# DOCTOR OF PHILOSOPHY

## Krylov Subspace Model Order Reduction for Nonlinear and Bilinear Control Systems

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# Abstract

The use of Krylov subspace model order reduction for nonlinear/bilinear systems, over the past few years, has become an increasingly researched area of study. The need for model order reduction has never been higher, as faster computations for control, diagnosis and prognosis have never been higher to achieve better system performance. Krylov subspace model order reduction techniques enable this to be done more quickly and efficiently than what can be achieved at present.

The most recent advances in the use of Krylov subspaces for reducing bilinear models match moments and multimoments at some expansion points which have to be obtained through an optimisation scheme. This therefore removes the computational advantage of the Krylov subspace techniques implemented at an expansion point zero.

This thesis demonstrates two improved approaches for the use of one-sided Krylov subspace projection for reducing bilinear models at the expansion point zero. This work proposes that an alternate linear approximation can be used for model order reduction. The advantages of using this approach are improved input-output preservation at a simulation cost similar to some earlier works and reduction of bilinear systems models which have singular state transition matrices.

The comparison of the proposed methods and other original works done in this area of research is illustrated using various examples of single input single output (SISO) and multi input multi output (MIMO) models.

# Acknowledgements

Firstly, I would like to express my gratitude towards my former director of studies and supervisor Prof Keith J. Burnham. Also I would like to thanks my supervisors and current director of studies Dr Olivier Haas, Dr Malgorzata Sumisławska, Dr Othman Maganga and Dr Dina Laila for their support over all the years of my study.

I would also like to acknowledge all the staff of the Control Theory and Applications Centre (CTAC) who have taught me over the years and have inspired me to work hard to achieve my goals.

# Dedication

This work is dedicated to my family. My dad Mr Festus Oni Agbaje, my mum Deaconess Oluyemi Agbaje. Also to my siblings Mrs Opeyemi Ovabore, Olumide Babafemi Agbaje and Eyitayo Omoniyi Agbaje.

I would also like to dedicate this work to God Almighty, who is a constant in my life.

# Glossary of nomenclature

## Abbreviations

| | |
|---|---|
| BIBO | Bounded Input Bounded Output |
| BIRKA | Bilinear Iterative Rational Krylov Algorithm |
| BT | Balanced Truncation |
| FA | Firefly Algorithm |
| FB | Feng and Benner |
| HOBM | Higher Order Bilinear Model |
| IAE | Integral of Absolute Error |
| IP | Improved Phillips |
| IRKA | Iterative Rational Krylov Algorithm |
| KSBA | Krylov Subspace Bilinear Approximation |
| MIMO | Multiple Input Multiple Output |
| MOR | Model Order Reduction |
| MSE | Mean Square Error |
| NIAE | Integral of Absolute Error Divided by Number of Samples |
| P | Phillips |
| PID | Proportional, Integral, Differential |
| PLA | Parametrised Linear Approximation |
| POD | Proper Orthogonal Decomposition |
| PSO | Particle Swarm Optimisation |

QA . . . . . . . . . . Quadratic Approximation
RC . . . . . . . . . . Resistance/Capacitance
ROM . . . . . . . . Reduced Order Model
RSS . . . . . . . . Residual Sum of Squares
SE . . . . . . . . . . Squared Error
SPARK . . . . . . Stability Preserving Adaptive Rational Krylov
SISO . . . . . . . . Single Input Single Output
SPA . . . . . . . . . Singular Perturbation Approximation
SPM . . . . . . . . Solar Panel Model
SQP . . . . . . . . . Sequential Quadratic Optimisation
SSE . . . . . . . . Sum of Square of Error
SSR . . . . . . . . . Sum of Squared Residuals
ST . . . . . . . . . . Simulation Time
SVD . . . . . . . . . Singular Value Decomposition
TBIRKA . . . . Truncated Bilinear Iterative Rational Krylov Algorithm
TBR . . . . . . . . . Truncated Balanced Realisation
TPWL . . . . . . . Trajectory Piece-wise Linear

# Notation

| | |
|---|---|
| $a_i$ | $i^{th}$ coefficient of a polynomial series |
| $c_i$ | $i^{th}$ member of a set of constants |
| $f(x)/f$ | Nonlinear function |
| $g/g(v)$ | Nonlinear resistor |
| $g(x)$ | Nonlinear input function |
| $h$ | Nonzero members of a Hessenberg matrix which is formed during the Arnoldi process |
| $h_n(\sigma_1, \ldots, \sigma_n)$ | Impulse response also known as a kernel of a Volterra series |
| $m$ | Number of columns of input matrix/Number of columns of a second starting matrix/Number of bilinear state matrices (bilinear terms) |
| $m(l)$ | Moment |
| $m(l_1, l_2, \ldots l_i)$ | Multimoment |
| $\hat{m}(l)$ | Moment of a reduced order model |
| $\hat{m}(l_1, l_2, \ldots l_i)$ | Multimoment of a reduced order model |
| $n$ | State space dimension/Number of states |
| $ns$ | Number of simulations |
| $ns$ | Number of samples |
| $p$ | Number of columns of output matrix |
| $p_2$ | Parameter which determines the amount of columns of $V^{\{1\}}$ are used for computing $V^{\{2\}}$ |
| $p_{(i)}$ | Appropriate parameters for achieving orthogonality |
| $q_i/q$ | Dimension of a Krylov subspace/Projection matrix algorithm parameter for computing $V^{\{i\}}$/Columns of the matrix $Q$ |
| $r/r_i$ | Real number/Columns of the matrix $R$ |
| $r_{(i)}$ | Appropriate parameters for achieving orthogonality |
| $r_{ij}$ | Members of the matrix $R$ when performing the QR factorisation |
| $s$ | Continuous time variable |
| $s_i$ | $i^{th}$ continuous time variable |
| $\bar{s}_i$ | Member of a subspace $\mathbb{S}$ |
| $t$ | Time |
| $u$ | System input |
| $u_i$ | $i^{th}$ system input |

| | |
|---|---|
| $v$ | Voltage |
| $v_i$ | column vectors which are member of $\mathbb{V}$ or $V$/Voltage at node $i$ |
| $w_i$ | Weights for weighted combination using Trajectory piece-wise linear model order reduction |
| $x(t)/x$ | State vector |
| $\hat{x}(t)/\hat{x}$ | Reduced order state vector |
| $\dot{x}(t)/\dot{x}$ | State derivative |
| $\dot{\hat{x}}$ | Reduced order derivative of system state |
| $\hat{x}_i$ | Expansion point for trajectory piecewise approximation |
| $x^{(i)}$ | Kronecker product of the state up to the $i^{th}$ term |
| $\dot{x}^{(i)}$ | Derivative of $x^{(i)}$ |
| $x$ | State vector of the Carleman bilinearisation process |
| $\dot{x}$ | State vector derivative of the Carleman bilinearisation process |
| $y$ | System output |
| $y_i$ | $i^{th}$ System output |
| $\hat{y}$ | System output of reduced order model |
| $A$ | System matrix |
| $\hat{A}$ | Reduced order system matrix |
| $\hat{A}_i$ | Reduced order linear approximations of a nonlinear function at multiple points |
| $A_i$ | $i^{th}$ Derivative of a nonlinear function |
| $A$ | Resulting system matrix of Carleman bilinearisation process |
| $A_{i,k}$ | Member of $A$ |
| $A_\eta$ | State transition matrix of a linear approximation of bilinear system for a constant input |
| $B$ | Input matrix |
| $B_\eta$ | Input matrix for parametrised linear approximation of a bilinear model |
| $\hat{B}$ | Reduced order input matrix |
| $B$ | Resulting input matrix of Carleman bilinearisation process |
| $B_i$ | $i^{th}$ derivative of the nonlinear function $B(x)$ |
| $B_{i,k}$ | Member of $N$ |
| $C$ | output matrix |
| $\hat{C}$ | Reduced order output matrix |
| $C$ | Resulting output matrix of carleman bilinearisation process |

| | |
|---|---|
| $G_i$ ........... | $i^{th}$ derivative of $g(x)$ |
| $H(s)$ ......... | Transfer function |
| $\hat{H}(s)$ ......... | Transfer function of reduced order model |
| $I$ ............. | Identity matrix of appropriate dimensions |
| $I_m$ ........... | Identity matrix of dimensions |
| $K_q$ ........... | $q^{th}$ Krylov subspace |
| $\mathbb{M}$ ............ | Second starting matrix of a Krylov subspace |
| $N$ ............ | Bilinear state matrix for SISO bilinear system/model |
| $\mathbb{N}$ ............ | First starting matrix of a Krylov subspace |
| $\hat{N}_i$ ........... | $i^{th}$ reduced order bilinear state matrix |
| $N_i$ ........... | $i^{th}$ bilinear state matrix |
| $N$ .......... | Resulting bilinear system state matrix of carleman bilinearisation process |
| $P$ ............ | Observability Gramian |
| $\bar{Q}$ ............ | Controllability Gramian |
| $Q$ ............ | Set of vectors which form part of the QR factorrisation process |
| $R$ ............ | A matrix with nonzero members which form part of the QR factorisation / Residual |
| $\mathbb{R}$ ............ | Set of real numbers |
| $\mathbb{S}$ ............. | Subspace |
| $U(s)$ ......... | Laplace transform of the input of a linear system |
| $V$ ............ | Right projection matrix |
| $V^{\{k\}}$ ......... | Basis of the $kth$ Krylov subspace used to compute the projection matrix $V$ for MOR of a bilinear model |
| $\mathbb{V}$ ............ | A set of linearly independent vectors in a subspace $\mathbb{S}$ |
| $W$ ........... | Projection matrix formed using second Krylov subspace for two sided projection |
| $\bar{W}$ ........... | Quadratic function of states |
| $X(s)$ ......... | Laplace transform of the state of a linear system |
| $Y(s)$ ......... | Laplace transform of the output of a linear system |
| $\mathbb{Z}$ ............ | Set of positive integers |
| $\infty$ ............ | Infinity |
| $\eta/\eta_i$ .......... | Linear approximation parameter/Set of linear approximation parameters |
| $\tau$ ............. | A small interval of time |
| $\eta$ ............. | Linear approximation parameter |

| | |
|---|---|
| $\delta_i$ ............. | Expansion points |
| $\pi$ .............. | pi |
| $\lambda_i(A)$ ......... | $i^{th}$ eigenvalue of $A$ |

The above notation stands except when stated otherwise within the text of this thesis.

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# Chapter 1

# Introduction

Models in their most fundamental form are considered approximations of the real world. These can be abstract or physical in form. Models can be elements or amalgamations of elements to describe a system or a process describing a phenomenon under scrutiny. Models can be classified broadly as linear and nonlinear but the vast majority of systems are nonlinear in nature.

The increasing complexity of systems in automotive/aeronautic vehicles, manufacturing and energy installations has led to the requirement of sophisticated management systems for control, condition monitoring, diagnostic and prognostic purposes especially for systems which require safety and economic viability. Not only is it important to accurately control these systems to deliver the expected performance but also identify and diagnose system faults to prevent avoidable breakdown scenarios. In addition, the need for increased reliability has put more emphasis on estimation of failure modes, state of health and remaining useful life. Therefore models for control, diagnostics and prognostics are now more than ever essential in order to meet these requirements. However, models for control diagnostics and prognosis are required to be computationally efficient for online implementation whilst retaining enough characteristics to describe the systems behaviour. This has led to the need for model order reduction techniques that allow the resulting models to run online and also contain sufficient accuracy for the purpose of control, diagnostics and prognostics.

There is no fixed method or rule for model order reduction for a specific system but rather several options or paths that can be taken. The methodologies used often depend on the particular application and availability of a priori information. For instance, one can derive a reduced order model from an existing high fidelity/high order model using mathematical manipulation based approaches or in the case where there is no first principle model available, one can identify a model from data driven approaches. In addition to the possibility of different approaches, there are also the questions of how accurate the model needs to be, how it will be implemented online and what structure it needs to have.

Classical methods for model order reduction (MOR) are based on mathematical manipulation of the higher order model. These methods are used to project a high dimensional (high fidelity) model to a low dimensional (low fidelity) model, while preserving necessary dynamics and reasonable accuracy of the original system. There are different approaches of obtaining reduced order model (ROM) via mathematical manipulation such as Krylov subspace based, truncation based and methods based on proper orthogonal decomposition (POD). MOR techniques by Arnoldi (Arnoldi 1951), Lanczos (Lanczos 1950) and Moore (Moore 1981) are some of the original methods proposed for linear time-invariant models. Model order reduction techniques for linear systems can be classified into two categories: moment matching and Singular Value Decomposition (SVD) based approaches. They include Balanced Truncation, Krylov subspace moment matching methods, H2-norm MOR (Gugercin, Antoulas & Beattie 2008) and singular perturbation approximation (SPA). These methods have been well researched, with various extensions as documented in (Tan & He 2007, Liu & Anderson 1989, Kumar, Tiwari & Nagar 2011, Lohmann & Salimbahrami 2000). MOR for nonlinear systems is still a relatively open area of research. Most of the methods developed for linear systems have been extended to nonlinear systems. The approaches most popular today have been proposed for weakly nonlinear systems. These approaches include quadratic (Chen, White et al. 2000), piecewise-linear MOR (Bond & Daniel 2007, Rewieński & White 2003) and Krylov subspace-based MOR for nonlinear systems via Carleman bilinerarisation (Phillips 2000). The advantages of energy function based approaches for reduction of linear models, such as Balanced Truncation and $H_2$-norm approaches, are not easily transferred to nonlinear cases.

In systems engineering, control engineering and automotive applications, research into reduced order models and methods for achieving them are increasingly popular. The main objective for reduced order modelling is to preserve the input-output characteristics of a higher order model. There has not been much emphasis placed on the amount of effort required to achieve this. At the end of the day, in most cases it is not of much concern. However, this is a problem which has been raised in certain literature (Aizad, Sumisławska, Maganga, Agbaje, Phillip & Burnham 2014, Baur, Benner & Feng 2014). In an ideal case, it will be useful to achieve reduced order models at minimum cost whilst achieving an optimum input-output criteria.

Most of the linear MOR methods discussed so far are readily applicable to bilinear models. Unfortunately the peculiar disadvantages which apply to linear cases are also carried over to bilinear cases. Krylov subspace techniques are often preferred for computational efficiency and their ability to compute very large matrices. Balanced Truncation, unlike Krylov subspace MOR is not suitable for models with very large system matrices. It is therefore intuitive to find a way of exploiting the advantages. The most recent advances in the use of Krylov subspaces for reducing bilinear models, match moments and multimoments at some frequencies which have to be obtained iteratively (Choudhary & Ahuja 2016, Breiten & Damm 2010, Benner & Breiten 2012*a*, Benner & Breiten 2012*a*). However, in (Breiten & Damm 2010), this approach has been said to require improvement. Hence the iterative nature of finding the expansion points. In (Benner & Breiten 2012*a*), the Bilinear Iterative Rational Krylov Algorithm (BIRKA) has been proposed. The algorithm solves the interpolation problem in an optimal way i.e. a search of expansion points that match a suitable tolerance. This therefore removes the computational advantage of the Krylov subspace techniques as implemented by (Phillips 2000, Feng & Benner 2007, Condon & Ivanov 2007, Bai & Skoogh 2006), especially, when solving very large matrices. Other variants of the BIRKA such as the Truncated Bilinear Iterative Rational Krylov Algorithm (TBIRKA) (Benner & Breiten 2012*a*) also have these limitations.

However, it is interesting to note that a combination of methods as has been discussed in (Tan & He 2007) brings about more possibilities for reduced order modelling practitioners. This has also been discussed in (Benner & Damm 2011) and provide good prospects for the future. For most work done on hybrid approaches, the TBR approaches are combined with Krylov subspace MOR techniques. Other aspects of hybrid approaches are the combination of data based approaches with classical approaches (Saragih 2014).

In this thesis, the focus is on the reduced order modelling for bilinear systems utilizing Krylov subspace multimoment matching methods which match multimoments at an expansion point of zero. Bilinear form is a subset of nonlinear models and is a good approximation of nonlinear behaviour (Rugh 1981, Phillips 2000).

## 1.1 Motivation and problem statement

The motivation of this work is inspired by a series of issues which have been raised over the years. In the work done by (Phillips 2000) and (Feng & Benner 2007), two one-sided projection techniques have been proposed to match the moments and multimoments bilinear models. Other authors (Breiten & Damm

2010, Wang & Jiang 2013) have proposed a two-sided technique for improving the input-output relationship of reduced models. This implies that two Krylov subspaces are used therefore utilising twice the effort for a one sided approach. In (Bai & Skoogh 2006), a matrix inversion approach was proposed to match more moments of a bilinear model. This was shown to produce a better input-output relationship when compared to the approach presented in (Phillips 2000). However, the matrix inversion produces an awkward projection which cannot be regarded as one-sided approach and adds the need for more computational effort. Also, due to the computation of Krylov subspaces in (Feng & Benner 2007), there is often a multiplication of system matrices with singular bilinear state matrices and this could lead to loss of information and effectiveness of these techniques.

Matrix inversions are not always possible and additionally, it is done at an extra computational effort. Also, projection of matrix dimensions is done in such a way that each moment and multimoment is matched at a point in the projection subspace and because this needs to match multimoments, it presents a lack of flexibilty in the application and therefore quality of reduced order models. The aim of this work is therefore to propose two techniques which are computationally efficient, promote flexibility and also enable the reduction of models with noninvertible system matrices. This will improve the input-output characteristics of the reduced order models and expand the scope of its implementation.

## 1.2 Methodology and thesis outline

### 1.2.1 Methodology

The thesis will be based on the original works done in this field and a mathematical analysis of those original works. Based on the analysis of these methods, new approaches are proposed. Also, the algorithms to be used will be similar in order

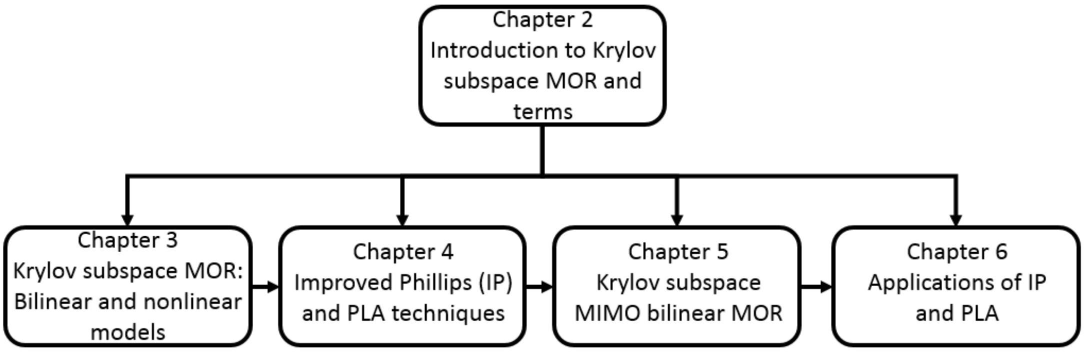



Figure 1.1: Schematic flow of thesis progression

to highlight the salient differences for comparison. An experiment which defines a problem, in this case a higher order model, is to be used for each method reviewed or proposed. Input and output data are collected for each simulation and each method is analysed with a predefined performance criteria. Also simulation time is of interest in each case study.

The chapters progress in such a way that each method is first applied to a single input single output (SISO) model, followed by their extension to multiple input multiple output (MIMO) models. The proposed methods are then applied in special cases which show the significance of the work done in this thesis.

### 1.2.2 Thesis outline

The thesis outline is as follows:

**Chapter 2:** This chapter forms the background from which all the concepts proposed in this study are based. A literature review is presented to provide a summary to the state of the art research in this field, and cover the basic ideas of model order reduction using Krylov subspaces considering a linear system model structure for illustration. The chapter reviews moment matching and the definition of moments and Krylov subspaces. Also the chapter revises the linear algebra concepts of subspaces, linear dependence and orthogonalisation. The Arnoldi process which is quite useful for computing projection matrices utilised in one sided Krylov subspace model order reduction is reported and the stability of Krylov subspace techniques is discussed.

A literature review of the utilization of Krylov subspaces for linear and nonlinear MOR is presented. An overview of the use of Krylov subspace techniques for the reduction of nonlinear models via bilinearisation, quadratic approximation and piecewise linear approximation is provided.

**Chapter 3:** An introduction to bilinear models is presented in this chapter. The chapter narrows this discussion into the approximation of nonlinear models through a bilinearisation process called Carleman bilinearisation. This process makes it possible to extend Krylov subspace model order reduction to nonlinear systems via its bilinear approximation. The Carleman bilinearisation process is discussed and this is followed by the review of model order reduction through projection for bilinear models as proposed in literature.

The various Krylov subspaces proposed over the years are discussed and this chapter also features the Taylor series expansion, uses of bilinear models and an algorithm for computing projection matrices. A literature review into the extension of Krylov subspace model order reduction techniques to MIMO bilinear models is presented. The chapter concludes with a discussion.

