automatica*, *46*(10), 1735–1738.

Chawda, V., Celik, O., & O'Malley, M. K. (2011). Application of Levant's differentiator for velocity estimation and increased z-width in haptic interfaces. In *World Haptics Conference (WHC), 2011 IEEE* (pp. 403–408).

Cieslak, J., Zolghadri, A., Goupil, P., & Dayre, R. (2016). A comparative study of three differentiation schemes for the detection of runaway faults in aircraft control surfaces. *IFAC-*

PapersOnLine, *49*(17), 70–75.
Cruz-Zavala, E., Moreno, J. A., & Fridman, L. M. (2011). Uniform robust exact differentiator. *IEEE Transactions on Automatic Control*, *56*(11), 2727–2733.
Dabroom, A. M., & Khalil, H. K. (1999). Discrete-time implementation of high-gain observers for numerical differentiation. *International Journal of Control*, *72*(17), 1523–1537.
Davila, J., Fridman, L., Levant, A., et al. (2005). Second-order sliding-mode observer for mechanical systems. *IEEE transactions on automatic control*, *50*(11), 1785–1789.
Davila, J., Fridman, L., & Poznyak, A. (2006). Observation and identification of mechanical systems via second order sliding modes. *International Journal of Control*, *79*(10), 1251–1262.
de Loza, A. F., Cieslak, J., Henry, D., Dávila, J., & Zolghadri, A. (2015). Sensor fault diagnosis using a non-homogeneous high-order sliding mode observer with application to a transport aircraft. *IET Control Theory & Applications*, *9*(4), 598–607.
De Loza, A. F., Rios, H., & Rosales, A. (2012). Robust regulation for a 3-dof helicopter via sliding-mode observation and identification. *Journal of the Franklin Institute*, *349*(2), 700–718.
DS1104, R. (2006). Controller board. *Manual, Features, RTI Reference, RTLib Reference, Release*, *5*.
Edwards, C., & Shtessel, Y. (2016). Adaptive dual-layer super-twisting control and observation. *International Journal of Control*, *89*(9), 1759–1766.
Efimov, D., Cieslak, J., Zolghadri, A., & Henry, D. (2013). Actuator fault detection in aircraft systems: Oscillatory failure case study. *Annual Reviews in Control*, *37*(1), 180–190.
Efimov, D., Levant, A., Polyakov, A., & Perruquetti, W. (2017). Supervisory acceleration of convergence for homogeneous systems. *International Journal of Control*(0), 1-11. doi:
Efimov, D., Polyakov, A., Levant, A., & Perruquetti, W. (2017). Realization and discretization of asymptotically stable homogeneous systems. *IEEE Transactions on Automatic Control*, *62*(11), 5962-5969.
Efimov, D. V., & Fridman, L. (2011). A hybrid robust non-homogeneous finite-time differentiator. *IEEE Transactions on Automatic Control*, *56*(5), 1213-1219.
Espinoza-Moreno, G., Begovich, O., & Sanchez-Torres, J. (2014). Real time leak detection and isolation in pipelines: A comparison between sliding mode observer and algebraic steady state method. In *World Automation Congress (WAC), 2014* (pp. 748–753).
Filippov, A. F. (1960). Differential equations with discontinuous right-hand side. *Matematicheskii Sbornik*, *93*(1), 99–128.
Furuhashi, T., Sangwongwanich, S., & Okuma, S. (1992). A position-and-velocity sensorless control for brushless DC motors using an adaptive sliding mode observer. *IEEE Transactions on Industrial electronics*, *39*(2), 89–95.
Ghanes, M., Barbot, J. P., Fridman, L., & Levant, A. (2017). A second order sliding mode differentiator with a variable exponent. In *2017 American Control Conference (ACC)* (p. 3300-3305).
Goebel, R., Sanfelice, R. G., & Teel, A. R. (2012). *Hybrid dynamical systems: modeling, stability, and robustness.* Princeton University Press.
Guerra, M., Efimov, D., Zheng, G., & Perruquetti, W. (2016). Avoiding local minima in the potential field method using Input-to-State Stability. *Control Engineering Practice*, *55*, 174–184.
Hongwei, W., & Heping, W. (2015). A comparison study of advanced tracking differentiator design techniques. *Procedia Engineering*, *99*, 1005–1013.
Hussain, S., Bhatti, A., Samee, A., & Qaiser, S. H. (2013). Estimation of reactivity and average fuel temperature of a pressurized water reactor using sliding mode differentiator observer. *IEEE Transactions on Nuclear Science*, *60*(4), 3025–3032.
Ibrir, S. (2003). Online exact differentiation and notion of asymptotic algebraic observers. *IEEE transactions on Automatic control*, *48*(11), 2055–2060.
Imine, H., Fridman, L., & Madani, T. (2015). Identification of vehicle parameters and estimation of vertical forces. *International Journal of Systems Science*, *46*(16), 2996–3009.