**Chapter 4:** This chapter presents and highlights the original contribution of this thesis. It contains an analysis of the one-sided projection techniques proposed in (Phillips 2000, Feng & Benner 2007) via their moment matching capability and the multimoments matched are analysed mathematically. This mathematical analysis forms part of the novelty of this thesis. This chapter improves on the work done in literature which has been reported in Chapter 3. A new method called an Improved Phillip type projection is presented followed by a mathematical analysis. Also a second approach for the application of Krylov subspaces to the model order reduction problem of bilinear models is presented. This is called the parametrised linear approximation (PLA) method.

A numerical simulation analysis of an alternate linear approximation for bilinear models is done and the results are discussed. This has been compared to the traditional linear approximation as used by the other Krylov subspace methods described in Chapter 3. This alternate linear approximation forms the foundation for the parametrised linear approximation approach which is the second proposal of this chapter.

This chapter also presents a simulation based study which helps to identify the parameters of an algorithm for computing Krylov subspace projection matrices. These parameters are then used to compute reduced order models from all the methods described in Chapters 3 and 4. Two examples from literature have been used to demonstrate these Krylov subspace model order reduction methods.

**Chapter 5:** More original contributions, focusing on bilinear systems, are presented in this chapter. The extension of the methods proposed in Chapter 4 are applied to MIMO case studies. The presentation starts with a literature review into the extension of Krylov subspace MOR techniques to MIMO bilinear models. Also, a mathematical analysis which is an extension of the mathematical analysis done in Chapter 4 are given for MIMO bilinear models.

This chapter contributes to the extension of the Feng and Benner (Feng & Benner 2007), Improved Phillips type projection and the parametrised linear approximation (PLA) approaches to the MIMO cases. Two arbitrary bilinear models have been used to illustrate the application of the proposed results. These are to be compared with the other reviewed methods and the advantages of the newly proposed methods are analysed.

**Chapter 6** This chapter provides two applications of the techniques proposed in Chapters 4 and 5. A hybrid MOR technique for SISO and MIMO bilinear systems/models is introduced. This combines the techniques for parameter estimation, artificial intelligence and optimisation to optimise the parameters used for computing the parametrised linear approximation for reduced order modelling via Krylov subspace MOR. The second application is the use of PLA for MOR of a pseudo-singular bilinear system. Pseudo-singular bilinear models have been defined therein.

Using these applications, two case studies have been presented to show the unique implications of the techniques proposed in this thesis. The numerical simulations have been analysed using plots and a set of performance criteria.

**Chapter 7:** This chapter provides an overall conclusion of the works reported in the thesis. The avenues for further work are also presented.

## 1.3 Contributions

In summary, the contributions of this research are given as follows:

1. The matching of a higher number of multimoments whilst avoiding the multiplication of nonsingular matrices. This has been called the Improved Phillip type projection (Chapter 3).

2. The proposal of a reduced order modelling approach using Krylov subspaces by applying a so called better linear approximation. This approach is called the Parametrised Linear Approximation (PLA).

3. The analysis of multimoment matching for the Feng and Benner type projection (Feng & Benner 2007), J. R. Phillip type projection (Phillips 2000) and the Improved Phillip type projection.

4. The extension of the Improved Phillip type projection, Parametrised Linear Approximation projection and the Feng and Benner type projection (Feng & Benner 2007) to MIMO cases.

5. The analysis of multimoment matching for MIMO bilinear model reduction using Krylov subspaces.

6. The use of PLA for the reduced order modelling of pseudo-singular bilinear systems to enable the reduction of systems with nonsingular system matrices.

7. The use of an optimisation scheme for finding parameters which form an alternate linear approximation of a bilinear system/model and the use of these parameters for model order reduction.

This thesis also served as a resource for understanding and practical implementation of reduced order modelling using Krylov subspaces.

## 1.4 List of publications

During the period of study, some publications have been made under the guidance of my supervisors and collaboration with other researchers. The publications cover a wide range of techniques for producing reduced order models via data based approaches and classical methods. These publications are listed below.

1. Agbaje, O, Kavanagh, D., Sumisławska, M., Howey, D., McCulloch, M. & Burnham, K., Estimation of temperature dependent equivalent circuit

parameters for traction-based electric machines, in *Hybrid and Electric Vehicles Conference 2013 (HEVC 2013)*, IET, pages 1-6, 2013.

2. Aizad, T., Sumisławska, M., Maganga, O., Agbaje, O., Phillip, N. & Burnham, K. J., Investigation of model order reduction techniques: A supercapacitor case study, in 'Advances in Systems Science', Springer, pages 795–804, 2014.

3. Sumisławska, M., Agbaje, O., Kavanagh, D. F. & Bumham, K. J., Equivalent circuit model estimation of induction machines under elevated temperature conditions, in 'UKACC International Conference on Control (CONTROL2014)', pages 413–418, 2014.

The knowledge gained from linear projection technique which has been investigated in (Aizad et al. 2014) has been expanded on for nonlinear systems and forms the focus of the research and original results presented in this thesis. Data based techniques have been used in (Agbaje, Kavanagh, Sumislawska, Howey, McCulloch & Burnham 2013, Sumisławska, Agbaje, Kavanagh & Burnham 2014) to estimate the parameters of a low order equivalent circuit model at extreme conditions. Some of these techniques such as the use of identifiability analysis, parameter estimation and weighted optimisation have been used to optimise the results presented in this thesis.

# Chapter 2

# Mathematical Background Preliminaries

## 2.1 Definition of terms

**Definition 2.1.1 (Moment)** *Moments have been defined as the coefficients of a Taylor series expansion (Tan & He 2007). Consider a continuous-time system with input $u$ and output $y$. The transfer function $H(s)$ describing the system behaviour is represented as*

$$H(s) = \frac{y(s)}{u(s)}. \tag{2.1}$$

*Then the Taylor series expansion of the transfer function at the expansion point $s = 0$ is defined as*

$$H(s) = \sum_{l=0}^{\infty} m(l)s^l \tag{2.2}$$

*where the moments, $m(l)$, are defined as*

$$m(l) = \frac{1}{l!} \times \frac{d^l H(s)}{ds^l}|_{s=0} \tag{2.3}$$

*i.e. the moments are defined from the corresponding derivatives of the transfer function with respect to $s$.*

Note that the moments can be defined for expansion points other than zero but in this thesis the methods described are only for moments for $s = 0$. Also in subsequent chapters, this same expansion point is considered when multimoments are discussed. However, moment matching for expansion point other than zero have been considered in literature and are quite easily derived (Salimbahrami & Lohmann 2002).

**Definition 2.1.2 (Multimoments)** *Multimoments are the coefficients of the Taylor series expansion for a multivariable transfer function.*

**Definition 2.1.3 (Krylov subspace)** *The $q^{th}$ Krylov subspace is defined as*

$$K_q(\mathbb{N}, \mathbb{M}) = span\{\mathbb{N}^0\mathbb{M}, \mathbb{N}^1\mathbb{M}, ..., \mathbb{N}^{q-1}\mathbb{M}\} \tag{2.4}$$

*where $\mathbb{N} \in \mathbb{R}^{n\times n}$, $\mathbb{M} \in \mathbb{R}^{n\times m}$ and $q, n, m \in \mathbb{Z}$. $\mathbb{N}$ and $\mathbb{M}$ are known as the starting matrices and they form the basis of the Krylov subspace. When considering a single input single output (SISO) system, $\mathbb{M}$ would be a vector and consequently, $m = 1$. Moreover, for multi input multi output (MIMO) systems, $m > 1$.*

**Definition 2.1.4 (Subspace)** *Given that an $n$-vector is an $n \times 1$ matrix with real numbers as components and all $n$-vectors belong to the subset of vectors defined by $\mathbb{R}^n$ called the $n$-space. A subspace can then be defined as a set of $\mathbb{R}^n$ within which the following properties are inherent*

1. *If $\bar{s}_1$ and $\bar{s}_2$ are in $\mathbb{S}$, then $\bar{s}_1 + \bar{s}_2$ is in $\mathbb{S}$*

$$\bar{s}_1 \in \mathbb{S}, \bar{s}_2 \in \mathbb{S}, \bar{s}_1 + \bar{s}_2 \in \mathbb{S} \tag{2.5}$$

2. *If $r$ is any real number, and $\bar{s}_i$ is any vector in $\mathbb{S}$, then $r \times \bar{s}_i$ is in $\mathbb{S}$*

$$r \times \bar{s}_i \in \mathbb{S} \tag{2.6}$$

*For these conditions to hold, it is expedient that $\mathbb{S}$ is nonempty, i.e. a set that contains at least one component.*

**Definition 2.1.5 (Linear Independence)** *If $\mathbb{V}$ is a set of $m$ vectors in the subspace $\mathbb{S}$, $\mathbb{V}$ can be said to be linearly independent if it is not possible to find constants, $c_1, c_2, \ldots, c_m$ such that*

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_m = 0 \tag{2.7}$$

*where $v_1$, $v_2$, ..., $v_m$ are the column vectors in $\mathbb{V}$. This means that the vectors are only linearly independent when all the constants are zero i.e.*

$$c_1 = c_2 = \ldots = c_n = 0 \tag{2.8}$$

**Definition 2.1.6 (Span)** *The vectors in $\mathbb{V}$ can be said to span $\mathbb{S}$ or $\mathbb{S}$ is said to be spanned by $\mathbb{V}$ if every vector in $\mathbb{S}$ is a linear combination of the vectors in $\mathbb{V}$.*

$$\mathbb{S} = span\{\mathbb{V}\} \tag{2.9}$$

*This makes $\mathbb{V}$ a unique subset of $\mathbb{S}$. The vectors of $\mathbb{V}$ are called the basis of $\mathbb{S}$.*

**Definition 2.1.7 (Bounded input bounded input (BIBO) stability)** *A system is BIBO stable iff, for any bounded input, the output is bounded at all times given zero initial conditions.*

## 2.2 Model order reduction (MOR)

Given a model (2.1) of any structure which describes the input-output behaviour of a system. The model order reduction problem is to find another model which can be used to replace the former, where this new model is of a lower dimensional space (vectors, matrices, equations), less storage requirements and low evaluation/simulation time, it is said to be a reduced order model. The process for getting this reduced order model is called model order reduction (MOR).

Some MOR techniques preserve the structure of the higher order model and some do not. MOR techniques can broadly be divided into data based techniques and classical techniques which are based on mathematical manipulation. The later takes advantage of the mathematical properties of the models to derive reduced order models. Data based techniques (Sumisławska et al. 2014, Agbaje et al. 2013) will require taking input-output data, structure selection, data manipulation and processing techniques to achieve a reduced order model.

The approach taken for MOR will depend on the type of system considered. Classical model order reduction techniques have been used for both linear and nonlinear models. MOR for linear models forms a background for extending MOR to other model structures.

### 2.2.1 MOR for linear systems

Linear systems have been defined in general as systems that obey the laws of superposition (Nise 2007) and have simple structures. This simple structure lends itself for implementation of control and diagnostic algorithms. To introduce Krylov subspace MOR, consider a state space form of linear model,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.10}$$

$$y(t) = Cx(t) \tag{2.11}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are called the system matrix, input matrix and output matrix respectively and $n, p, m \in \mathbb{Z}$. For a single input single output (SISO) system model, $p$ and $m$ are equal to one. Several electrical circuits and some classes of mechanical systems can be represented in this form as described in (Feng & Benner 2007, Silveira, Kamon, Elfadel & White 1997, Freund 2000). The solutions derived here are easily realizable in other linear system formations.

Moment matching and Gramian based model order reduction methods are the most popular for reducing the order of large linear systems/models. Gramian based approaches such as Balanced Truncation (BT) guarantee stability and provide an error bound for the higher order model and reduced order model. Other

MOR techniques which are regarded as Gramian based are singular perturbation approximation (SPA) (Benner, Quintana-Orti & Quintana-Ortí 2000, Liu & Anderson 1986, Varga 1991, Aizad et al. 2014), Balanced stochastic truncation (Benner, Quintana-Ortí & Quintana-Ortí 2001), Frequency weighted balanced truncation (Gawronski & Juang 1990) and Hankel norm approximation (Glover 1984, Benner, Quintana-Ortí & Quintana-Ortí 2004). These approaches are referred to because they involve the balancing of the higher order model by computing the observability and controllability Gramians, $P$ and $Q$, respectively. For a linear system, they are defined mathematically as

$$P = \int_0^\infty e^{At} BB^T e^{A^T t} dt \tag{2.12}$$

$$\bar{Q} = \int_0^\infty e^{A^T t} CC^T e^{At} dt. \tag{2.13}$$

The observability and controllability Gramians can be computed by solving two Lyapunov equations

$$AP + PA^T + BB^T = 0 \tag{2.14}$$

$$A^T \bar{Q} + \bar{Q}A + C^T C = 0. \tag{2.15}$$

This forms the first step for the Gramian based approaches. Using the Gramians, a reduced order model can be obtained by using a balancing procedure followed by truncation. This is referred to as BT. Other variations of BT can be found in (Phillips, Daniel & Silveira 2003, Phillips & Silveira 2005, Reis & Stykel 2010, Benner 2010). As part of the procedure within this method, a set of linear equations need to be solved. The size of this set is the same as the dimension of the higher order model, thus the computational complexity of Gramian based approaches is $O(n^3)$, whilst the required storage is of order $O(n^2)$. Due to this disadvantage of BT, new and more efficient methods for solving large Lyapunov equations have been proposed (Jaimoukha, Kasenally & Limebeer 1992, Ahmad, Jaimoukha & Frangos 2010). These approaches use

Krylov subspaces to overcome the computational complexity of Gramian based approaches.

Moment matching methods refer to MOR approches which match moments of the higher order model and reduced order model such that the reduced order model is accurate up to a certain degree depending on the amount of moments matched. These are also known as Krylov subspace based approaches. Together with Gramian based approches (Antoulas & Sorensen 2001, Tan & He 2007), they have been classified as projection based MOR for linear systems in (Tan & He 2007).

## 2.3 Krylov subspace MOR methods

In cases where very large system matrices which are not suitable for certain applications are considered, it is necessary to reduce the order of the system states using techniques that are appropriate to meet a predetermined criteria. Krylov subspace algorithms introduced in (Arnoldi 1951) and (Lanczos 1950) are some of the original results proposed in this area. Improvements to these techniques and associated difficulties in their implementation are discussed in detail in (Lohmann & Salimbahrami 2000, Silveira et al. 1997, Odabasioglu, Celik & Pileggi 1997, Kerns, Wemple & Yang 1995). Krylov subspace based approaches have been described as being quite closely related to other projection techniques (Tan & He 2007). In order to introduce MOR via Krylov subspaces, it is useful to consider linear systems as a case study. Linear systems have been represented using the transfer function in the s domain where a transfer function is defined as the ratio of the input to the output, $H(s) = Y(s)/U(s)$. This can be derived from obtaining the Laplace transform of the state space representation (2.10) - (2.11). Using the relations $\dot{x}(t) \mapsto sX(s)$, $y(t) \mapsto Y(s)$ and $u(t) \mapsto U(s)$, we can write

$$sX(s) = AX(s) + BU(s) \tag{2.16}$$

$$(sI - A)X(s) = BU(s) \tag{2.17}$$

$$X(s) = (sI - A)^{-1}BU(s). \tag{2.18}$$

Therefore, the output $Y(s)$ can be expressed as

$$Y(s) = CX(s) \tag{2.19}$$

$$Y(s) = C(sI - A)^{-1}BU(s). \tag{2.20}$$

Furthermore, for SISO systems we can write

$$Y(s) = [C(sI - A)^{-1}B]U(s) \tag{2.21}$$

$$Y(s)/U(s) = C(sI - A)^{-1}B \tag{2.22}$$

$$H(s) = C(sI - A)^{-1}B. \tag{2.23}$$

By using the Taylor series expansion, the transfer function can be expressed as a polynomial function. The coefficients of this expansion at zero are called the moments of the transfer function

$$H(s) = \sum_{l=1}^{\infty} m(l)s^{l-1} \tag{2.24}$$

where $m(l)$ are the moments. When the expansion point is at infinity, the moments are called Markov parameters (Salimbahrami & Lohmann 2002). It is possible to find moments at different values of $s$. For the case described here, the moments are

$$m(l) = -CA^{-l}B. \tag{2.25}$$

Krylov subspace methods aim to obtain reduced order models in such a way that makes the moments of the reduced and higher order models equivalent.

### 2.3.1 One-sided projection

There are other projection methods. But this literature review focuses on Krylov subspace MOR which is closely related to other projection techniques. Krylov subspace MOR is known for its fast computational time when compared to other methods. Krylov subspace methods are well reported in (Celik, Pileggi & Odabasioglu 2002, Tan & He 2007).

In one sided projection, two techniques are generally used. In (Feng & Benner 2007) they have been refered to as the first projection technique and the second projection technique. They will be discussed here in that order. In the first projection technique as discussed in (Phillips 2000), the system matrix is not inverted prior to projection and the approximation $x = V\hat{x}$ is used, where $V$ is an orthonormal matrix of dimension $n \times q$ such that $V^TV = I$, where $q$ is much smaller than $n$, $q << n$. Substituting the approximation $x \approxeq V\hat{x}$, where $\hat{x}$ is the state of the reduced order model, and premultiplying (2.10) by $V^T$, results in a reduced order system with matrices of the form

$$\hat{A} = V^TAV, \hat{B} = V^TB, \hat{C} = CV. \tag{2.26}$$

The alternative method i.e. the second projection technique premultiplies both sides of the equation (2.10) with $A^{-1}$ prior to approximating the states $x$

$$A^{-1}\dot{x} = x + A^{-1}Bu. \tag{2.27}$$

Applying projection matrix $V$ to (2.27) and (2.11) by substituting the approximation $x \approxeq V\hat{x}$ and premultiplying (2.27) by $V^T$ gives

$$V^TA^{-1}V\dot{\hat{x}} = \hat{x} + V^TA^{-1}Bu \tag{2.28}$$

$$\hat{y} = CV\hat{x} \tag{2.29}$$

and premultiplying both sides of (2.28) by the inverse of $V^TA^{-1}V$, the resultant reduced order matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ are obtained as

$$\hat{A} = (V^TA^{-1}V)^{-1}, \hat{B} = (V^TA^{-1}V)^{-1}V^TA^{-1}B, \hat{C} = CV. \tag{2.30}$$

The first projection technique has also been used in (Odabasioglu et al. 1997) to preserve the stability and passivity of a system. It also matches as many moments as the second projection technique (Odabasioglu et al. 1997). However, the second technique cannot be said to be one sided due to the awkward definition of the projection matrices when compared to the first projection technique. Notice that the right projection matrix is not actually a transpose of the left projection matrix.

### 2.3.2 Moment matching

The moment matching properties of the one sided projection has been discussed in detail in (Tan & He 2007, Lohmann & Salimbahrami 2000). (Tan & He 2007) state that the one sided projection methods match $q$ moments whilst two sided methods match $2q$ moments. From Taylor series expansion of the reduced order model transfer function given below,

$$\hat{H}(s) = \sum_{l=1}^{\infty} \hat{m}(l) s^{l-1} \tag{2.31}$$

the moments of the reduced order model can be defined as

$$\hat{m}(l) = -\hat{C}\hat{A}^{-l}\hat{B}. \tag{2.32}$$

For the **first moment** of the reduced order model $\hat{m}(1) = -\hat{C}\hat{A}^{-1}\hat{B}$, note that $A^{-1}B$ belongs to the Krylov subspace $K_q(A^{-1}, A^{-1}B)$ and therefore can be written as $A^{-1}B = Vr_{(1)}$ and $B = AVr_{(1)}$ with $r_{(i)} \in \mathbb{R}^{q\times 1}$, $i = 1 \ldots q$, being a vector with parameters which make the statement true. From the definition of the moments for the higher and lower order models, moment matching can be proved as follows. Substituting the matrices (2.26) into (2.32), for $l = 1$, results in

$$\hat{m}(1) = -\hat{C}\hat{A}^{-1}\hat{B} \tag{2.33}$$

$$\hat{m}(1) = -CV(V^TAV)^{-1}V^TB. \tag{2.34}$$

Since $A^{-1}B = Vr_{(1)}$ and $B = AVr_{(1)}$. Substituting this into (2.34) yields

$$\hat{m}(1) = -CV(V^TAV)^{-1}V^TAVr_{(1)}. \tag{2.35}$$

Also, $(V^TAV)^{-1}V^TAV = I$ and $Ir_{(1)} = r_{(1)}$ therefore

$$\begin{aligned}\hat{m}(1) &= -CVr_{(1)}\\ &= -CA^{-1}B\\ \hat{m}(1) &= m(1).\end{aligned} \tag{2.36}$$

Likewise for the **second moment**, $m(2)$, the moment of the reduced order model can be defines as

$$\begin{aligned}\hat{m}(2) &= -\hat{C}(\hat{A}^{-1})\hat{A}^{-1}\hat{B}\\ &= -CV(V^TAV)^{-1}(V^TAV)^{-1}V^TB\\ &= -CV(V^TAV)^{-1}(V^TAV)^{-1}V^TAVr_{(1)}\\ &= -CV(V^TAV)^{-1}r_{(1)}.\end{aligned} \tag{2.37}$$

Using the orthogonality of the matrix V and some manipulations of matrix algebra, we write:

$$\begin{aligned}\hat{m}(2) &= -CV(V^TAV)^{-1}V^TVr_{(1)}\\ &= -CV(V^TAV)^{-1}V^TA(A^{-1})A^{-1}B.\end{aligned} \tag{2.38}$$

Defining $Vr_{(2)} := (A^{-1})A^{-1}B$, we obtain

$$\begin{aligned}\hat{m}(2) &= -CV(V^TAV)^{-1}V^TAVr_{(2)}\\ &= -CVr_{(2)}\\ &= -C(A^{-1})A^{-1}B\\ \hat{m}(2) &= m(2).\end{aligned} \tag{2.39}$$

Continuing further for the **third moment**, $m(3)$. The third moment of the reduced order model is

$$\begin{aligned}\hat{m}(3) &= -\hat{C}(\hat{A}^{-2})\hat{A}^{-1}\hat{B}\\ \hat{m}(3) &= -CV(V^TAV)^{-2}(V^TAV)^{-1}V^TB.\end{aligned} \tag{2.40}$$

Since $B = AVr_{(1)}$,

$$\begin{aligned}
\hat{m}(3) &= -CV(V^TAV)^{-2}(V^TAV)^{-1}V^TAVr_{(1)} \\
&= -CV(V^TAV)^{-2}r_{(1)} \\
&= -CV(V^TAV)^{-2}V^TVr_{(1)} \\
\hat{m}(3) &= -CV(V^TAV)^{-2}V^TA(A^{-1})A^{-1}B.
\end{aligned} \tag{2.41}$$

Also $A^{-2}B = Vr_{(2)}$, therefore,

$$\begin{aligned}
\hat{m}(3) &= -CV(V^TAV)^{-2}V^TAVr_{(2)} \\
&= -CV(V^TAV)^{-1}r_{(2)} \\
&= -CV(V^TAV)^{-1}V^TVr_{(2)} \\
\hat{m}(3) &= -CV(V^TAV)^{-1}V^TAA^{-1}Vr_{(2)}.
\end{aligned} \tag{2.42}$$

Since $Vr_{(2)} = A^{-2}B$ and $A^{-3}B = Vr_{(3)}$, therefore,

$$\begin{aligned}
\hat{m}(3) &= -CV(V^TAV)^{-1}V^TA(A^{-1})(A^{-1})A^{-1}B \\
&= -CV(V^TAV)^{-1}V^TAVr_{(3)} \\
&= -CVr_{(3)} \\
&= -C(A^{-2})A^{-1}B \\
\hat{m}(3) &= m(3).
\end{aligned} \tag{2.43}$$

As can be observed, the crucial step in proving the matching of moments is that the vector $(A^{-1})^qB = Vr_{(q)}$ belongs to the Krylov subspace $K_q(A^{-1}, A^{-1}B)$. From the proof of moment matching shown above, the following theorem suffices.

**Theorem 2.3.1** *Given a projection matrix, $V$, for the Krylov subspace $span(V) = K_q(A^{-1}, A^{-1}B)$, where $A$ is the system matrix and $B$ is the input vector, $q$ moments of the higher order model are matched by the reduced order model if the reduced order model matrices are formed such that $\hat{A} = V^TAV$, $\hat{B} = V^TB$, $\hat{C} = CV$ (Tan & He 2007).*

Note that the case described here is for moments at an expansion point zero where the matrix $V$ spans the Krylov subspace and can be computed using algorithms proposed by (Arnoldi 1951) and (Lanczos 1950). When $V$ is computed using one Krylov subspace and $V^TV = I$, this is referred to as a one-sided projection. There exists/are methods which use two Krylov subspaces such that $W^TV = I$ where $W$ spans the Krylov subspace, $K_q(A^{-T}, A^{-T}C)$. This implies that more computational effort is needed for computing the projection matrices.

Moments can also be matched at $\delta = \infty$. In this case, they are referred to as Markov parameters. The work done in (Grimme 1997) matches the first two moments using a two sided approach at multiple expansion points, $\delta_1, \delta_2, \ldots, \delta_k$, $k \in Z$, by using a rational Krylov algorithm. Improvements to this approach for selecting expansion points in an optimal manner have been reported in (Frangos & Jaimoukha 2007$a$).

The most recent work done in this area is inspired by (Grimme 1997) where rational Krylov algorithms are used for computing the projection matrices $V$ and $W$ where

$$Span(V) = K_q((A-\delta_1 I)^{-1}B, (A-\delta_2 I)^{-1}B, \ldots, (A-\delta_n I)^{-1}B) \tag{2.44}$$

$$Span(W) = K_q((A^T-\delta_1 I)^{-1}C^T, (A^T-\delta_2 I)^{-1}C^T, \ldots, (A^T-\delta_n I)^{-1}C^T) \tag{2.45}$$

In (Gugercin et al. 2008), an iterative rational Krylov algorithm (IRKA) for an optimal selection of the expansion points was proposed. The method is reported to be $H_2$ optimal as it is based on satisfying the $H_2$ error of the higher and reduced order model given by

$$||H-\hat{H}|| = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(jw) - \hat{H}(jw)|^2 dw}. \tag{2.46}$$

However, the preservation of stability is not guaranteed as the initial selection of the interpolation points is not clearly defined. The IRKA has been extended to several directions (Flagg, Beattie & Gugercin 2013, Druskin &

Simoncini 2011, Panzer, Jaensch, Wolf & Lohmann 2013). In (Flagg et al. 2013), a nearly optimal approach is developed. It uses Krylov subspaces for solving linear system of equations which are required for the IRKA and therefore is more efficient. An adaptive shift computation for selecting the expansion points has been developed in (Druskin & Simoncini 2011). This approach has been reported to be less accurate when compared to the IRKA but has less computational cost. In (Panzer et al. 2013), a stability preserving, adaptive rational Krylov (SPARK) algorithm was developed to guarantee stability and is $H_2$ optimal at the cost of being more expensive computationally when compared to the IRKA. Generally, methods based on the Krylov subspaces (2.44) - (2.45), match only one moment at each expansion point and lack flexibility for matching more moments. The $H_2$ optimal approach has been extended to apply to MIMO linear systems in (Van Dooren, Gallivan & Absil 2008, Van Dooren et al. 2008).

## 2.4 Algorithms

In this section, some algorithms that enable the computation of the projection matrices are presented. It is advantageous to build an orthogonal basis for the Krylov subspace by using numerically stable processes as the computation of V can be unstable as the dimension of the Krylov subspace $q$ gets large (Tan & He 2007). Here the processes for orthogonalisation and the Arnoldi process are discussed. The variations and genealogy of these algorithms are well reported (see for instance (Saad 2003)). Note that in this section, $q_i,\ i = 1, 2, \ldots,$ denotes the elements of the matrix $Q$ in the $QR$ factorisation.

### 2.4.1 Orthogonalisation

Two vectors are said to be orthogonal if the result of their product is zero. i.e.

$$v_1^T v_2 = 0. \tag{2.47}$$

For a set of vectors, this same rule is applied. The set is said to be orthogonal if all the component vectors follow this same rule

$$v_i^T v_j = 0, \tag{2.48}$$

where $i \neq j$ and $i, j \in \mathbb{Z}$. The set of vectors is orthonormal if, whilst being orthogonal, each vector component has a 2-norm of 1.

An orthonormal basis can be obtained by taking the basis of a subspace and orthonormalising them. This process of orthonormalisation and orthogonalisation is called the Gram-Schmidt process e.g. (Tan & He 2007, Saad 2003, Salimbahrami & Lohmann 2002, Daniel, Gragg, Kaufman & Stewart 1976).

For a set of $n$-vectors which are linearly independent, $(v_1, v_2, \ldots, v_n)$, the initial step of the Gram-Schmidt process is to normalize the first vector of this set by dividing it by its 2-norm. The resulting vector, $q_1$ is of norm 1. The following linearly independent vector, $v_2$ is then orthogonalised against $q_1$. This can be achieved by subtracting a multiple of $q_1$ from $v_2$ which makes the resulting vector orthogonal to $q_1$

$$v_2 \leftarrow v_2 - (v_2^T q_1) q_1 \tag{2.49}$$

The resulting vector is then normalised to obtain a vector $q_2$. This means that the Gram-Shmidt process orthonormalises any vector $v_j$ in the set against any previous vector $q_{j-1}$.

The Gram-Schmidt algorithm is presented below. In the algorithm, it is necessary that the matrices are linearly independent to prevent a break down of the process.

**Algorithm 2.1 (Gram-Schmidt process)**

1. ***Compute:*** $r_{11} = ||v_1||_2$, if $r_{11} = 0$ ***Stop,*** else ***Compute*** $q_1 = v_1/r_{11}$

2. for $j = 2 : n$
3. $r_{ij} = v_j^T q_i$ for $i = 1, 2, \ldots, j-1$
4. $\hat{q} = x_j - \sum_{i=1}^{j-1} r_{ij} q_i$
5. $r_{jj} = ||\hat{q}||_2$
6. if $r_{jj} = 0$ then ***Stop,*** else $q_j = \hat{q}/r_{jj}$
7. ***end***

There exist other versions of this algorithm which have been proposed by other authors to deal with loss of orthogonality in the process (Daniel et al. 1976). The Modified Gram-Schmidt algorithm is one of them and has been reported to have better numerical properties (Saad 2003) and is used in all the algorithms in this work.

**Algorithm 2.2 (Modified Gram-Schmidt process)**

1. ***Compute:*** $r_{11} = ||v_1||_2$, if $r_{11} = 0$ ***Stop,*** else ***Compute*** $q_1 = v_1/r_{11}$
2. for $j = 2 : n$
3. $\hat{q} = x_j$
4. for $i = 1, \ldots, j-1$
5. $r_{ij} = \hat{q}^T q_i$
6. $\hat{q} - r_{ij} q_i$
7. end

8. **Compute:** $r_{jj} = ||\hat{q}||_2$

9. if $r_{jj} = 0$ then **Stop,** else $q_j = \hat{q}/r_{jj}$

10. **end**

An alternative orthogonalisation method which uses a factorisation approach is the Householder's method (Golub & Van Loan 2012).

### 2.4.2 QR factorisation

As can be observed from Algorithm 2.1 steps 4 and 5, the relationship between the normalised and orthogonalised vectors at every step of the algorithm is

$$v_j = \sum_{i=1}^{j} r_{ij} q_i, \tag{2.52}$$

or in the matrix form:

$$\mathbb{V} = QR. \tag{2.53}$$

This is known as the $QR$ decomposition of the matrix $\mathbb{V}$, where $\mathbb{V}$ is a set of linearly independent vectors $[v_1, v_2, \ldots, v_n]$, $Q = [q_1, q_2, \ldots, q_n]$ and $R$ are non-zero elements, $r_{ij} \in \mathbb{R}$. The $QR$ factorisation is an inbuilt function in MATLAB and can be accessed by using the command ***orth***. This command has been used in Krylov subspace algorithms proposed in (Bai & Skoogh 2006, Lin, Bao & Wei 2009) and will also be used in this thesis.

### 2.4.3 Arnoldi process

To compute projection matrix $V$ where $V^T V = I$, Krylov subspace methods are utilised. The starting vectors of the Krylov subspace correspond to the system matrices and input vectors of the linear system. This is the case for one sided projection. In two sided projection, two Krylov subspaces are utilised. In (Lohmann & Salimbahrami 2000, Arnoldi 1951), the Arnoldi and Lanczos algorithms have been discussed in details.

Using the starting vectors of the Krylov subspace, the original Arnoldi algorithm (Arnoldi 1951) iteratively formulates a set of vectors with norm 1 which are orthogonal to each other. The result of the algorithm is the matrix $V$. Which is orthonormal. The algorithm as presented in (Tan & He 2007) is as follows:

**Algorithm 2.3 (Arnoldi algorithm)**

1. ***Input:*** $A, B, C, q$
2. ***Compute:*** $r = A^{-1}B$
3. ***Compute:*** $v_1 = r/||r||_2$
4. for $i = 1 : q - 1$
5. $\quad r = A^{-1}v_i$
6. $\quad h = (V_{[i]})^T r$
7. $\quad r = r - V_{[i]}h$
8. $\quad$ if $||r||_2 = 0$, ***end***
9. $\quad v_{i+1} = r/||r||_2$
10. ***end***
11. ***return*** $V$

The algorithm is easily modified to a multiple input multiple output case as documented in (Tan & He 2007). The outcome of the algorithm is the projection matrix $V$ where the vectors $v_i$ are the columns of $V$. A two sided Arnoldi algorithm has been developed in (Lohmann & Salimbahrami 2000) to match $2q$ moments at zero. Rational Arnoldi algorithms which match moments at multiple expansion points have been presented in (Frangos & Jaimoukha 2007*b*, Ruhe 1994).

## 2.5 Stability of Krylov subspace techniques

As mentioned earlier, MOR can sometimes results in unstable reduced models. While it is possible to simply discard unstable poles (Odabasioglu et al. 1997), several algorithms have been proposed to guarantee stability of the resulting models (Silveira et al. 1997, Kerns et al. 1995). In (Silveira et al. 1997), the system is said to be stable if all its eigenvalues have nonpositive real parts. Also given that the system matrix $A \in \mathbb{R}^{n\times n}$ is negative semidefinite, i.e.

$$p^T A p \leq 0, \tag{2.55}$$

where in this section, $p$ is an arbitrary non-zero vector of appropriate dimensions, it can be shown that the Arnoldi algorithm produces stable reduced order systems as follows:

$$p^T \hat{A} p \leq 0 \tag{2.56}$$

$$p^T V^T A V p \leq 0 \tag{2.57}$$

$$(Vp)^T A V p \leq 0. \tag{2.58}$$

In (Bond & Daniel 2007), an algorithm that utilises a Lyapunov function was proposed to ensure stable reduced order models for linear piecewise MOR approaches. For linear systems, a natural choice of a Lyapunov function is a quadratic function of the states

$$\bar{W} = x^T P x, \tag{2.59}$$

where $P \in \mathbb{R}^{n\times n}$ is some symmetric positive definite matrix that solves the algebraic equation

$$PA + A^T P = -Q \tag{2.60}$$

where $A$ is the state matrix of the system and $Q \in R^{n\times n}$ is a positive definite matrix, which often is chosen as the identity matrix. In order to achieve this, the left projection matrix and the matrices of the reduced linear model are defined as

$$U^T = (V^T PV)^{-1} V^T P \tag{2.61}$$

$$\hat{A} = U^T AV, \hat{B} = U^T B, \hat{C} = CV. \tag{2.62}$$

For the reduced order model (2.62), there exists a Lyapunov function, $\hat{W} = x^T \hat{P} x$ satisfying

$$\hat{P}\hat{A} + \hat{A}^T \hat{P} = -\hat{Q} \tag{2.63}$$

$$\hat{P} = V^T PV \tag{2.64}$$

which can be used to prove that the formulations (2.61) and (2.62) produce stable reduced order models. From (2.61), (2.62) and (2.64), we can write:

$$\begin{aligned}\hat{P}\hat{A} &= V^T PVU^T AV\\ &= V^T PV[(V^T PV)^{-1} V^T P]AV\\ \hat{P}\hat{A} &= V^T PAV.\end{aligned} \tag{2.65}$$

Using $\hat{A} = U^T AV$ from (2.62) and (2.64), $\hat{A}^T \hat{P}$ from (2.63) can be expressed as

$$\hat{A}^T \hat{P} = (U^T AV)^T V^T AV.$$

Based on the definition of $U^T$ in (2.61),

$$\begin{aligned}\hat{A}^T \hat{P} &= [(V^T PV)^{-1} V^T PAV]^T V^T PV\\ &= (V^T PAV)^T [(V^T PV)^{-1}]^T V^T PV\\ &= (V^T PAV)^T\\ \hat{A}^T \hat{P} &= V^T A^T PV\end{aligned} \tag{2.67}$$

Thus,

$$\begin{aligned}\hat{P}\hat{A} + \hat{A}^T\hat{P} &= V^T(PA + A^TP)V = -\hat{Q}\\ \hat{P}\hat{A} + \hat{A}^T\hat{P} &= -V^TQV\end{aligned} \tag{2.68}$$

Therefore, (2.63) is satisfied for a positive definite matrix $\hat{Q} = V^TQV$.

(Silveira et al. 1997) utilised a congruence argument to guarantee stability. Their work provides improvement to other work done earlier in (Kerns et al. 1995). In (Silveira et al. 1997) a computationally efficient Arnoldi algorithm is proposed for arbitrary and stable reduced order systems.

Some Krylov subspace approaches as proposed in (Bai & Freund 2001, Freund & Feldmann 1996, Freund & Feldmann 1997, Freund & Feldmann 1998, Kerns et al. 1995, Silveira et al. 1997) do guarantee stability and passivity (Odabasioglu et al. 1997). However, there have been cases where reduced order models derived from Krylov subspaces have resulted in unstable models (Bai & Skoogh 2006) which might be due to numerical approximations from solvers rather than from the method used.

### 2.5.1 MOR for nonlinear systems

Most of the techniques for reducing linear systems can be extended to apply to nonlinear systems by taking advantage of the Taylor series expansion of nonlinear models. Examples are the Carleman bilinearisation (Phillips 2000) and quadratic approximation (QA) as have been discussed in (Chen et al. 2000, Chen 1999). More recently, a trajectory piece-wise linear (TPWL) method and all its variants (Aizad et al. 2014, Rewieński & White 2003) have been proposed for the reduction of nonlinear systems. These approaches (QA and TPWL) have been used to reduce nonlinear models of the form

$$\dot{x} = f(x) + Bu \tag{2.69}$$

$$y = Cx \tag{2.70}$$

where $f$ is a nonlinear function such that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. The TPWL is quite unique when compared to other approaches

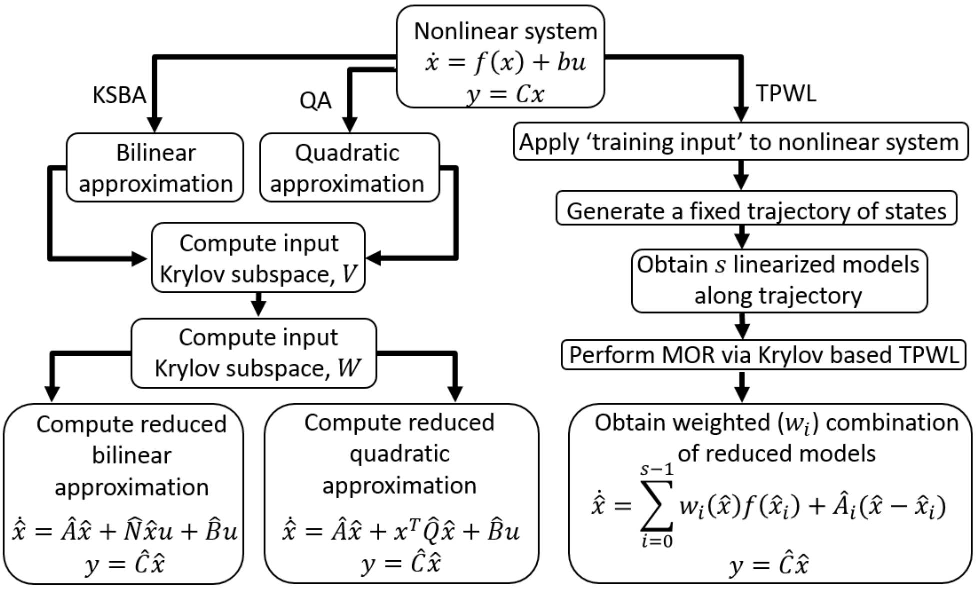



Figure 2.1: Procedure for MOR of nonlinear systems using Krylov subspace approaches.

which depend on the Taylor series expansion about a point of a nonlinear model (Carleman bilinearization and QA). The TPWL method was first proposed to overcome this limitation by using multiple linearisation points. However, it has been reported to have limited small-signal distortion and interpolation fidelity (Dong & Roychowdhury 2003). In (Dong & Roychowdhury 2003, Dong & Roychowdhury 2008) a method which takes advantage of the TPWL and polynomial representation of models which further increases its complexity was therefore presented. These nonlinear MOR methods are either based on moment-matching or BT. This is because the structure provided by bilinearisation, quadratic approximations and piece-wise linear approximation of the nonlinear model allows linear MOR techniques to be readily applicable. The MOR procedure for nonlinear systems using Krylov subspace approaches is outlined in Figure 2.1. Some aspects of this will be discussed further in Chapter 3.

## 2.6 Conclusion

Krylov subspace MOR methods have been described as one of the most important algorithms developed in this century (Dongarra & Sullivan 2000). They have been found to be very useful when dealing with systems of very high order.

The use of Krylov subspaces has initially been proposed for linear systems, however the advantages they provide have driven research into their use for reduction of nonlinear models. In this chapter all the basic ideas which inform the reader of the hows and the whys of the use of Krylov subspaces have been discussed. The concepts such as moment matching, orthogonalisation, normalisation and the Gram-Schmidt process and their corresponding algorithms have been presented. Also, a linear model structure has been used to described projection techniques. Krylov subspace model order reduction which is a type of projection based reduction method has been shown to produce reduced order models which match the moments of the higher order model. This is achieved by projection bases which are computed using the Arnoldi process as shown in Algorithm 2.3.

The processes discussed in this chapter, such as subspaces, orthogonalisarion and projection form a basic framework for which Krylov subspace projection can be extented to bilinear systems and in some ways, nonlinear systems which will be the focus of discussion in Chapter 3.