Iqbal, M., Bhatti, A. I., Ayubi, S. I., & Khan, Q. (2011). Robust parameter estimation of nonlinear systems using sliding-mode differentiator observer. *IEEE Transactions on industrial electronics*, *58*(2), 680–689.

Kalman, R. E. (1960). A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, *82*(1), 35–45.

Kaplan, B., Gabay, I., Sarafian, G., & Sarafian, D. (2008). Biological applications of the filtered Van der Pol oscillator. *Journal of the Franklin Institute*, *345*(3), 226–232.

Khalil, H. K. (2017). High-gain observers in feedback control: Application to permanent magnet synchronous motors. *IEEE Control Systems*, *37*(3), 25–41.

Landau, I. D., Bouziani, F., Bitmead, R. R., & Voda-Besançon, A. (2008). Analysis of control relevant coupled nonlinear oscillatory systems. *European Journal of Control*, *14*(4), 263–282.

Levant, A. (1998). Robust exact differentiation via sliding mode technique. *Automatica*, *34*(3), 379–384.

Levant, A. (2003). Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, *76*(9-10), 924–941.

Levant, A. (2005). Homogeneity approach to high-order sliding mode design. *Automatica*, *41*(5), 823-830.

Levant, A. (2014). Globally convergent fast exact differentiator with variable gains. In *Control Conference (ECC), 2014 European* (pp. 2925–2930).

Levant, A., Livne, M., & Lunz, D. (2016). On discretization of high-order sliding modes. *Recent Trends in Sliding Mode Control, Fridman L, Barbot JP, Plestan F(eds). IET, Stevenage, UK*, 177–205.

Lopez-Ramirez, F., Efimov, D., Polyakov, A., & Perruquetti, W. (2017). Fixed-time output stabilization of a chain of integrators. In *55th Conference on Decision and Control (CDC)* (p. 3886-3891).

Lopez-Ramirez, F., Polyakov, A., Efimov, D., & Perruquetti, W. (2018). Finite-time and fixed-time observer design: Implicit Lyapunov function approach. *Automatica*, *87*, 52–60.

Madani, T., & Benallegue, A. (2007). Sliding mode observer and backstepping control for a quadrotor Unmanned Aerial Vehicles. In *American Control Conference (ACC), 2007.* (pp. 5887–5892).

Moreno, J. A., & Osorio, M. (2012). Strict Lyapunov functions for the super-twisting algorithm. *IEEE transactions on Automatic Control*, *57*(4), 1035–1040.

Perruquetti, W., Floquet, T., & Moulay, E. (2008). Finite-time observers: application to secure communication. *IEEE Transactions on Automatic Control*, *53*(1), 356–360.

Polyakov, A., Efimov, D., & Perruquetti, W. (2014). Homogeneous differentiator design using Implicit Lyapunov function method. In *Control Conference (ecc), 2014 European* (pp. 288–293).

Poznyak, A. (2017). Stochastic super-twist sliding mode controller. *IEEE Transactions on Automatic Control*, doi: 10.1109/TAC.2017.2755594.

Poznyak, A. S. (2018). Stochastic sliding mode control and state estimation. In *Advances in variable structure systems and sliding mode control-theory and applications* (pp. 57–100). Springer.