# Chapter 3

# Bilinear Systems

Bilinear systems form a set of nonlinear systems which are closely related to linear systems. Bilinear models are particularly important because they are suitable for approximating the dynamics of nonlinear systems and models whilst retaining a well structured mathematical framework within which linear systems co-exist. They have been used to approximate a wide range of physical/electrical (Bai & Skoogh 2006, Phillips 2003), chemical (Espana & Landau 1978), biological (Mohler & Barton 1978), social (Breiten & Damm 2010) and engineering systems (Mohler 1973), as well as manufacturing processes (Mula, Peidro, Díaz-Madroñero & Vicens 2010). Reduced order modeling is only one among many areas where bilinear models have gained interest. Bilinear models have been utilized for control system design (Schelfhout 1996, Martineau, Burnham, Haas, Andrews & Heeley 2004, Goodhart, Burnham & James 1994), fault detection (Yu & Shields 1996) and system analysis (Younis, Abdel-Rahman & Nayfeh 2003). In (Martineau et al. 2004) a bilinear PID control strategy has been proposed. Their approach comprises of a standard linear PID cascaded with a bilinear compensator. This has been used to control an industrial furnace where the bilinear PID has been observed to reduce power consumption. In (Goodhart et al. 1994) a bilinear self-tuning pole-placement strategy is proposed. This control strategy has been applied to an industrial heat treatment furnace. Comparisons made with an industrial PID controller show encouraging results and indicate that adopting adaptive bilinear approaches can provide significant improvements. Also, in (Yu & Shields 1996) a diagnostic observer for a bilinear system with unknown inputs is proposed. By using a bilinear fault detection observer, residuals with a high sensitivity to a larger class of faults can be achieved.

In the following subsections, a bilinear model structure and model order reduction techniques for bilinear models will be discussed. This will be followed by nonlinear models and model order reduction approaches proposed for this more general class of nonlinear systems.

## 3.1 Definition of bilinear systems

In literature, bilinear models can be found in different forms (Zajıc 2013) but in this chapter and subsequent ones, we focus on those of the form:

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u_i + Bu \tag{3.1}$$

$$y = Cx, \tag{3.2}$$

where $A \in \mathbb{R}^{n\times n}$, $N_i \in \mathbb{R}^{n\times n}$ for $i = 1, 2, \ldots, m$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and zero initial condition, $(x_0 = 0)$, is assumed. For a single input single output (SISO) model, $m$ and $p$ are one. Otherwise if they are both greater than one, the bilinear system is of multi input multi output (MIMO). Other variations of this configuration exist such as multi-input-single-output (MISO) and single-input-multi-output (SIMO). Generally, the bilinearity is defined by a product of the system states and inputs (Mohler 1973). Therefore, for a fixed input, the bilinear model is linear in state. Also for a fixed state, it is linear in the input.

In (Phillips 2000, Rugh 1981, Bai & Skoogh 2006), bilinear models have been used to approximate nonlinear models of the form

$$\dot{x} = f(x) + Bu \tag{3.3}$$

$$y = Cx \tag{3.4}$$

where $f$ is a nonlinear function such that $f : \mathbb{R}^n \to \mathbb{R}^n$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$.

## 3.2 Volterra series representation of bilinear and nonlinear systems

Using the Volterra series functional, the input $u(t)$ and output $y(t)$ relationship of nonlinear systems can be mapped. This is done by using an infinite polynomial sum of homogenous terms in the form of

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} h_n(\sigma_1, \ldots, \sigma_n) u(t-\sigma_1) \ldots u(t-\sigma_n) d\sigma_1 \ldots d\sigma_n. \tag{3.5}$$

Some cases of static nonlinear systems described by a polynomial series in the state are also part of these sets of nonlinear systems

$$\dot{x} = a_1 x + \ldots + a_\infty x^\infty \tag{3.6}$$

$$\dot{x} = \sum_{n=1}^{\infty} a_n x^n. \tag{3.7}$$

For example, consider a system of differential equations which describes a nonlinear system as given in (3.3)-(3.4). The differential equations can be represented in an infinite polynomial form with convergence properties that retain the input-output relationship of the nonlinear system. One useful polynomial expansion for describing the behaviour of nonlinear systems is the Taylor series expansion. The Taylor series is a representation of a nonlinear/linear function $f(x)$ as an infinite sum of derivative terms calculated from the function at a single point.

At a finite derivative term, the Taylor series is defined mathematically as

$$f(x) = A_0 + A_1(x-a) + A_2((x-a)\otimes(x-a)) + A_3((x-a)\otimes(x-a)\otimes(x-a)) + \ldots + A_n((x-a)\otimes \ldots \otimes (x-a)), \quad (3.8)$$

where the coefficient of the $i^{th}$ term, $A_i, i = 1, 2, 3, \cdots n$ is the $i^{th}$ derivative of the function $f(x)$ evaluated at the point $a$ and $\otimes$ is the Kronecker product. Due to time and space considerations, it is quite common and efficient to use the truncated form of the Taylor series expansion. This concept was formally introduced by Brook Taylor in 1715 (Taylor 1715) although it was first discovered by James Gregory (Roy 1990). It is important to note that the form presented here is around a point zero. The Taylor series centred at zero, is also called a Maclaurin series:

$$f(x) = A_1(x) + A_2((x)\otimes (x)) + A_3((x)\otimes (x)\otimes (x)) + \ldots + A_n((x)\otimes \ldots \otimes (x)). \quad (3.9)$$

The Taylor series provides a framework for the reduction of nonlinear systems. Some authors have proposed direct techniques applied to the truncated expansion. The quadratic approximation and bilinear approximation are linked to the Taylor series and will be discussed in the next section.

### 3.2.1 Multimoment for bilinear systems

The input-output relationship of systems are often represented using the convolution theorem (Rugh 1981, Bai & Skoogh 2006). Consider a SISO bilinear model. This can be described by using an infinite sum of convolution integrals to describe the input-output characteristics of a bilinear model (3.1) - (3.2),

$$y(t) = \sum_{k=1}^{\infty} y_k(t), \quad (3.10)$$

where $y_k(t)$ is the output of the $k^{th}$ subsystem and can be represented as

$$y_k(t) = \int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} h(t_1, t_2, \ldots, t_k)u(t - t_1 - t_2 - \ldots - t_k) \ldots \\ \times u(t - t_k)dt_k \ldots dt_1, \quad (3.11)$$

where $h(t_1, t_2, \ldots, t_k)$ is the kernel also known as the impulse response and can be represented as

$$h(t_1, t_2, \ldots, t_k) = Ce^{At_k - 1}N \ldots e^{At_2}Ne^{At_1}B. \quad (3.12)$$

A multivariable Laplace transform of the kernels can be used to define a transfer function for $h(t_1, t_2, \ldots, t_k)$ as given below.

$$H(s_1, s_2, \ldots, s_k) = C(s_kI - A)^{-1}N(s_{k-1}I - A)^{-1}N \ldots \\ (s_2I - A)^{-1}N(s_1I - A)^{-1}B. \quad (3.13)$$

Also, the concept of transfer functions for time-invariant bilinear systems/models has been discussed in (Bai & Skoogh 2006). $H(s_1, s_2, \ldots, s_k)$ is referred to as the transfer function of the $k^{th}$ subsystem and it can be expanded in a multivariable Maclaurin series such that

$$H(s_1, \ldots, s_k) = \sum_{l_k=1}^{\infty} \ldots \sum_{l_1=1}^{\infty} m(l_1, l_2, \ldots, l_k)s_1^{l_1-1}s_2^{l_2-1} \ldots s_k^{l_k-1} \quad (3.14)$$

with

$$m(l_1, \ldots, l_k) = (-1)^k CA^{-l_k}N \ldots A^{-l_2}NA^{-l_1}B \quad (3.15)$$

the multimoments of the $k^{th}$ subsystem. Consider the first system transfer function and its Maclaurin series expansion

$$H(s_1) = C(s_1I - A)^{-1}B, \quad (3.16)$$

$$H(s_1) = \sum_{l_1=1}^{\infty} m(l_1)s_1^{l_1-1}, \quad (3.17)$$

respectively and its moments

$$m(l_1) = -CA^{-l_1}B. \tag{3.18}$$

Also consider the second subsystem transfer function and expansion are represented as

$$H(s_1, s_2) = C(s_2I - A)^{-1}N(s_1I - A)^{-1}B \tag{3.19}$$

$$H(s_1, s_2) = \sum_{l_2=1}^{\infty}\sum_{l_1=1}^{\infty} m(l_1, l_2)s_1^{l_1-1}s_2^{l_2-1} \tag{3.20}$$

and the associated multimoments are

$$m(l_1, l_2) = CA^{-l_2}NA^{-l_1}B. \tag{3.21}$$

The aim of Krylov subspaces model order reduction for bilinear systems/models is to match as many multimoments, i.e. $m(l_1) = \hat{m}(l_1)$ and $m(l_1, l_2) = \hat{m}(l_1, l_2)$ of the original model in a reduced order model. As will be shown in subsequent sections, the order of the reduced order model increases with the number of multimoments matched.

### 3.2.2 Multimoment for MIMO bilinear systems

As has been discussed in Section 3.2.1, the input-output relationship of a bilinear system can be represented in a Volterra series. The output $y_k(t)$ of the $k^{th}$ subsystem is given as

$$y_k(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} h(t_1, t_2, \dots, t_k).\big(u\big(t - \sum_{i=1}^{k} t_i\big) \otimes \dots \otimes u(t - t_k)\big)dt_k \dots dt_1 \tag{3.22}$$

In (3.22), the degree $k$ kernel, $h(t_1, t_2, \ldots, t_k)$ of the MIMO bilinear model is given as

$$h(t_1, t_2, \ldots, t_k) = Ce^{At_k} N(I_m \otimes e^{At_{k}-1})(I_m \otimes N) \ldots (\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}2} \otimes e^{At_2})$$
$$(\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}2} \otimes N).(\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}1} \otimes e^{At_1})(\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}1} \otimes B), \quad (3.23)$$

where $A$, $B$ and $C$ are the state matrix, input and output matrices respectively. $N$ consists of the bilinear state matrices

$$N = [N_1, N_2, \ldots, N_m]. \quad (3.24)$$

Consequently, the $k^{th}$ transfer function can be derived from a multivariable Laplace transform and in given as

$$H(s_1, s_2, \ldots, s_k) = C(s_k I - A)^{-1} N[I_m \otimes (s_{k-1} I - A)^{-1}](I_m \otimes N) \ldots$$
$$[\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}2} \otimes (s_2 I - A)^{-1}](\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}2} \otimes N).$$
$$[\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}1} \otimes (s_1 I - A)^{-1}](\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}1} \otimes B) \quad (3.25)$$

$$= C(s_k I - A)^{-1} N[I_m \otimes (s_{k-1} I - A)^{-1} N]) \ldots$$
$$.[\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}2} \otimes (s_2 I - A)^{-1} N].[\underbrace{I_m \otimes \ldots \otimes I_m}_{k\text{-}1} \otimes (s_1 I - A)^{-1} B]. \quad (3.26)$$

Given (3.25), its Maclaurin series expansion is derived as

$$H(s_1, s_2, \ldots, s_k) = \sum_{l_k=1}^{\infty} \ldots \sum_{l_1=1}^{\infty} m(l_1, l_2, \ldots, l_k) s_1^{l_1 - 1} s_2^{l_2 - 1} \ldots s_k^{l_{k-1}}, \quad (3.27)$$

where $m(l_1, l_2, \ldots, l_k)$ is the multimoments of the $k^{th}$ subsystem such that

$$m(l_1, l_2, \ldots, l_k) = (-1)^k C A^{-l_k} N(I_m \otimes A^{-l_{k-1}} N) \ldots (I_m \otimes \ldots \otimes A^{-l_2} N)$$
$$(I_m \otimes \ldots \otimes I_m \otimes A^{-l_1} B). \quad (3.28)$$

For example, the transfer function and the Taylor series expansion of the second subsystem are represented respectively as

Therefore, the corresponding multimoments of the MIMO bilinear model is

$$m(l_1, l_2) = CA^{-l_2}N(I_m \otimes A^{-l_1}B). \tag{3.29}$$

The first transfer function is the same as that of the SISO case except for $C$ and $B$ being matrices.

## 3.3 Bilinearization of nonlinear systems

The bilinearization process described here has been described in (Rugh 1981) and is called the Carleman bilinearization. We consider several classes of nonlinear systems for which the Carleman bilinearization can be applied.

### 3.3.1 Input affine nonlinear system with constant input matrix

Consider the input affine nonlinear system with constant input matrix (3.3) - (3.4). The resulting bilinear system is an approximation of a nonlinear model of the form (3.1) - (3.2). Using the Taylor series expansion of the nonlinear function

$$f(x) \approx A_1x^{(1)} + A_2x^{(2)} + A_3x^{(3)} \ldots + A_ix^{(i)} \tag{3.30}$$

and a definition of new states $x$ ,

$$x^{(\ )} = [x^{(1)}x^{(2)}x^{(3)} \ldots x^{(i)}]^T, \tag{3.31}$$

a bilinear approximation is achieved where $x^{(1)} = x, x^{(2)} = x \otimes x, x^{(3)} = x \otimes x \otimes x,$ and so on. Generally, $x^{(i)} = x \otimes x \otimes \ldots \otimes x \in \mathbb{R}^{n^i}$.

During the bilinearization process, it is crucial to note the relationship between each state element of current state in $x$ , i.e $x^{(i)}$, the time derivative of the original systems state, $\dot{x}$, and the previous state in $x$ , $x^{(i-1)}$. The next example illustrates this relationship.

**Example 3.3.1** *Consider a state definition where* $x \;\; = [x^{(1)}x^{(2)}]$. *For* $x^{(1)}$,

$$\dot{x}^{(1)} = \dot{x} = A_1 x^{(1)} + Bu. \tag{3.32}$$

*For* $x^{(2)}$,

$$\begin{aligned}
\dot{x}^{(2)} &= \frac{d}{dy}[x \otimes x] \\
&= \frac{d}{dt}[x^{(1)} \otimes x^{(1)}] \\
&= \dot{x} \otimes x^{(1)} + x^{(1)} \otimes \dot{x} \\
&= (A_1 x^{(1)} + Bu) \otimes x^{(1)} + x^{(1)} \otimes (A_1 x^{(1)} + Bu) \\
&= A_1 x^{(1)} \otimes x^{(1)} + x^{(1)} \otimes A_1 x^{(1)} + Bu \otimes x^1 + x^1 \otimes Bu \\
&= A_1 (I \otimes I)(x^1 \otimes x^1) + (I \otimes I) A_1 (x^1 \otimes x^1) + (B \otimes I + I \otimes B) x^1 u \\
&= [(A_1 \otimes I) + (I \otimes A_1)](x^{(1)} \otimes x^{(1)}) + (B \otimes I + I \otimes B) x^{(1)} u.
\end{aligned} \tag{3.33}$$

*Since* $\dot{x} \;\; = [\dot{x}^{(1)}\dot{x}^{(2)}]$, *a bilinear model can be defined as*

$$\dot{x} \;\; = A \;\; x \;\; + N \;\; x \;\; u(t) + B \;\; u \tag{3.34}$$

$$y = C \;\; x \;\; , \tag{3.35}$$

*where,*

$$A \;\; = \begin{bmatrix} A_1 & A_2 \\ 0 & [A_1 \otimes I + I \otimes A_1] \end{bmatrix}, \; N \;\; = \begin{bmatrix} 0 & 0 \\ [B \otimes I + I \otimes B] & 0 \end{bmatrix} \tag{3.36}$$

$$B = \begin{bmatrix} B \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} C & 0 \end{bmatrix} \tag{3.37}$$

*with* $A \in \mathbb{R}^{(n+n^2)\times(n+n^2)}$, $N \in \mathbb{R}^{(n+n^2)\times(n+n^2)}$, $B \in \mathbb{R}^{(n+n^2)\times 1}$, $C \in \mathbb{R}^{1\times(n+n^2)}$.

For a state definition where the higher order Taylor series expansions are used, the system matrices ($A$ and $N$), output and input vectors ($C$, $B$) as defined in (Phillips 2000) are given below

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & \\ 0 & A_{21} & A_{22} & 0 & \cdots \\ \vdots & 0 & A_{31} & A_{32} & \ddots \\ & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \; N = \begin{bmatrix} 0 & 0 & \cdots & & \\ B_{20} & 0 & \cdots & & \\ & B_{30} & 0 & \cdots & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{3.38}$$

$$B = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} C & 0 & \cdots & 0 \end{bmatrix}, \tag{3.39}$$

where $A_{ki} = A_i \otimes I \otimes \cdots \otimes I + I \otimes A_i \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes A_i$ for $k > 1$ and $A_{1i} = A_i$. Note that there are $k$ terms and $k-1$ Kronecker products. Likewise $B_{k0} = B \otimes I \otimes \cdots \otimes I + I \otimes B \otimes \cdots \otimes I + I \otimes I \otimes \cdots \otimes B$. The dimension of the state matrices increases exponentially $A$, $N \in \mathbb{R}^{(n+n^2+\cdots+n^i)\times(n+n^2+\cdots+n^i)}$.

Similarly, $B \in \mathbb{R}^{(n+n^2+\cdots+n^i)\times 1}$, $C \in \mathbb{R}^{1\times(n+n^2+\cdots+n^i)}$. In this formulation, the dimensions of zero matrices, 0, are as required in order to achieve the correct dimensions.

### 3.3.2 Input affine nonlinear systems

Another class of nonlinear systems which can be bilinearized using the Carleman bilinearization procedure is the general input affine nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{3.40}$$

$$y = Cx. \tag{3.41}$$

In this case, the elements of the input matrix $g(x)$ are nonlinear functions of states. This has been discussed in (Breiten & Damm 2010, Rugh 1981). Using the same state definition as has been used in Subsection 3.3.1, the bilinearization process is possible as the nonlinear function $g(x)$ can also be expressed as a Taylor series expansion

$$g(x) = G_0 + G_1 x + \ldots + G_2(x \otimes x) + G_3(x \otimes x \otimes x) + \ldots + G_n(x \otimes \ldots \otimes x). \tag{3.42}$$

An example of a nonlinear system of this class being bilinearized is presented in Chapter 4.

The Carleman bilinearization process is quite useful in engineering applications. Control design (Sanchez & Collado 2010), system identification (Juang & Lee 2012), filtering (Germani, Manes & Palumbo 2005*b*, Germani, Manes & Palumbo 2005*a*), motion tracking (Sayem, Braiek & Hammouri 2010, Sayem, Braiek & Hammouri 2013) are some of the applications for which it has been found to be extensively used. Its application in the use for model complexity reduction (Ghasemi, Ibrahim, Gildin et al. 2014) has been researched widely.

One of the limitations of this process of approximating nonlinear models is the resulting exponentially increasing, high dimensions of the bilinear models.

This brings about the need for model order reduction. The time efficiency of Krylov subspace projection techniques makes them ideal for solving this problem. When compared to methods such as balanced truncation and $H_2$ model reduction for bilinear systems, where the computation of Lyapunov equations of high dimensions is necessary, the Krylov subspace projection techniques prove to be of great advantage. In some cases, the use of other model order reduction techniques are practically impossible.

## 3.4 Stability of bilinear models

There exist different definitions of stability for bilinear systems as discussed in (Dunoyer 1996), some of which are related to stability as defined for linear systems. However, in order to sufficiently guaranty the stability of bilinear models, it is convenient to consider a bounded input bounded output stability BIBO. In (Bose & Chen 1995, Kotsios 1995, Bibi 2004, Siu & Schetzen 1991) sufficient conditions have been given on the input of bilinear models to ensure BIBO stability. The following definition of BIBO stability for bilinear models can be found in (Zhang & Lam 2002)

**Definition 3.4.1** *The bilinear system model of the form* (3.1) - (3.2) *is said to be BIBO stable if for a bounded input, the output is bounded on* $[0, \infty)$.

The following theorem for BIBO stability can be found in (Siu & Schetzen 1991, Flagg 2012, Zhang & Lam 2002)

**Theorem 3.4.1** *: For a bilinear system model of the form* (3.1) - (3.2)*, suppose there exists an* $M > 0$ *so that the input* $||u|| = \sqrt{\sum_{i=1}^{m} |u_i|^2}$ *satisfies* $||u|| \leq M$ *for all* $t$ *greater than zero. Let* $\Gamma < \sum_{i=1}^{m} ||N_i||$*. Then the output,* $y$*, given from the inputs,* $u_i$*, is bounded on* $[0, \infty]$ *if there exist scalars* $\beta > 0$ *and* $0 < \alpha \leq -max_i(Re(\lambda_i(A)))$*, such that* $||e^{At}|| \leq \beta e^{-\alpha t}, t \geq 0$ *and* $\Gamma < \alpha / M\beta$

From this theorem, it can be seen that the system represented by (3.1) - (3.2) is BIBO stable if $A$ is stable and $N_i, i = 1, \ldots, m$ are sufficiently bounded (Zhang & Lam 2002).