Puvsenjak, R. R., Tivcar, I., & Oblak, M. M. (2014). Self-excited oscillations and fuel control of a combustion process in a Rijke tube. *International journal of nonlinear sciences and numerical simulation*, *15*(2), 87–106.

Reichhartinger, M., & Horn, M. (2009). Application of higher order sliding-mode concepts to a throttle actuator for gasoline engines. *IEEE Transactions on Industrial Electronics*, *56*(9), 3322–3329.

Ríos, H., Punta, E., & Fridman, L. (2016). Fault detection and isolation for nonlinear non-affine uncertain systems via sliding-mode techniques. *International Journal of Control*, 1–13.

Ríos, H., & Teel, A. R. (2018). A hybrid fixed-time observer for state estimation of linear systems. *Automatica*, *87*, 103–112.

Rodriguez, A., De Leon, J., & Fridman, L. (2009). Synchronization in reduced-order of chaotic systems via control approaches based on high-order sliding-mode observer. *Chaos, Solitons & Fractals*, *42*(5), 3219–3233.

Ruscak, S., & Singer, L. (1995). *Using histogram techniques to measure a/d converter noise* (Vol. 29) (No. 2).

Ryzhii, E., & Ryzhii, M. (2014). A heterogeneous coupled oscillator model for simulation of ECG signals. *Computer Methods and Programs in Biomedicine*, *117*(1), 40–49.

Salgado, I., Camacho, O., Yáñez, C., & Chairez, I. (2014). Proportional derivative fuzzy control supplied with second order sliding mode differentiation. *Engineering Applications of Artificial Intelligence*, *35*, 84–94.

Salgado, I., Chairez, I., Camacho, O., & Yañez, C. (2014). Super-twisting sliding mode differentiation for improving pd controllers performance of second order systems. *ISA transactions*, *53*(4), 1096–1106.

Salgado, I., Cruz-Ortiz, D., Camacho, O., & Chairez, I. (2017). Output feedback control of a skid-steered mobile robot based on the super-twisting algorithm. *Control Engineering Practice*, *58*, 193–203.

Shtessel, Y. B., & Poznyak, A. S. (2005). Parameter identification of affine time varying systems using traditional and high order sliding modes. In *Proceedings of the American Control Conference* (Vol. 4, p. 2433).

Sidhom, L., Brun, X., Smaoui, M., Bideaux, E., & Thomasset, D. (2015). Dynamic gains differentiator for hydraulic system control. *Journal of Dynamic Systems, Measurement, and Control*, *137*(4), 041017.

Simon, D. (2006). *Optimal state estimation: Kalman, H infinity, and nonlinear approaches.* John Wiley & Sons.

Sinha, M., Dorfler, F., Johnson, B. B., & Dhople, S. V. (2016). Synchronization of Lienard-type oscillators in uniform electrical networks. In *2016 American Control Conference (ACC)* (p. 4311-4316).

Spurgeon, S. K. (2008). Sliding mode observers: a survey. *International Journal of Systems Science*, *39*(8), 751–764.

Vasiljevic, L. K., & Khalil, H. K. (2008). Error bounds in differentiation of noisy signals by high-gain observers. *Systems & Control Letters*, *57*(10), 856–862.

Vázquez, C., Aranovskiy, S., Freidovich, L. B., & Fridman, L. M. (2016, Sept). Time-varying gain differentiator: A mobile hydraulic system case study. *IEEE Transactions on Control Systems Technology*, *24*(5), 1740-1750. doi:

Wu, J., Long, J., Liang, Y., & Wang, J. (2004). Numerical differentiation based algorithms for frequency estimation of multiple signals. In *Southeastcon, 2004. Proceedings. IEEE* (pp. 251–254).

Yan, X., Primot, M., & Plestan, F. (2014). Comparison of differentiation schemes for the velocity and acceleration estimations of a pneumatic system. *IFAC Proceedings Volumes*, *47*(3), 49–54.

Yan, X.-G., & Edwards, C. (2008). Adaptive sliding-mode-observer-based fault reconstruction for nonlinear systems with parametric uncertainties. *IEEE Transactions on Industrial Electronics*, *55*(11), 4029–4036.