## 3.5 MOR for bilinear models

Linear model order reduction approaches such as Krylov subspace projection (Lohmann & Salimbahrami 2000), balanced truncation (Aizad et al. 2014) and $H_2$ model reduction (Gugercin et al. 2008) have been extended to bilinear models (Phillips 2000, Bai & Skoogh 2006, Breiten & Damm 2010, Condon & Ivanov 2007, Hartmann, Zueva & Schäfer-Bung 2010, Benner & Breiten 2012*a*, Zhang & Lam 2002, Couchman, Kerrigan & Böhm 2011) with all their disadvantages and advantages.

Gramian based model order reduction tecniques have been proposed for bilinear systems as described in (Al-Baiyat & Bettayeb 1993, Benner & Damm 2011, Condon & Ivanov 2005, Couchman et al. 2011, Hartmann et al. 2010). The use of balanced truncation was first proposed in (Al-Baiyat & Bettayeb 1993). Similar to MOR for linear systems, the computation of observability and controllability Gramians via two Lyapunov equations of the bilinear model is essential. Different notions of these Lyapunov equations have been described in (Condon & Ivanov 2005). These Gramians can then be used for balancing followed by truncation of the system matrices. The limitations of balanced truncation are even more prominent for bilinear models derived from Carleman bilinearization due to the exponential increase in model dimensions which in turn increases the computational complexity of solving the two Lyapunov equations.

## 3.6 Krylov subspace MOR for bilinear models

In 2000, Phillips proposed a method which matches the multimoments of a bilinear model (Phillips 2000). This approach has influenced most of the work done so far (Bai & Skoogh 2006, Feng & Benner 2007, Breiten & Damm 2010). (Bai & Skoogh 2006) proposed a method which tries to match the moments and multimoments of the bilinear model. Feng and Benner discuss a one-sided approach which they claim is equivalent to the work done by Bai and Scoogh. A slightly different approach proposed by (Condon & Ivanov 2007) which uses a linear approximation of the bilinear model about a small constant input over a finite time period can also be explored for one sided projection.

Comparative studies of these approaches have been done in (Baur et al. 2014, Feng & Benner 2007, Bai 2002). In this section, these methods are to be discussed in some detail with numerical simulations done to compare their input-output preservation qualities using predefined performance criteria. An improved approach for moment matching is also proposed. The approach proposed in (Bai & Skoogh 2006) cannot be referred to as one-sided because of the awkward formulation of the reduced system matrices, therefore it will not be considered here. Note that in this section, only the methods which have been proposed for SISO model structures are discussed.

### 3.6.1 Petrov-Galerkin projection for bilinear models

Considering a bilinear system of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u + Bu \tag{3.43}$$

$$y = Cx. \tag{3.44}$$

In this subsection we focus on the case for $m = 1$, $A \in \mathbb{R}^{n\times n}$, $N_1 = N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and we assume zero initial condition, $(x = 0)$, is assumed. Approximating system states will result in a system of lower dimension. Projection methods try to achieve this using the approximation $x \approx V\hat{x}$, where $\hat{x}$ is the new set of states. Hence, (3.43) - (3.44) can be rewritten as

$$V\dot{\hat{x}} = AV\hat{x} + NV\hat{x}u + Bu \tag{3.45}$$

$$\hat{y} = CV\hat{x}. \tag{3.46}$$

Premultiplying (3.45) by the transpose of $V$, results in a set comprising of a new system matrix, input and output vectors

$$V^T V\dot{\hat{x}} = V^T AV\hat{x} + V^T NV\hat{x}u + V^T Bu \tag{3.47}$$

$$\hat{y} = CV\hat{x} \tag{3.48}$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{N}\hat{x}u + \hat{B}u \tag{3.49}$$

$$\hat{y} = \hat{C}\hat{x} \tag{3.50}$$

of lower dimensions. Of particular importance is the condition that $V^T V = I$. This is because orthogonal matrix computations contain less numerical noise (Tan & He 2007). The reduced system is of $q$ states, $q << n$, $q \in \mathbb{Z}$, with system matrices, input and output vectors labelled $\hat{A}$, $\hat{N}$, $\hat{B}$ and $\hat{C}$, with

$$\hat{A} = V^T AV \in \mathbb{R}^{q\times q} \tag{3.51}$$

$$\hat{N} = V^T NV \in \mathbb{R}^{q\times q} \tag{3.52}$$

$$\hat{B} = V^T B \in \mathbb{R}^{q\times 1} \tag{3.53}$$

$$\hat{C} = CV \in \mathbb{R}^{1\times q}. \tag{3.54}$$

This is often referred to as the Petrov-Galerkin projection (Flagg 2012). In another analogy, the Petrov-Galerkin projection is derived by defining the state approximation $x \approx V\hat{x}$ such that $\hat{x} \in \mathbb{R}^q$ and enforcing the Petrov-Galerkin condition $W^T R = 0$, i.e. requiring $R$ to be orthogonal, where $R$ is the residual

$$R = Ax + Nxu + Bu - \dot{x} \tag{3.55}$$

and $W$ and $V$ are matrices with columns that span suitable subspaces. Premultiplying (3.55) by $W^T$ and substituting $x$ with $V\hat{x}$ results in

$$W^T R = W^T AV\hat{x} + W^T NV\hat{x}u + W^T Bu - W^T V\dot{\hat{x}} \tag{3.56}$$

$$0 = W^T AV\hat{x} + W^T NV\hat{x}u + W^T Bu - \dot{\hat{x}}. \tag{3.57}$$

The reduced order model is defined as in (3.49) and (3.50) where $\hat{A} = W^T AV \in \mathbb{R}^{q\times q}$, $\hat{N} = W^T NV \in \mathbb{R}^{q\times q}$, $\hat{B} = W^T B \in \mathbb{R}^{q\times 1}$, $\hat{C} = CV \in \mathbb{R}^{1\times q}$.

In both analogies, it is required to find appropriate matrices V and/or W. This can be achieved by using Krylov subspace techniques. For one-sided Krylov subspace projection for bilinear systems, $W = V$.

### 3.6.2 Phillips type projection

In (Phillips 2000), a multimoment matching approach has been proposed by using the Krylov subspaces

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B) \tag{3.58}$$

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, NV^{\{k-1\}}) \tag{3.59}$$

$$span\{V\} = span\{\bigcup_{k=1}^{k} span\{V^{\{k\}}\}\}, \tag{3.60}$$

where $V^{\{k\}}$ is the basis of the $q_k^{th}$ Krylov subspace $K_{q_k}(\mathbb{M}, \mathbb{N})$. The Krylov subspace (3.58), as defined, matches $q_1 - 1$ moments of the first subsystem of the bilinear model.

In the numerical studies which will be presented in this thesis, only $V^{\{1\}}$ and $V^{\{2\}}$ are used for computing $V$, i.e. the Krylov subspaces and projection matrices are defined as

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B) \tag{3.61}$$

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}}) \tag{3.62}$$

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^{\{k\}}\}\}. \tag{3.63}$$

The projection matrix, $V$, is computed as a union of $V^{\{1\}}$ and $V^{\{2\}}$. The dimension of $V$ is therefore $n \times (q_2 q_1 + q_1)$, where $n$ is the dimension of $A$, $q_1$ and $q_2$ refer to the Krylov subspaces $K_{q_1}(A^{-1}, B)$ and $K_{q_2}(A^{-1}, NV^{\{1\}})$ respectively. The formulation of $V$ and its dimension is the same for the other types of projection types to be discussed in this section.

### 3.6.3 Feng and Benner type

In the work influenced by the approach of (Phillips 2000) and (Bai & Skoogh 2006), Feng and Benner have proposed the Krylov subspaces

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{3.64}$$

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, A^{-1}NV^{k-1}) \tag{3.65}$$

$$span\{V\} = span\{\bigcup_{k=1}^{k} span\{V^k\}\} \tag{3.66}$$

for matching the maximum amount of moments. These sets of Krylov subspace bases are said to match the same amount of moments as in (Bai & Skoogh 2006). In this case, the Krylov subspaces $span\{V^{\{1\}}\}$ and $span\{V^{\{2\}}\}$ are

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{3.67}$$

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}}) \tag{3.68}$$

and the corresponding projection vector is formed as

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^k\}\}. \tag{3.69}$$

In (Bai & Skoogh 2006) an approach which multiplies the system equation by the inverse of the state transition matrix is used. In (Breiten & Damm 2010) a generalization of the methods discussed in (Bai & Skoogh 2006) has been proposed. They proposed a multiplication of the system matrix by an appropriate non-singular matrix of the same dimensions. In this case, the computation of the reduced order matrices is different from the projection methods described in this section.

### 3.6.4 Condon type Krylov subspace projection

Using a slightly different approach from the other authors discused in this section is a method proposed by (Condon & Ivanov 2007). Consider a method which matches only moments of a single variable expansion of the bilinear model about a small input $u = \eta$ (Condon & Ivanov 2007). If a bilinear model is analysed over a finite time interval $t \in [0, \tau]$, then it is possible to analyse it as a linear model. The validity of this linear approximation has been discussed extensively in (Condon & Ivanov 2005). The resulting system is

$$\dot{x} = Ax + Nx\eta + Bu \tag{3.70}$$

$$y = Cx. \tag{3.71}$$

Applying the definition of the Krylov subspace (2.4) (defined in Chapter 2), $q$ moments of (3.70) - (3.71) can be matched using the Krylov subspace

$$span\{V\} = K_{q_1}(A_\eta^{-1}, A_\eta^{-1}B), \tag{3.72}$$

where $A_\eta = [A + N\eta]$ and $H_\eta(s) = C(sI - A_\eta)^{-1}B$. Making use of the Petrov-Galekin projection procedure, the reduced system with, $\hat{A} = V^TAV$, $\hat{N} = V^TNV$, $\hat{B} = V^TB$, $\hat{C} = CV$, is achieved. In this thesis, we will refer to this as a Condon type, as it was used in (Condon & Ivanov 2007). Condon & Ivanov have used a two-sided approach to improve the output of the reduced model, but in this thesis it will be implemented using a one-sided approach.

Note that the major difference between the projection types discussed in this section is the definition of the Krylov subspaces. These slight differences, as will be shown in subsequent subsections and sections, can have a significant effect on the input-output relationship preservation for the reduced order model.

Since the works proposed by (Phillips 2000, Feng & Benner 2007, Condon & Ivanov 2007), there has been a lot of interest in the reduction of bilinear models using Krylov subspaces. The works done in (Benner & Breiten 2015, Flagg 2012, Breiten & Damm 2010, Benner & Breiten 2012*a*) show multimoment matching at some frequencies. Using Krylov subspaces, (Breiten & Damm 2010) show multimoment matching using this approach and demonstrated their work using two numerical simulations. In their conclusion, it has been noted that this approach needs improvement. An extension of the IRKA, namely the bilinear IRKA was first proposed in (Benner & Breiten 2012*a*). This approach applies a two sided rational Krylov iterative procedure for computing a reduced order model. The reduced order model is said to satisfy the conditions for $H_2$ optimality and methods of this form are regarded as $H_2$ model order reduction. The bilinear IRKA (BIRKA) has been described to be very expensive in (Choudhary & Ahuja 2016). This is because the algorithm solves the projection matrices by using the solutions of two generalised Sylvester equations. The truncated BIRKA (TBIRKA) (Flagg 2012) was proposed for this purpose. Using Krylov subspace methods for solving the Sylvester equations, the computational efficiency of the $H_2$ norm approach has been improved. However, this is at the cost of solving the Sylvester equations to some tolerance which can deteriorate the quality of the resulting reduced order model.

## 3.7 Krylov subspace MOR for MIMO bilinear models

Many applications of bilinear models are MIMO systems. Similar to the case of moment matching for SISO linear models as discussed in Chapter 2, multimoment matching and other classical model order reduction techniques (Phillips 2000, Feng & Benner 2007, Hartmann et al. 2010) proposed for bilinear models can be extended to MIMO models. In (Hartmann et al. 2010), different balanced truncation approaches were proposed for MIMO bilinear models. Likewise, in

(Benner & Breiten 2012*a*, Zhang & Lam 2002), $H_2$ methods were proposed and implemented. The structural differences between SISO and MIMO models also pose a set of different challenges one of which is multiple $N$ matrices. Intrinsically, MIMO models have input and output matrices as opposed to row and column vectors respectively. Also, they might possess multiple system matrices. Some of the problems with extending multimoment matching to MIMO bilinear models have been dealt with in linear cases. In (Tan & He 2007) a block Arnoldi algorithm which is capable of computing orthonormal basis for two starting matrices is presented.

Krylov subspace model order reduction techniques for MIMO bilinear models of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u_i + Bu \tag{3.73}$$

$$y = Cx, \tag{3.74}$$

where $A \in \mathbb{R}^{n\times n}$, $N_i \in \mathbb{R}^{n\times n}$ for $i = 1, 2, \ldots, m$, and $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ with $n \in \mathbb{Z}, m \in \mathbb{Z}$, and $p \in \mathbb{Z}$ was first proposed in (Lin, Bao & Wei 2007) wherein the Phillips (Phillips 2000) type projection was extended to bilinear MIMO models. In a subsequent paper written by the same authors (Lin et al. 2009), the Bai type projection framework and algorithm was extended to MIMO models. In their work, they compared both approaches using arbitrary bilinear models and showed superior reduced order models using the Phillips type (Phillips 2000) approach.

The application of the Petrov-Galerkin condition and state approximation $x = V\hat{x}$ to (3.74) yields a reduced order model of the form

$$\dot{\hat{x}} = \hat{A}\hat{x} + \sum_{i=1}^{m} \hat{N}_i \hat{x} u_i + \hat{B}u \tag{3.75}$$

$$\hat{y} = \hat{C}\hat{x} \tag{3.76}$$

such that if $V \in \mathbb{R}^{n\times q}$, then, $\hat{A} \in \mathbb{R}^{q\times q}$, $\hat{N}_i \in \mathbb{R}^{q\times q}$, $\hat{B} \in \mathbb{R}^{q\times m}$,$\hat{C} \in \mathbb{R}^{p\times q}$, where

$\hat{A} = V^T AV$, $\hat{N}_i = V^T N_i V$, $\hat{B} = V^T B$, $\hat{C} = CV$. Both analogies discussed in this chapter for SISO bilinear systems also apply here.

### 3.7.1 Krylov subspace for MIMO Phillips type projection

In the work presented by Lin, Bao and Wei (Lin et al. 2007) an algorithm that computes the projection bases for MIMO bilinear was proposed. The following Krylov subspaces were said to match the multimoments of the MIMO bilinear model.

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B) \tag{3.77}$$

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, N_i V^{\{1\}}) \tag{3.78}$$

$$i = 1, 2. \dots, m \tag{3.79}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}. \tag{3.80}$$

The algorithm presented was shown to be able to preserve the input output characteristics of the high order bilinear model. In (Baur et al. 2014) the work done by Lin, Bao and Wei has been identified as an extension of the Phillips type projection of SISO bilinear models.

### 3.7.2 Bai type projection

An alternative to the projection formulation for the reduced order models as proposed for one-sided and two-sided projections for bilinear systems has been utilised in (Bai & Skoogh 2006). This alternative formulation has been achieved by premultiplying the bilinear system equation by $A^{-1}$

$$A^{-1}\dot{x} = x + A^{-1}Nxu + A^{-1}Bu. \tag{3.81}$$

Utilizing the change in state approximation, $x = V\hat{x}$, such that $\hat{x} \in \mathbb{R}^{q\times q}$.

$$A^{-1}\dot{x} = V\hat{x} + A^{-1}NV\hat{x}u + A^{-1}Bu \tag{3.82}$$

$$\hat{y} = CV\hat{x}. \tag{3.83}$$

Premultiplying (3.82) with $V^T$ results in a reduced order system.

$$V^TA^{-1}V\dot{\hat{x}} = V^TV\hat{x} + V^TA^{-1}NV\hat{x}u + V^TA^{-1}Bu \tag{3.84}$$

$$\hat{y} = CV\hat{x}. \tag{3.85}$$

Note that due to the orthogonality of V, the system matrix for the reduced order system is identity, $V^TV = I$. However, an equivalent definition of reduced order matrices can be achieved using the inverse of $V^TA^{-1}V$. Therefore

$$\dot{\hat{x}} = (V^TA^{-1}V)^{-1}\hat{x} + (V^TA^{-1}V)^{-1}V^TA^{-1}NV\hat{x}u + V^TA^{-1}V^TA^{-1}Bu \tag{3.86}$$

$$\hat{y} = CV\hat{x} \tag{3.87}$$

The new definition of the reduced system matrix $\hat{A}$ is $(V^TA^{-1}V)^{-1}$. This new definition can be made computationally effective by using $\hat{A}$ for computing the other system matrices as follows:

$$\hat{N} = \hat{A}V^TA^{-1}NV \tag{3.88}$$

$$\hat{B} = \hat{A}V^TA^{-1}B \tag{3.89}$$

$$\hat{C} = CV. \tag{3.90}$$

According to (Bai & Skoogh 2006), if it is assumed that $VV^T = I$, then it can be proved, for the Bai type projection, that for $k = 1, 2$, the $q^k$ moments of the reduced order model $\hat{m}(l_1, l_2)$ matches $q^k$ moments of the higher order model $m(l_1, l_2)$ by utilising the Krylov subspaces

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{3.91}$$

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}N_iV^{\{1\}})\ i = 1, 2. \dots, m \tag{3.92}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}, \tag{3.93}$$

where $l_1, l_2 = 1, 2, \ldots, q$. Numerical studies carried out by the authors of (Bai & Skoogh 2006) have shown that using this method, the input-output relationship of higher order bilinear models is preserved in the reduced order models.

## 3.8 Algorithms for computing projection matrices

For computing the projection matrix, $V$, a modified version of the algorithm proposed in (Bai & Skoogh 2006) is described in this section. The algorithm has been shown to be useful for matching moments and multimoments (Bai & Skoogh 2006, Breiten & Damm 2010). The required inputs to the algorithm are the starting matrices for the first Krylov subspace $\mathbb{M}$, $\mathbb{N}$ and $q_1$ for computing $V^{\{1\}}$. Another parameter, $p_2$, is required to select the number of columns of $V^{\{1\}}$ to be used for computing $V^{\{2\}}$. $q_2$ is used for computing the second Krylov subspace $spanV^{\{2\}}$. Note that the bases, $\mathbb{M}$, $\mathbb{N}$ will vary depending on the method used. The outline of this algorithm is as follows:

**Algorithm 3.1 (Computation of projection base, $V$)**

1. ***Input:*** $\mathbb{N}, \mathbb{M}, q_1, p_2, q_2, N$
2. ***Compute:*** $r = \mathbb{M}$
3. ***Compute:*** $v_1^{\{1\}} = r/||r||_2$
4. for $i = 1 : q_1 - 1$
5. $\quad r = \mathbb{N} v_i^{\{1\}}$
6. $\quad h = (V_{[i]}^{\{1\}})^T r$

7. $\quad r = r - V^{\{1\}}_{[i]} h$

8. $\quad$ if $||r||_2 = 0$, ***end***

9. $\quad v^{\{1\}}_{i+1} = r/||r||_2$

10. ***end***

11. $G = \mathbb{N} N V^{\{1\}}_{[p_2]}$

12. $V^{\{2\}} =$ ***orth***$(G)$

13. for $i = 1 : p_2(q_2 - 1)$

14. $\quad r = \mathbb{N} v^{\{2\}}_i$

15. $\quad h = (V^{\{2\}}_{[p_2+i-1]})^T r$

16. $\quad r = r - V^{\{2\}}_{[p_2+i-1]} h$

17. $\quad$ if $||r||_2 = 0$, ***end***

18. $\quad v^{\{2\}}_{p_2+i} = r/||r||_2$

19. ***end***

20. $V =$ ***orth***$([V^{\{1\}} V^{\{2\}}])$

The routine from step 3 to step 10 and step 12 to step 19 can be recognised as the Arnoldi process which has been used to compute $V^{\{1\}}$ and $V^{\{2\}}$ respectively. Step 11 is used to define the base for $V^{\{2\}}$ which is dependent on $V^{\{1\}}$. Using the in-built MATLAB function ***orth*** in steps 12 and 20, a QR decomposition/Gram-Schmidt process is carried out to generate an orthonormal basis for the range of $G$ and $[V^{\{1\}} V^{\{2\}}]$. The output of the algorithm is $V$ which is to be used for computing the reduced order model as discussed in Section 3.6.1.

The projection matrix computed here using Algorithm 3.1 is for the Feng and Benner (Feng & Benner 2007) type projection where $\mathbb{N} = A^{-1}$ and $\mathbb{M} = A^{-1}B$ as at step 11, $NV^{\{1\}}_{[p_2]}$ is multiplied by $\mathbb{N}$. This is not required in Phillips type projection (Phillips 2000), as only $NV^{\{1\}}_{[p_2]}$ is used instead. The subscript $[p_2]$ denotes the number of columns of $V^{\{1\}}$ which are used in forming $V^{\{2\}}$.

## 3.9 MIMO projection algorithms

In order to construct a projection matrix for matching multimoments of a MIMO bilinear model, a series of algorithms need to be put in place. Firstly, a block Arnoldi process as described in (Tan & He 2007, Lin et al. 2009) is needed in order to compute an orthonormal basis for a block Krylov subspace $K_q(\mathbb{N}, \mathbb{M})$ to handle the matrix $\mathbb{M}$. The Block Arnoldi algorithm is presented below:

**Algorithm 3.2 (Block Arnoldi for MIMO models)**

1. ***Input:*** $\mathbb{N}$, $\mathbb{M}$, $q$
2. $Q =$ ***orth***$(\mathbb{M})$
3. $W = Q$
4. for $i = 1 : q - 1$
5. $\quad R = \mathbb{N}Q$
6. $\quad R = R - W(W^T R)$
7. $\quad Q =$ ***orth***$(R)$
8. $\quad W = [W, Q]$

9. ***end***

10. ***Return*** $W$

At the end of the algorithm, a matrix $W$ is returned where

$$W = [W^{\{1\}}, W^{\{2\}}, \ldots, W^{\{q\}}] \tag{3.96}$$

and the columns of $W$ form basis for the Krylov subspace spanned by $K_q(\mathbb{N}, \mathbb{M})$.

Algorithm 3.2 can be used for computing the projection matrix $V$ for a bilinear model using the procedure in the following algorithm proposed in (Lin et al. 2009). A similar version of this algorithm proposed for the Phillips type projection method has been presented in (Lin et al. 2007).

**Algorithm 3.3 (Computation of $V$ for MIMO bilinear models)**

1. ***Input:*** $A, B, N_1, \ldots, N_m, m, q_1, q_2, p_2$
2. ***Compute an orthonormal basis,*** $V^{\{1\}}$***, for the Krylov subspace:*** $K_{q_1}(A^{-1}, A^{-1}B)$***, using Algorithm 3.2***
3. ***for*** $i = 1 : m$***, compute an orthonormal basis,*** $V_i^{\{2\}}$***, for the Krylov subspace:*** $K_{q_2}(A^{-1}, A^{-1}NV_{[p_2]}^{\{1\}})$***, using Algorithm 3.2***
4. ***end***
5. $V =$ ***orth***$([V^{\{1\}}, V_1^{\{2\}}, \ldots, V_m^{\{2\}}])$
6. ***Return*** $V$

Observing steps 2 and 4, the Krylov subspaces utilised here are for the Feng and Benner (Feng & Benner 2007) type and the Bai (Bai & Skoogh 2006) type projections. For the other projection types, the Krylov subspace starting vectors should be changed accordingly. The output of the algorithm can be used for computing reduced order MIMO bilinear models for one-sided projection using the following algorithm.

**Algorithm 3.4 (Computation of matrices for MIMO models)**

1. ***Input:*** $V$
2. $\hat{A} = V^T A V$
3. $\hat{N}_i = V^T N_i V$
4. $\hat{B} = V^T B$
5. $\hat{C} = C V$

The Bai type projection (Bai & Skoogh 2006) reduced order matrices can be computed by using the following algorithm.

**Algorithm 3.5 (Computation of matrices for Bai type)**

1. ***Input:*** $V$
2. $\hat{A} = (V^T A^{-1} V)^{-1}$
3. $\hat{N}_i = \hat{A} V^T A^{-1} N_i V$
4. $\hat{B} = \hat{A} V^T A^{-1} B$
5. $\hat{C} = C V$

The computation of the reduced order matrices for Bai (Bai & Skoogh 2006) type projection has been explained in Subsection 3.7.2.

## 3.10 Discussion

Krylov subspace model order reduction techniques for bilinear systems have been quite useful for reduction of bilinear and nonlinear models. They have been successfully applied to many nonlinear systems. In (Bai & Skoogh 2006) a nonlinear transmission line model and an electrostatic gap-closing actuator have been reduced using Krylov subspaces. Also (Benner & Damm 2011) used Krylov subspaces to reduce the order of a heat transfer model. In (Breiten & Damm 2010) a flow model which can be used for modelling engineering problems such as traffic flow and gaseous systems has been reduced for control applications.

When compared to other methods such as balanced truncation and $H_2$ model order reduction for bilinear systems, it has been reported that the Krylov subspace approaches are more desirable due to the ease of implementation (Baur et al. 2014). This is because the computation of controllability and observability Gramians for high dimensional systems are very costly and in some cases, solving Lyapunov equations is impossible (Damm 2008). In (Benner & Damm 2011) a hybrid approach which combines the Gramian computation approaches and the Krylov subspace MOR has been proposed. This implements the methods by reducing the dimensions of the bilinear model using Krylov subspaces before the computation of the Gramians of a reduced bilinear model to a much smaller dimension.

One limitation that Krylov subspace projection methods have is the need for inversion of the system matrix. There exist systems whose matrices are not invertible and this poses a problem for the discussed methods. In (Mach, Pranić & Vandebril 2013) and (Chu, Lai & Feng 2008), Krylov subspace approaches which do not require explicit matrix inversions have been proposed. However in

(Mach et al. 2013) they have been reported not to deliver good results for the application therein and further investigation is needed.

The Carleman bilinearization process often produces sparse matrices and the system matrix $N$ is in some cases singular. This is likely to pose some numerical issues for the Krylov subspaces where the multiplication of $N$ with a matrix occurs. This is an issue which has not been discussed in literature previously and is an interesting prospect for discussion as model order reduction using Krylov subspaces have been reported to have numerical issues (Choudhary & Ahuja 2016).

## 3.11 Conclusion

In this chapter, model order reduction of nonlinear systems via bilinearization has been discussed focusing on the Krylov subspace based model order reduction methods for bilinear systems. By using the Taylor series expansion, the approximation of nonlinear systems is made possible not only via bilinearization but also quadratic approximation. A second derivative truncation of the Taylor series is used to explain the bilinearization process. This is followed by examples of the different applications of bilinearization. Other model order reduction approaches for nonlinear systems have also been highlighted.

In literature, the input-output behaviour of reduced order models is determined by the amount of multimoments matched. Within the following chapter, a detailed analysis of the multimoment matching capacity of the original works done in this field is presented. This analysis which has not been seen in this forms for multimoment matching form part of the contributions of this work. The next chapter introduces two new approaches for MOR using Krylov subspaces and solves some of the problems which arise when using Krylov subspace MOR.

# Chapter 4

# Improved Phillips and Parametrised Linear Approximation for MOR of Bilinear Systems

## 4.1 Introduction

Some bilinear models result in non-invertible $A$ and/or $N$ matrices (Bai & Skoogh 2006, Breiten & Damm 2010, Couchman et al. 2011). This means that the model order reduction of these models is not possible using the methods discussed in the previous chapters that involve the use of matrix inversion. Also, when the $N$ matrix is singular, there is likely to be an irreversible loss of information when multiplying matrices. The method proposed in (Feng & Benner 2007) uses a routine which multiplies two matrices of the same dimensions ($A^{-1}N$). However, if one of these matrices is singular, then the loss of information, such as loss in rank, affects the input-output preservation of the reduced order model. These are questions which have not been considered in other work and an investigation of this effect is presented in this chapter.

As mentioned earlier, the overall desire of model order reduction is to preserve the input-output relationship of the high order/fidelity model. Research in this topic that exploits the use of Krylov subspace projection at the expansion point $s = 0$, is seemingly exhaustive. In this chapter, a new method for improving the input-output preservation is proposed based on using a so called better linear approximation of the bilinear model. Some simulation studies to illustrate the usefulness of this proposed method are provided.

In this chapter, an analysis of multimoment matching for the Phillips (Phillips 2000) type projection and the Feng and Benner (Feng & Benner 2007) type projection are presented. This forms part of the original contributions of this thesis. Based on this analysis, a new approach is presented which is called the Improved Phillips (IP) type projection. A multimoment matching analysis for this new approach is also presented. This is followed by a proposal for using alternate linear approximations for MOR using Krylov subspaces. This is called the parametrised linear approximation (PLA) for Krylov subspace MOR.

## 4.2 Multimoment matching

### 4.2.1 Multimoment matching for Phillips type projection

By using the Krylov subspace $K_{q_1}(A^{-1}, B)$ for computing $V^{\{1\}}$, as has been shown in Section 2.3.2, this only matches $q_1 - 1$ moments of the first transfer function of the bilinear model. This consequently affects the multimoment matching when $V^{\{1\}}$ is used for computing $V^{\{2\}}$. However, this formulation of Krylov subspaces presented by Phillips (Phillips 2000) can be shown to match multimoments of the multivariable transfer function $H(s_1, s_2)$ of the bilinear model.

**Theorem 4.2.1** *The Krylov subspaces* $K_{q_1}(A^{-1}, B)$ *and* $K_{q_2}(A^{-1}, NV^{\{1\}})$ *when* *used to compute projection vectors,* $V^T$ *and* $V$ *as defined in* (3.61) - (3.63) *match the multimoments of a bilinear model* (3.1) - (3.2) *and a reduced order bilinear model such that* $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1 - 1$, $l_2 = 1, \ldots, q_2 - 1$, *if the reduced order model matrices are computed as* $\hat{A} = V^T AV$, $\hat{N} = V^T NV$, $\hat{B} = V^T B$ *and* $\hat{C} = CV$, *where* $V$ *spans the Krylov subspaces* $K_{q_1}(A^{-1}, B)$ *and* $K_{q_2}(A^{-1}, NV^{\{1\}})$ *and* $V^T V = I$.

Proof. This multimoment matching property can be shown by first substituting the reduced order matrices (3.51) - (3.54) as given in Subsection 3.6.1 into the multimoments of the reduced order model, $\hat{m}(l_1, l_2)$. From (3.15) the multimoment of the reduced order bilinear model can be defined as

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}. \tag{4.1}$$

Now, using the definition of the reduced order matrices, $\hat{A}$, $\hat{N}$, $\hat{B}$ and $\hat{C}$, this multimoment equation can be rewritten as

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^T AV)^{-l_2}(V^T NV)(V^T AV)^{-l_1}V^T B. \tag{4.2}$$

Because $A^{-(q_1-1)}B$ belongs to the Krylov subspace $K_{q_1}$, it can be written that $A^{-(q_1-1)}B = V^{\{1\}}r_{(q_1)}$. Then $B = A^{(q_1-1)}V^{\{1\}}r_{(q_1)}$. Also because $V^{\{1\}} \in V$, then $V^{\{1\}}r_{(q_1)} = Vp_{(q_1)}$, where $r_{(i)}$ and $p_{(i)}$ are appropriate parameters and dimensions, where $r_{(i)} \in \mathbb{R}^{q_1}$, $p_{(i)} \in \mathbb{R}^{q_1+q_1q_2}$ for $i \leq q_1$ and $p_{(i)} \in \mathbb{R}^{(q_1+q_1q_2)\times q_1}$ for $i > q_1$. From the routine of moment matching as shown in (2.40) - (2.43), for any value of $q_1 \in \mathbb{Z}|q_1 > 0$, when $l_1 = q_1 - 1$

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^T AV)^{-l_2}V^T NVp_{(q_1)}. \tag{4.3}$$

Since $Vp_{(q_1)} = V^{\{1\}}r_{(q_1)}$ then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^T AV)^{-l_2}V^T NV^{\{1\}}r_{(q_1)}. \tag{4.4}$$

Moreover, from (3.62) - (3.63), $NV^{\{1\}} \in V$, therefore $NV^{\{1\}} = Vp_{(q_1+1)}$ and

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^T AV)^{-l_2}V^T Vp_{(q_1+1)}r_{(q_1)}\\ &= CV(V^T AV)^{-l_2}V^T AA^{-1}Vp_{(q_1+1)}r_{(q_1)}.\end{aligned} \tag{4.5}$$

Further, since $A^{-1}NV^{\{1\}} \in V$ and $A^{-1}NV^{\{1\}} = A^{-1}Vp_{(q_1+1)} = Vp_{(q_1+2)}$, therefore,

$$\begin{aligned} \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^TAV)^{-l_2}V^TAVp_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+1}p_{(q_1+2)}r_{(q_1)}. \end{aligned} \tag{4.6}$$

Using this routine iteratively until $q_2 = (l_2 + 1)$, (4.6) becomes

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CVp_{(q_1+l_2+2)}r_{(q_1)}. \tag{4.7}$$

Now $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2+2)}$, so

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NV^{\{1\}}r_{(q_1)}. \tag{4.8}$$

Since $V^{\{1\}}r_{(q_1)} = A^{-(q_1-1)}B$ and $l_1 = q_1 - 1$, then,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B. \tag{4.9}$$

Therefore,

$$\hat{m}(l_1, l_2) = m(l_1, l_2), \tag{4.10}$$

where $l_1 = 1, \ldots, q_1 - 1$ and $l_2 = 1, \ldots, q_2 - 1$. □

This proof forms an extension of the proof of moment matching as done is (Tan & He 2007) to multimoment matching. A proof of multimoment matching for (3.61) - (3.63) has been done in (Phillips 2000). However, the author has stated that $l_1$ can be less than or equal to $q_1$. This has been shown not to be the case here. Also unique to this proof is the relationship between $l_2$ and $q_2$.

Consequently, this proof can be generalised for cases where the transfer function has more than two variables i.e. for $H(s_1, s_2, \ldots, s_k)$,

$$\hat{m}(l_1, l_2, \ldots, l_k) = m(l_1, l_2, \ldots, l_k), \tag{4.11}$$

such that $l_1 = 1, 2, \ldots, q_1 - 1$, $l_2 = 1, 2, \ldots, q_2 - 1$, … $l_k = 1, 2, \ldots, q_k - 1$ where $V^{\{k\}}$ for $k = 1, 2, \ldots,$ as defined in (3.58)-(3.59) have been used for computing the projection matrix $V$ (3.60).

### 4.2.2 Multimoment matching for Feng and Benner type projection

**Theorem 4.2.2** *Given the Krylov subspaces as proposed by Fend and Benner (Feng & Benner 2007), $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$, multimoments of the bilinear model* (3.1) - (3.2) *and a reduced order model can be matched such that $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2$, when the reduced order model matrices are computed as $\hat{A} = V^TAV$, $\hat{N} = V^TNV$, $\hat{B} = V^TB$ and $\hat{C} = CV$ with $V$ spans the Krylov subspaces $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$ and $V^TV = I$.*

PROOF. To match the multimoments of the multivariable transfer function, the Krylov subspaces proposed by Feng and Benner (Feng & Benner 2007) can be used following a similar procedure as in Section 4.2.1. The multimoments of the reduced order model are defined as

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}. \tag{4.12}$$

Using the definition of the reduced order matrices we have, $\hat{A} = V^TAV$, $\hat{N} = V^TAV$, $\hat{B} = V^TB$ and $\hat{C} = CV$. Substituting these matrices into (4.12), gives

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB. \tag{4.13}$$

Because $A^{-q_1}B = V^{\{1\}}r_{(q_1)}$, then

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)} &(4.14)\\ &= CV(V^TAV)^{-l_2}V^TAA^{-1}NV^{\{1\}}r_{(q_1)}. &(4.15)\end{aligned}$$

From (3.68), $A^{-1}NV^{\{1\}} \in V$, therefore $A^{-1}NV^{\{1\}} = Vp_{(q_1+1)}$ and

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^TAV)^{-l_2}V^TAVp_{(q_1+1)}r_{(q_1)}\\ &= CV(V^TAV)^{-l_2+1}p_{(q_1+1)}r_{(q_1)} \qquad (4.16)\\ &= CV(V^TAV)^{-l_2+1}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}.\end{aligned}$$

Moreover, $A^{-2}NV^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$ therefore,

$$\begin{aligned} \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^TAV)^{-l_2+1}V^TAVp_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+2}p_{(q_1+2)}r_{(q_1)}. \end{aligned} \tag{4.17}$$

Following this routine results in

$$\begin{aligned} \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CVp_{(q_1+l_2)}r_{(q_1)} \\ &= CA^{-l_2}NV^{\{1\}}r_{(q_1)} \end{aligned} \tag{4.18}$$

for any value of $l_2$. Note that $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2)}$. Since $V^{\{1\}}r_{(q_1)} = A^{-q_1}B$,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B. \tag{4.19}$$

Therefore,

$$\hat{m}(l_1, l_2) = m(l_1, l_2), \tag{4.20}$$

where $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2$ $p_{(i)}$, $i = 1, \ldots, (q_1 + q_2)$ and $r_{(i)}, i = 1, \ldots, q_1$ are appropriate parameters for achieving orthogonality. □

The same approach used in (Tan & He 2007) for linear systems moment matching has been extended to multimoment matching for bilinear models. In the proof for multimoment matching shown in (Feng & Benner 2007), it has been assumed that $VV^T = I$. This is not the case in this thesis. The condition for computing the projection matrices, i.e. $V^TV = I$ has been used for showing multimoment matching. This proof can also be generalised for cases where the transfer function has more than two variables. For $H(s_1, s_2, \ldots, s_k)$,

$$\hat{m}(l_1, l_2, \ldots, l_k) = m(l_1, l_2, \ldots, l_k), \tag{4.21}$$

such that $l_1 = 1, 2, \ldots, q_1$, $l_2 = 1, 2, \ldots, q_2$, $\ldots$ $l_k = 1, 2, \ldots, q_k$ where $V^{\{k\}}$ for $k = 1, 2, \ldots,$ as defined in (3.64)-(3.65) have been used for computing the projection matrix $V$ (3.66).

## 4.3 Improved Phillips type projection

The method proposed by Phillips (Phillips 2000) which has been discussed in Subsection 3.6.2 can be readily improved because it has been shown that the Krylov subspace used to compute $V^{\{1\}}$ matches only $q_1 - 1$ moments of its linear approximation. Further observation of the Phillips (Phillips 2000) type projection shows that in order to match more moments of the linear approximation of the bilinear model, the Krylov subspace bases (3.58) to compute $V^{\{1\}}$ should be replaced by

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B). \tag{4.22}$$

This formulation for computing $V^{\{1\}}$ would match $q_1$ moments, i.e. if we are only considering the linear approximation of the bilinear model. This implies that the Krylov subspace multi-moment matching of the bilinear model can be improved by using (4.22). Also, since the methods so far use the system matrix $A$ for the linear approximation of the bilinear model, exchanging $A$ with a so called better linear approximation and using it for computing $V^{\{1\}}$ will result in improved input-output preservation for the reduced bilinear model. This approach can be demonstrated numerically as we will show in Section 4.4 and Section 4.5.

In Subsection 4.3.1, an analysis of multimoment matching will be shown. The approach presented here is an extension of the proof for moment matching as shown in (Tan & He 2007). This approach has also been used to analyse matched multimoments for methods proposed by Phillips (Phillips 2000) as well as Feng and Benner (Feng & Benner 2007) in Subsections 3.6.2 and 3.6.3 respectively. An improved parametrised linear approximation for MOR of the bilinear model will also be discussed in Section 4.4.

### 4.3.1 Multimoment matching for Improved Phillips projection

For matching the multimoments of the multivariable transfer function, the following Krylov subspaces is proposed in this thesis:

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{4.23}$$

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}}) \tag{4.24}$$

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^k\}\}. \tag{4.25}$$

**Theorem 4.3.1 (Improved Phillips projection)** *For a bilinear system/model as defined in* (3.1) - (3.2)*, a reduced order model of dimensions* $q_1q_2 + q_1$ *can be constructed by using projection matrices,* $V$ *and* $V^T$*,* $V^TV = I$*, If* $V$ *is computed using the Krylov subspaces* (4.23) *and* (4.24)*. This formulation of Krylov subspaces matches multimoments of the multivariable transfer function* $H(s_1, s_2)$ *of the bilinear model such that* $\hat{m}(l_1, l_2) = m(l_1, l_2)$*,* $l_1 = 1, \ldots, q_1$*,* $l_2 = 1, \ldots, q_2 - 1$*. This has been called the Improved Phillips type projection projection.*

Proof. From (3.15) the multimoment of the reduced order bilinear model can be defined as

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}. \tag{4.26}$$

Also, using the definition of the reduced order matrices, $\hat{A} = V^TAV$, $\hat{N} = V^TNV$, $\hat{B} = V^TB$ and $\hat{C} = CV$, and substituting the matrices into (4.26), gives

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB. \tag{4.27}$$

Because, $A^{-q_1}B = V^{\{1\}}r_{(q_1)}$, where $r_{(i)}$ appropriate parameters for achieving orthogonality, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)} \tag{4.28}$$

From (4.24), $NV^{\{1\}} \in V$, therefore $NV^{\{1\}} = Vp_{(q_1+1)}$, where $p_{(i)}$ are appropriate parameters for achieving orthogonality, thus

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^TAV)^{-l_2}V^TVp_{(q_1+1)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}.\end{aligned} \tag{4.29}$$

Since $A^{-1}NV^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^TAV)^{-l_2}V^TAVp_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+1}p_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+1}V^TAA^{-1}Vp_{(q_1+2)}r_{(q_1)}.\end{aligned} \tag{4.30}$$

Moreover, as $A^{-2}NV^{\{1\}} = Vp_{(q_1+3)} = A^{-1}Vp_{(q_1+2)}$

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CV(V^TAV)^{-l_2+1}V^TAVp_{(q_1+3)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+2}p_{(q_1+3)}r_{(q_1)}.\end{aligned} \tag{4.31}$$

Using this iterative scheme, it can be derived that

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} &= CVp_{(q_1+l_2+1)}r_{(q_1)} \\ &= CA^{-l_2}NV^{\{1\}}r_{(q_1)}\end{aligned} \tag{4.32}$$

for any value of $l_2$, where it is true that $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2+1)}$. Since $V^{\{1\}}r_{(q_1)} = A^{-q_1}B$, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B. \tag{4.33}$$

Note that $p_{(i)}$ are matrices for $i > q_1$ otherwise they are vectors and $r_{(i)}$ are vectors for SISO bilinear models. Using this analysis, it can be observed that the Krylov subspaces (4.23) - (4.24) match the multi-moments, $\hat{m}(l_1, l_2)$ and $m(l_1, l_2)$, such that $l_1 = 1 \ldots q_1$, and $l_2 = 1 \ldots q_2 - 1$. □

This proof of multimoment matching for the Improved Phillip approach is unique to this thesis.

### 4.3.2 For higher order subsystem of the bilinear model

In order to compute the projection matrices for higher order subsystems of the bilinear model, the following definition of Krylov subspaces is then used:

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{4.34}$$

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, NV^{\{k-1\}}) \tag{4.35}$$

$$span\{V\} = span\{\bigcup_{k=1}^{k} span\{V^{\{k\}}\}\} \tag{4.36}$$

where $k$ is the number of Krylov subspaces computed. This formulation matches the multimoments with all the indices $l_1, l_2, l_3, \ldots, l_k$.

The new results for the proof of multimoment matching using the Improved Phillips approach can also be generalised for a higher order subsystem of the bilinear model i.e. for $H(s_1, s_2, \ldots, s_k)$,

$$\hat{m}(l_1, l_2, \ldots, l_k) = m(l_1, l_2, \ldots, l_k), \tag{4.37}$$

such that $l_1 = 1, 2, \ldots, q_1$, $l_2 = 1, 2, \ldots, q_2 - 1$, ... $l_k = 1, 2, \ldots, q_k - 1$ where $V^{\{k\}}$ for $k = 1, 2, \ldots,$ as defined in (4.34)-(4.35) have been used for computing the projection matrix $V$ (4.36).

**Remark 4.3.1** *Note that the formulation of $V$ and its dimension are the same for all the types of projection discussed in this section. The only difference between these projection types is the definition of the matrices configuration at the initial setting of the Krylov subspaces. These seemingly slight differences as will be shown in subsequent subsections and sections have a significant effect on the input-output relationship preservation for the reduced order model.*

The advantage of using the Improved Phillips approach is that it is a compromise between the Phillips type projection (Phillips 2000) and the Feng and Benner (Feng & Benner 2007) type projection. It improves the matched linear moments when computing $V^{\{1\}}$ and reduces the loss of information which occurs by multiplying the inverse of the system matrix $A$ with $N$. This is for the case where $N$ is a singular matrix.

## 4.4 Parametrised linear approximation for multimoment matching

The formulation of the projection matrix $V^{\{1\}}$ for the methods, i.e. Phillips (Phillips 2000) type projection, Feng and Benner (Feng & Benner 2007) type projection and the newly proposed Krylov subspaces for multimoment matching (4.23) - (4.25), aattempts to match moments of the linear approximation of the bilinear model. In this subsection, we propose a new and improved approach, allowing any linear approximation of the bilinear model to satisfy the condition for matching multimoments of the bilinear system. This approach promises to achieve 'better' preservation of the input-output propertied of the bilinear model. This is demonstrated here by utilising the linear approximation of

$$\dot{x} = Ax + Nxu + Bu \tag{4.38}$$

for a constant input $u = \eta$ as given below

$$\dot{x} = Ax + Nx\eta + B\eta \tag{4.39}$$

$$y = Cx, \tag{4.40}$$

where $A_\eta = [A + N\eta]$ and $B_\eta = B \times \eta$, the linear approximation of the bilinear system can be written as

$$\dot{x} = A_\eta x + B_\eta u \tag{4.41}$$

$$y = Cx. \tag{4.42}$$

This forms a linear approximation of the bilinear model for a constant input $\eta$ (Flagg 2012). This will be referred to as the parametrised linear approximation of the bilinear model. With this linear approximation, a Krylov subspace can be defined such that

$$span\{V^{\{1\}}\} = K_{q_1}(A_\eta^{-1}, A_\eta^{-1}B_\eta) \tag{4.43}$$

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, NV^{\{k-1\}}) \tag{4.44}$$

$$span\{V\} = span\{\bigcup_{k=1}^{k} span\{V^{\{k\}}\}\}. \tag{4.45}$$

This formulation a parametrised linear approximation of the bilinear system allow the computation of the projection matrices and subsequently the reduced order models that can be described as being one-sided. The parametrised linear approximation has been applied to the Improved Phillips type projection (4.43) - (4.45).

All the methods discussed in Section 3.5 use a conventional linear approximation, with transition matrix $A$, of the bilinear model for achieving this. Note that the linear approximation approach can be applied to the other moment matching approaches such as Phillips type projection (Phillips 2000) and Feng and Benner type projection (Breiten & Damm 2010). A comparison of these methods can be carried out numerically. When reducing the order of a bilinear model via Krylov subspaces, the computation of $V^{\{1\}}$ is done first. As there exists a linear approximation of the bilinear model for a constant input applied to the bilinear systems over a finite time, consider using a bilinear model as described in Example 4.4.1 where the outputs of both linear approximations are compared using input-output plots over time.

**Example 4.4.1** ***Linear approximation of bilinear models***

*Consider the bilinear model presented in (Flagg 2012):*

$$A = \begin{bmatrix} -1 & 0 & 0.1667 & 0 \\ 0 & -2 & 0 & 0.25 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \tag{4.46}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \tag{4.47}$$

*where* $A \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ *and* $n = 4$.

A constant input of $\eta = 0.5$ is chosen to achieve a linear approximation of the system. This is compared to a linear approximation of the bilinear model where the linear approximation uses only the system matrix $A$. All three models are excited using a sinusoidal input $u = \sin(t)$. The plotted output of the models can be observed in Figure 4.1, where the parametrised linear approximation, tends

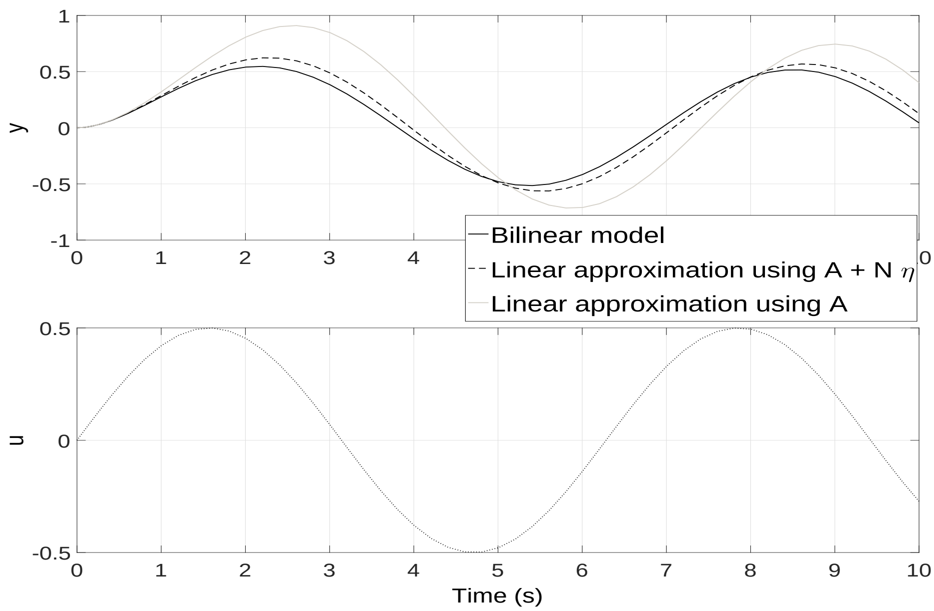



Figure 4.1: Comparison of linear approximation using the conventional state transition matrix and the parametrised linear approximation

to provide a better approximation of the bilinear model.

Similar linear approximations have been used in various applications. In (Juang & Lee 2012) a bilinear system identification algorithm was developed by introducing constant inputs at designated sampling points. A linear approximation for bilinear systems of this kind was also used to compute observability and controllability Gramians for bilinear systems in (Condon & Ivanov 2005).

## 4.5 Case studies

### 4.5.1 Simulation based study

A form of this simulation based study as described in this subsection has been used in (Baur et al. 2014) to understand the effect of the parameters on the reduction process. Here, using the input defined as

$$u = (\cos(2\pi t/10) + 1)/2, \tag{4.48}$$

the effect of the parameters $q_1$, $q_2$ and $p_2$ on the reduced order model are investigated. It is expected that the accuracy of reduced order model will improve when the order increases and vice versa as this corresponds to the multimoments matched. However, this is not always the case as it has been reported that Krylov subspace methods lose accuracy as the subspace dimension increases. The parameters, $q_1$, $q_2$ and $p_2$ affect the dimensions of the reduced order model and will also determine the input-output preservation of the reduced order model. However, it is more desirable to achieve a much lower order at a reasonably high accuracy.

Using $V^{\{1\}}$ and $V^{\{2\}}$ to compute the projection bases, $V$, an experiment is carried out by defining four cases. Each case possesses different values of $q_1$, $q_2$ and $p_2$ for the Algorithm 3.1.

- **Case 1:** $q_1 = 20$, $q_2 = 1$ and $p_2 = 1$
- **Case 2:** $q_1 = 11$, $q_2 = 2$ and $p_2 = 5$
- **Case 3:** $q_1 = 5$, $q_2 = 16$ and $p_2 = 1$

- **Case 4:** $q_1 = 17$, $q_2 = 2$ and $p_2 = 2$

This is done for all the multimoment matching methods discussed in Chapter 3 (Phillip type projection (Phillips 2000) and Feng and Benner type projection (Breiten & Damm 2010)), and the Improved Phillip type projection discussed in Section 4.3.

All the cases presented here are expected to achieve the same reduced order dimensions. The results are to be assessed using performance criteria which will be predefined, graphic output and absolute error plots.

### 4.5.2 Projection procedure

After selecting the parameters, $q_1$, $q_2$ and $p_2$, The algorithm used to compute the projection matrices $V$ and $V^T$ is given in Algorithm 3.1 which requires a slight modification for the Phillips type projection (Phillips 2000) and Improved Phillips Type projection. For the Phillips type projection, the input vectors are $\mathbb{N} = A^{-1}$ and $\mathbb{M} = B$. For the Improved Phillips type projection, the input vectors are $\mathbb{N} = A^{-1}$ and $\mathbb{M} = A^{-1}B$. Also in step 11, $G = NV^{\{1\}}_{[p_2]}$ for the Phillips (Phillips 2000) and Improved Phillips type projection. Note that for the Condon type projection, only $V^{\{1\}}$ has been used as the right projection base, i.e $V = V^{\{1\}}$. This means that an algorithm which matches only moments is sufficient. An algorithm for matching moments has been presented in (Tan & He 2007). This process for constructing $V^{\{1\}}$ is know as the Arnoldi process and has been discussed in Chapter 2. After getting the projection bases, the reduced order models are computed as described in Subsection 3.6.1. An algorithm for this is given below:

**Algorithm 4.1 (Reduced order matrix computation)**

1. ***Input:*** $V$

2. $\hat{A} = V^T A V$

3. $\hat{N} = V^T N V$

4. $\hat{B} = V^T B$

5. $\hat{C} = C V$

The implementation of the Algorithm and simulation of models has been carried out using Matlab and Simulink. Two examples of bilinear models are presented as discussed next. The first example demonstrates the use of case studies for determining parameters $q_1$, $q_2$ and $p_2$. This is followed by the MOR of a flow model. These two examples have been used by (Bai & Skoogh 2006) and (Breiten & Damm 2010) respectively to demonstrate Carleman bilinearization and Krylov subspace MOR. In order to compare the different reduced order models, some performance criteria have been used as discussed next.

### 4.5.3 Performance criteria

The quality of a reduced model using Krylov subspaces is determined by how many moments are matched. Reduced models are only useful when pre-set criteria are reached.

C.1 Integral of absolute error: In order to assess the goodness of fit of the reduced order model in the time domain, a quantitative efficacy index is required. In this thesis the integral of absolute error (IAE) of the difference in the output responses between the high order bilinear model and the reduced order bilinear model is used, which is calculated as

$$\text{IAE} = \int_0^{n_s} |y(t) - \hat{y}(t)| dt \tag{4.50}$$

where $y$ and $\hat{y}$ are the outputs of the higher order bilinear model and the reduced order bilinear model respectively. Terms $n_s$ refer to the number of the samples collected in the time sequence and $t$ is the time sequence. These notations are also used for the remaining performance criteria to be discussed.

C.2 Coefficient of determination: The coefficient of determination ($RT^2$ is mathematically defined as

$$RT^2 = 100 \times \frac{||\hat{y} - y||_2^2}{||y - y_{mean}||_2^2}, \tag{4.51}$$

where $y_{mean}$ is the mean of bilinear/nonlinear system output. The desire is to keep the coefficient of determination as high as possible. Generally, a model with an $RT^2 = 90$ is regarded as highly acceptable.

C.3 Mean square error: The mean square error (MSE) computes the average of the squared deviations of the reduced order model from the high order model

$$\text{MSE} = \frac{1}{n_s} \sum_{i=1}^{n_s} (\hat{y} - y)^2 \tag{4.52}$$

and is analogous to mean squared deviation (MSD). It is widely used in statistics, regression analysis and parameter estimation.

C.4 Sum of square of error: The sum of square of error (SSE) is a mathematical function which computes the sum of the squared errors of the reduced order model and is mathematically defined as:

$$\text{SSE} = \sum_{i=1}^{n_s} (\hat{y} - y)^2 \tag{4.53}$$

The SSE is also known as the residual sum of squares (RSS) and the sum of squared residuals (SSR). The desire is to keep the SSE as small as possible. This criterion is widely used in parameter and model selection.

C.5 IAE divided by number of samples (NIAE): The NIAE is the IAE divided by the number of samples

$$\mathrm{NIAE} = \frac{1}{n_s}\int_0^{n_s} |y(t) - \hat{y}(t)|dt. \tag{4.54}$$

C.6 Simulation time in seconds (ST) During the simulations, the time for each algorithm for computing the projection matrices is measured,

$$\mathrm{ST} = \frac{1}{ns}\sum_{i=1}^{ns} \mathrm{ST}_i \tag{4.55}$$

where $ns$ is the number of simulations carried out.

These criteria have been used to determine the accuracy of the reduced order models. Also, the magnitude error of each reduced order model with respect to the input $u$ and the absolute error with respect to time depicted in graphic plots have been used to analyse the reduced order models in Subsections 4.5.4 and 4.5.5. Note that the time series used in computations used in this thesis have not been equally spaced as a variable time step has been used for simulations. However, both high order and low order model output values which have been analysed using the performance criteria are identical in size and sampling points.

### 4.5.4 Case study 1: A nonlinear RC circuit

The nonlinear model used in this paper to compare the methods discussed is a transmission line model of $20^{th}$ order, i.e. $n = 20$ as illustrated in Figure 4.2.

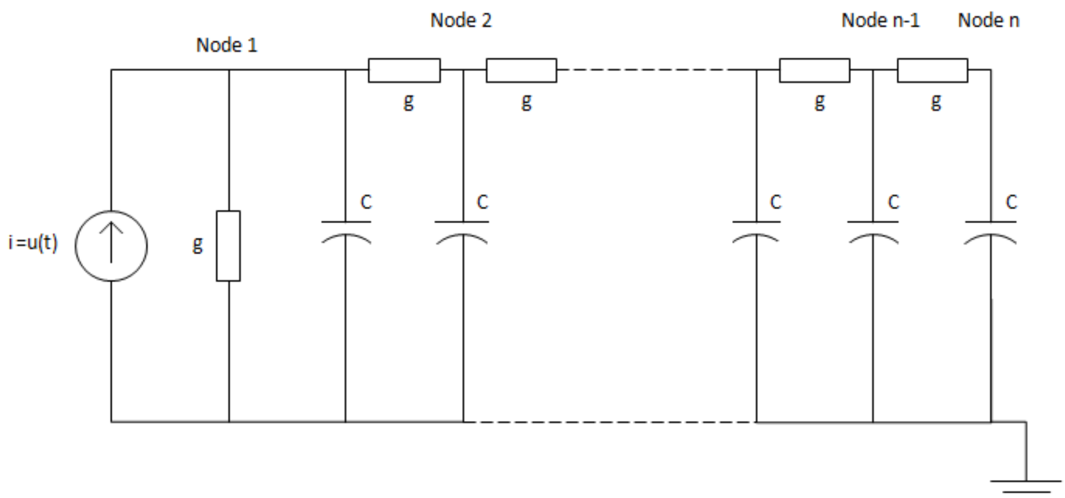



Figure 4.2: Nonlinear circuit

The nonlinear circuit model is of the form (3.3) - (3.4) discussed in Chapter 3, where $f(x) = f(v)$, input $B$ and output $C$ matrices are given as

$$f(v) = f_k v = \begin{bmatrix} -g(v_1) - g(v_1 - v_2) \\ g(v_1 - v_2) - g(v_2 - v_3) \\ \vdots \\ g(v) - g(v_{k-1} - v_k) \\ \vdots \\ g(v) - g(v_{n-1} - v_n) \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad (4.56)$$

The model is composed of linear capacitors which are assumed to have capacitance values of unity, i.e. $C = 1$, and nonlinear resistors where the resistance

$g(v)$ is a function of voltage:

$$g(v) = \exp(40v) + v - 1 \tag{4.57}$$

As the definition of the output vector indicates, the output of the nonlinear circuit is the voltage between node 1 and the ground.

A state vector $x = [x^{(1)} x^{(2)} x^{(3)}]$ has been defined for the Carleman bilinearization of the nonlinear model. This results in system matrices, input and ouput vectors with the following dimensions, $A \in \mathbb{R}^{8420 \times 8420}$, $N \in \mathbb{R}^{8420 \times 8420}$, $B \in \mathbb{R}^{8420 \times 1}$, $C \in \mathbb{R}^{1 \times 8420}$.

**Results:**

First of all, the results of the Phillips (Phillips 2000) type projection for the case studies is presented. The outputs $y$ of the different cases are shown in the first row of Figure 4.3. The middle row of this figure shows the input $u$ whilst the absolute error values are shown in the bottom row. In Table 4.1, the performance criteria values of the different cases are shown.

Table 4.1: Performance criteria for different experimental cases of the Phillips (Phillips 2000) type method.

| Phillip type | $RT^2$ | MSE | IAE | NIAE | SSE | NSSE |
|---|---|---|---|---|---|---|
| Case 1 | 96.98 | 9.6002e-07 | 0.4470 | 7.4617e-04 | 5.7505e-04 | 9.6002e-07 |
| Case 2 | 99.87 | 4.0522e-08 | 0.0799 | 1.3338e-04 | 2.4273e-05 | 4.0522e-08 |
| Case 3 | 99.56 | 1.3929e-07 | 0.1528 | 2.5504e-04 | 8.3435e-05 | 1.3929e-07 |
| Case 4 | 99.90 | 3.0749e-08 | 0.0697 | 1.1632e-04 | 1.8419e-05 | 3.0749e-08 |

As presented, the results suggest that Case 4 with parameter values $q_1 = 17$, $q_2 = 2$, and $p_2 = 2$, produce the best results. Next, the experimental results of the Feng and Benner (Feng & Benner 2007) type projection are presented. Figure 4.4 compares the output of the reduced order model to that of the nonlinear model in the top figure. In the middle is the input $u$ and the absolute error values

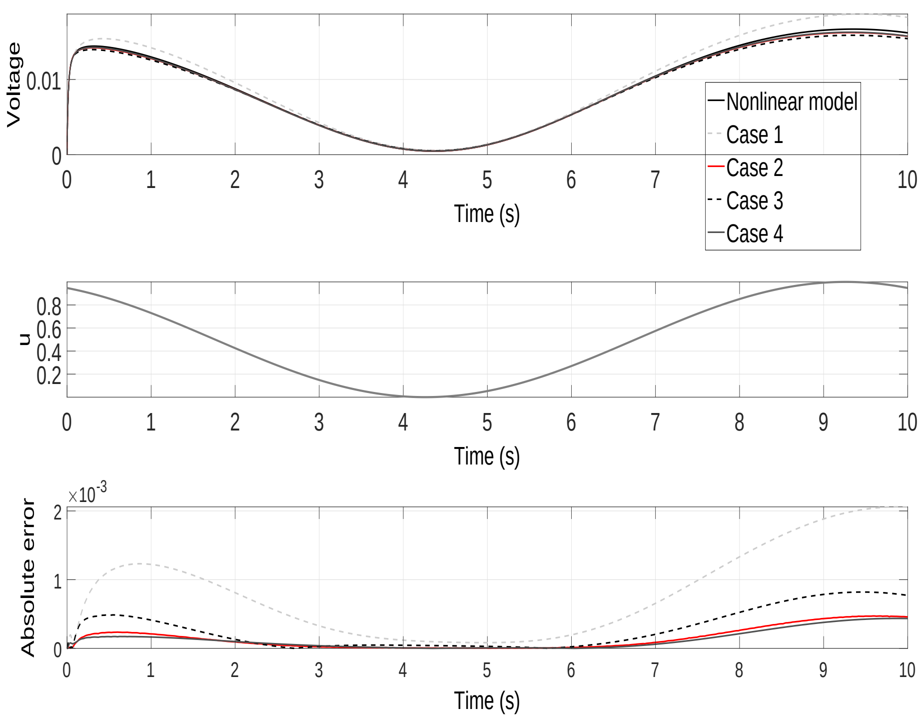



Figure 4.3: Time response $y$ of bilinear model and reduced order of different cases, input $u$ and absolute error values for all the cases using the Phillips (Phillips 2000) type projection.

are shown in the bottom figure. In Table 4.2, the performance criteria values of the different cases 1 to 4 are shown. For the Feng and Benner projection, the, Case 2 ($q_1 = 11$, $q_2 = 2$, $p_2 = 5$) and Case 4 ($q_1 = 17$, $q_2 = 2$, $p_2 = 2$) show similar results. However, Case 2 is slightly better with $RT^2 = 99.86$, 0.11 more than Case 4. The experimental results of the Improved Phillips type projection are presented in Figure 4.5. This compares the output of the reduced order model to that of the nonlinear model in the first row. In the second row is the input ($u$) and the absolute error values are shown in the third row. In Table 4.3, the performance criteria values of the different cases 1 to 4 are shown. Also in the experimental values for the Improved Phillips type projection show improved

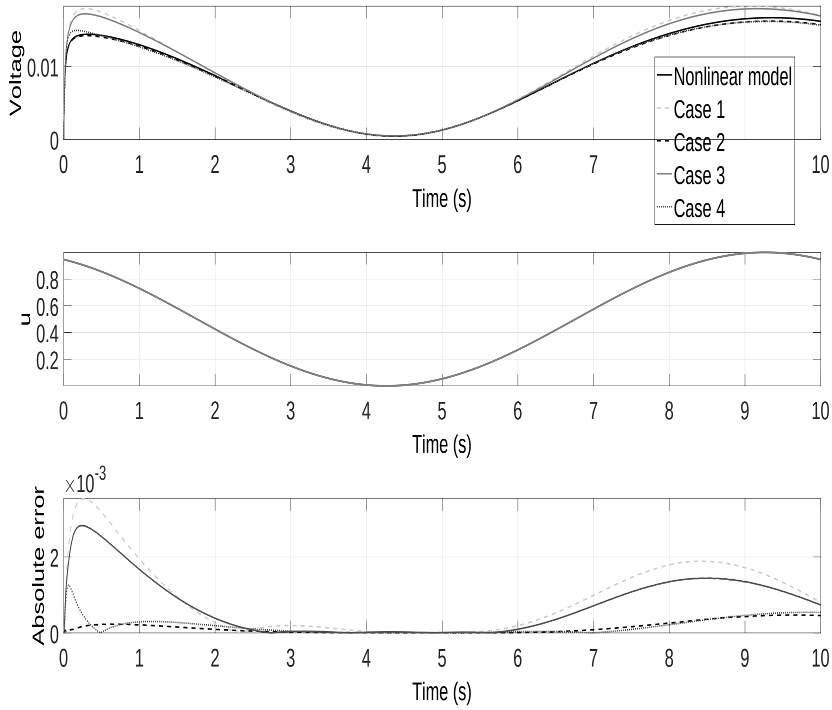



Figure 4.4: Time response $y$ of bilinear model and reduced order of different cases, input $(u)$ and absolute error values for all the cases using the Feng and Benner (Feng & Benner 2007) type projection.

values for the Cases 2 and 4 when compared to the other cases. The results for the experiment suggest that choosing $q_1$ too high does not necessarily improve the output of the reduced order model. Likewise, selecting a high value of $q_2$ does not improve the results. However, values of $p_2$ higher than 1 is likely to increase the input-output preservation capacity of the reduced order model. These experimental results suggest that the computation of a reduced order model that preserves input-output relationship which satisfies a set of performance criteria is highly dependent on $V^{\{1\}}$ and therefore the correct selection of the parameters $q_1$ and $p_2$ is critical. This knowledge has been used to manually derive a much lower reduced order model for the Phillips type projection (Phillips 2000), Feng

Table 4.2: Performance criteria for different experimental cases of the Feng and Benner type method.

| Feng and Benner | $RT^2$ | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|---|
| Case 1 | 94.51 | 1.7247e-06 | 0.4939 | 9.3022e-04 | 9.1583e-04 |
| Case 2 | 99.86 | 4.4082e-08 | 0.0764 | 1.4385e-04 | 2.3408e-05 |
| Case 3 | 96.46 | 1.1107e-06 | 0.3932 | 7.4053e-04 | 5.8980e-04 |
| Case 4 | 99.75 | 7.7999e-08 | 0.0964 | 1.8163e-04 | 4.1418e-05 |

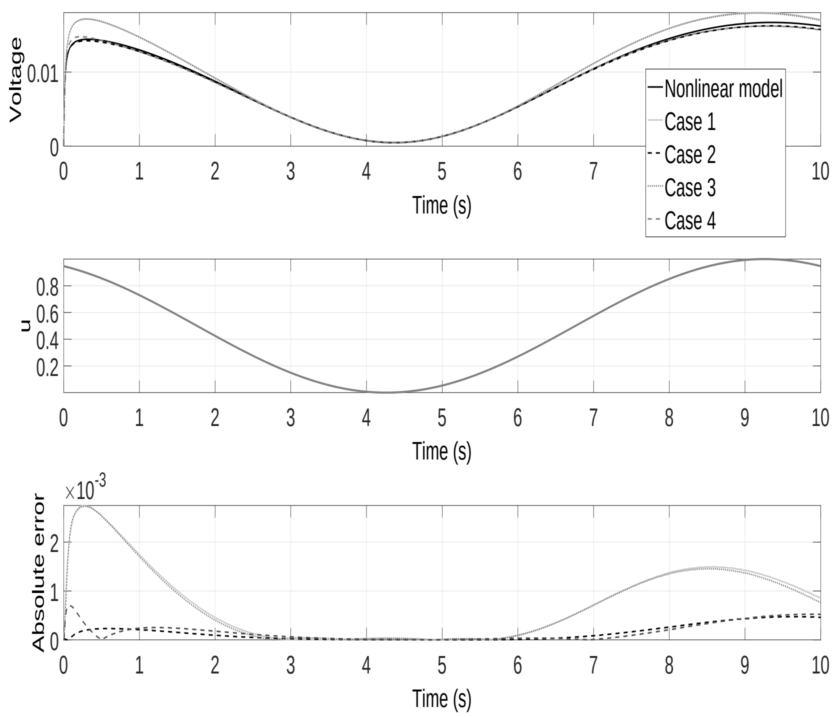


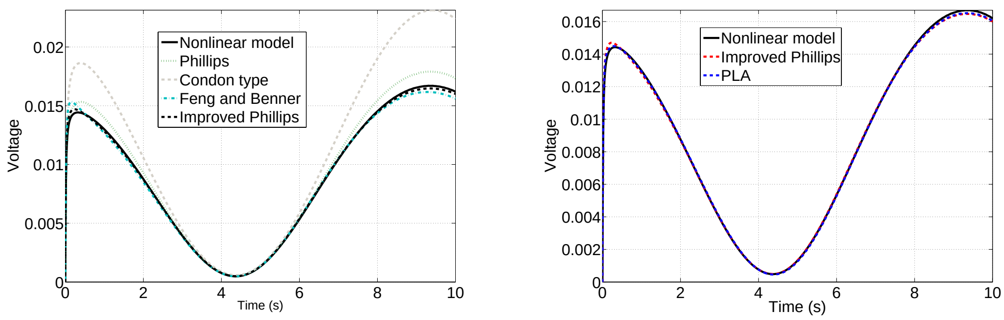

Figure 4.5: Time response $y$ of bilinear model and reduced order of different cases, input $(u)$ and absolute error values for all the cases using the Improved Phillips type projection.

and Benner (Feng & Benner 2007), Improved Phillips and Parametrised Linear Approximation projection (PLA) techniques whose results will be shown next. The results of using a one-sided Condon (Condon & Ivanov 2007) approach are also included. The computation of $V^{\{1\}}$ for the PLA type projection uses the

Table 4.3: Performance criteria for different experimental cases of the Improved Phillips type method.

| Improved Phillips | $RT^2$ | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|---|
| Case 1 | 96.38 | 1.1453e-06 | 0.4118 | 7.5701e-04 | 6.2306e-04 |
| Case 2 | 99.86 | 4.3794e-08 | 0.0770 | 1.4160e-04 | 2.3864e-05 |
| Case 3 | 96.53 | 1.0976e-06 | 0.4001 | 7.3544e-04 | 5.9710e-04 |
| Case 4 | 99.82 | 5.5529e-08 | 0.0874 | 1.6058e-04 | 3.0208e-05 |

parameter $\eta = 0.522$. The results presented in here are for reduced order models of $7^{th}$ order where the parameters $q_1 = 5$, $p_2 = 2$ and $q_2 = 1$.




(a) Plot of reduced order bilinear models via Condon (Condon & Ivanov 2007), Feng and Benner (Feng & Benner 2007) (FB), Phillips (Phillips 2000) (IP) and nonlinear model.

(b) Plot of nonlinear model, Improved Phillips (IP) and parametrise linear approximation (PLA) reduced order bilinear models.

Figure 4.6: Plots of outputs $\hat{y}$ for reduced order models and nonlinear model output $y$.

Figure 4.6(a) shows a graphic comparison of the simulated outputs for the Condon type projection (Condon & Ivanov 2007), Phillips type projection (Phillips 2000), Feng and Benner type projection (Feng & Benner 2007), the Improved Phillips type projection and the nonlinear circuit model. As can be observed, the approach proposed by Condon et. al. (Condon & Ivanov 2007), when applied to a one sided projection, is not as effective as the other methods. This is because only moments are matched. Figure 4.6(b) compares the output for the Improved Phillips type projection with that of the PLA output. The effect of the so called better linear approximation of the bilinear model can be observed. This trend can also be observed in Table 4.4 which shows the $RT^2$, SSE, IAE, NIAE and MSE values of the reduced order models. Comparing the results presented in Table 4.4, IP improves, in terms of $RT^2$, the results found using Phillips (Phillips 2000) and Fend and Benner (Feng & Benner 2007) by 1.4% and 0.28% respectively. Applying PLA for MOR further improves on the results from IP in terms of $RT^2$ by 0.02% and IAE by 20%. These results confirm the effectiveness of using an alternate linear approximation for computing a reduced order system model, see Section 4.4.

The simulation time values for computing the projection matrices are 271 seconds, 292 seconds, 424 seconds and 293 seconds for Phillips (Phillips 2000), Improved Phillips, Feng and Benner (Feng & Benner 2007) and PLA respectively. These values were computed using the average of 6 simulation runs. This shows similar simulation times for IP, Phillips (Phillips 2000) and PLA, with Phillips being the fastest, as expected. Feng and Benner (Feng & Benner 2007) is about 1.5 times slower than IP due to the additional matrix inversion required.

Table 4.4: Performance criteria for 7th order reduced order models.

| | $RT^2$ | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|---|
| Condon | 65.01 | 1.0982e-05 | 1.3207 | 0.0025 | 00.58 |
| Phillips | 98.50 | 4.6991e-07 | 0.2846 | 5.3504e-04 | 2.4999e-04 |
| Feng and Benner | 99.66 | 1.0718e-07 | 0.1121 | 2.1063e-04 | 5.7020e-05 |
| Improved Phillips | 99.94 | 1.9686e-08 | 0.0482 | 9.0529e-05 | 1.0473e-05 |
| PLA | 99.96 | 1.1669e-08 | 0.0385 | 7.2371e-05 | 6.2077e-06 |

According to (Rugh 1981) the error, $y - \hat{y}$, of bilinear approximations is a function of the input and number of Taylor series expansion terms. In Figures 4.7, 4.8 and 4.9 the reduced order model errors corresponding to the increasing input is shown. Comparing the reduced order model of Feng and Benner type projection with the Phillips type projection in Figure 4.7, the Feng and Benner type projection shows less error for most of the input values. This can also be observed in the $RT^2$ values. However, this is not the case when it is compared the Improved Phillips type projection in Figure 4.8.

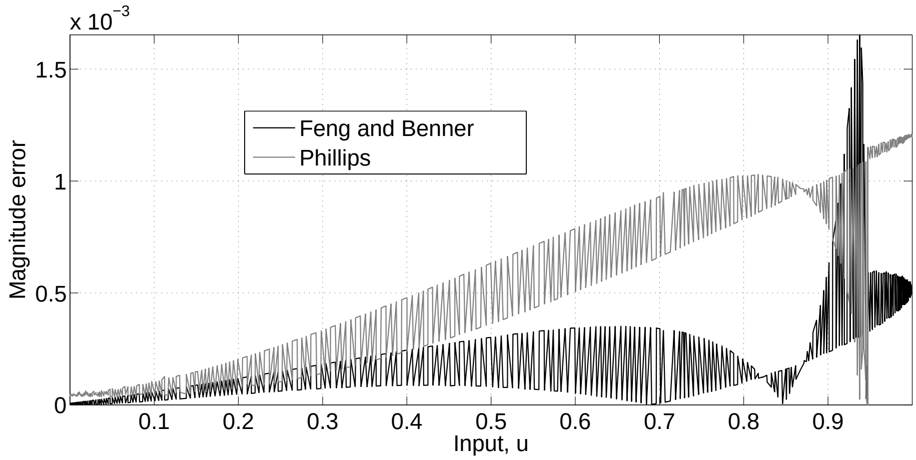



Figure 4.7: Comparison of Phillips (Phillips 2000) type projection with Feng and Benner (Feng & Benner 2007) (FB) type projection using plot of magnitude error against corresponding ascending input values.

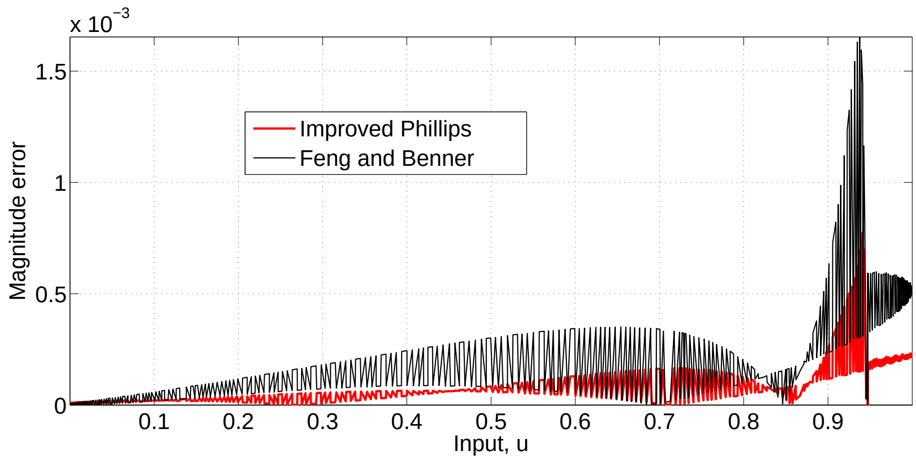



Figure 4.8: Comparison of Improved Phillips type projection with Feng and Benner (Feng & Benner 2007) (FB) using plot of magnitude error against corresponding ascending input values.

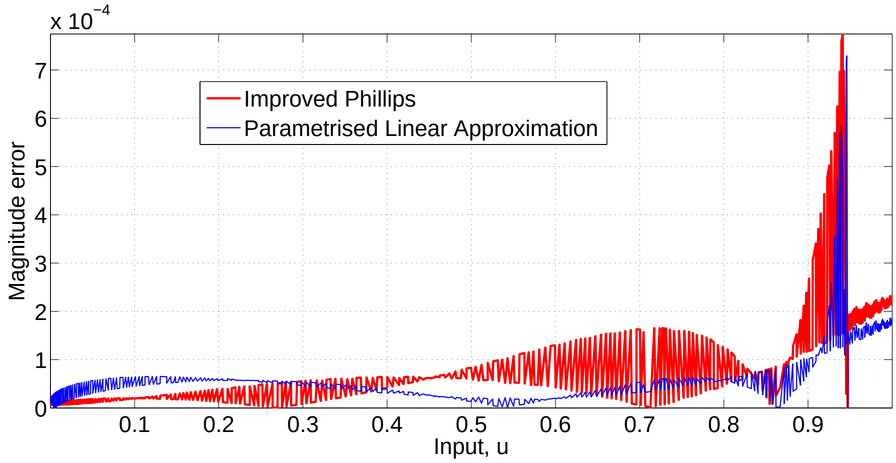



Figure 4.9: Comparison of Improved Phillips (IP) type projection with Parametrised Linear Approximation (PLA) approach using plot of magnitude error against corresponding ascending input values.

As can be observed in Figure 4.9, the Improved Phillips type projection shows less errors for smaller inputs but as the input gets bigger, the PLA maintains its error range whilst the first method becomes worse. Overall, the parametrised linear approximation approach shows a better input-output preservation for the nonlinear model.

### 4.5.5 Case study 2: Flow model

Another model which can be used to demonstrate bilinearization and Krylov subspace model order reduction for nonlinear models is the flow model which has been used in (Breiten & Damm 2010). The system presented therein is a one-dimensional Burgers equation:

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = \frac{\partial}{\partial t}\left(v\frac{\partial w}{\partial x}\right), \quad \text{for } (x,t) \in (0,L) \times (0,T), \tag{4.58}$$

with

$$w(x,0) = p(x) \text{ for } x \in (0,L) \tag{4.59}$$

$$w(0,t) = u(x) \text{ for } t \in (0,T) \tag{4.60}$$

$$w(L,t) = q(x) \text{ for } t \in (0,T), \tag{4.61}$$

where $x$ is a point at time, $w(x,t)$ is the velocity, and $v$ is the viscosity coefficient which also depends on space and time. In order to reduce the model order, (Breiten & Damm 2010) assumed a constant viscosity coefficient. A zero initial condition is also imposed on the system. Only the left boundary condition is controlled while the right boundary condition is 0. A spacial discretization of (4.58) results in a nonlinear control system with nonlinear functions of system states.

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} -\frac{w_1 w_2}{2h} + \frac{v}{h^2}(w_2 - 2w_1) \\ -\frac{w_2}{2h}(w_3 - w_1) + \frac{v}{h^2}(w_3 - 2w_2 + w_1) \\ \vdots \\ -\frac{w_i}{2h}(w_{i+1} - w_{i-1}) + \frac{v}{h^2}(w_{i+1} - 2w_i + w_{i-1}) \\ \vdots \\ \frac{w_n w_{n-1}}{2h} + \frac{v}{h^2}(-2w_n + w_{n-1}) \end{bmatrix} + \begin{bmatrix} \frac{w_1}{2h} + \frac{v}{h^2} \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \tag{4.62}$$

This nonlinear control system is of the form

$$\dot{w} = f(w) + g(w)u \tag{4.63}$$

$$y = Cw, \tag{4.64}$$

where $f(w)$ and $g(w)$ are nonlinear functions with Taylor series expansions

$$f(w) = A_1 w + A_2(w \otimes w) \tag{4.65}$$

and

$$B(w) = G_0 + G_1 w, \tag{4.66}$$

where $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{n\times n^2}$ are the first and second derivatives of $f(w)$. $G_0 \in \mathbb{R}^n$ and $G_1 \in \mathbb{R}^{n\times n}$ are the solution of $g(w)$ at $w = 0$ and the first derivative of $g(w)$ respectively.

A Carleman bilinearization is then carried out on the nonlinear control system by introducing a state vector

$$x = \begin{bmatrix} w \\ w \otimes w \end{bmatrix}. \tag{4.67}$$

This results in the following bilinear system matrices,

$$A = \begin{bmatrix} A_1 & \frac{1}{2}A_2 \\ 0 & A_1 \otimes I + I \otimes A_1 \end{bmatrix}, \quad N = \begin{bmatrix} G_1 & 0 \\ G_0 \otimes I + I \otimes G_0 & 0 \end{bmatrix} \tag{4.68}$$

$$B = \begin{bmatrix} G_0 \\ 0 \end{bmatrix}, \quad C = \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{4.69}$$

The simulations have been done with the parameters of the nonlinear model $L = 1$, $v = 0.1$ and with state space order, $n = 50$.

Using Algorithms 3.1 and 4.1, the high order bilinear model is reduced to a 3rd order model using the Phillips (Phillips 2000) type, Improved Phillips type and Feng and Benner (Feng & Benner 2007) type projection methods. The parameters of Algorithms 3.1 have been set as $q_1 = 1$, $q_2 = 1$ and $p_2 = 1$. These parameters have been obtained experimentally. Simulation results using input function,

$$u = \sin((2\pi/10) \times 50t + 50) \tag{4.70}$$

are presented.

**Results:**

The model reduction outcomes of this case study uses performance criteria $RT^2$, MSE, SSE, IAE and NIAE to show the goodness of fit for each reduced order model which have been derived by the Phillips (Phillips 2000) type projection, Feng and Benner (Feng & Benner 2007), Improved Phillips and PLA type projections. Graphical plots have also been used to show the system outputs and error values.

Table 4.5 shows the performance criteria values of the reduced order models. The numerical figures for Phillips (Phillips 2000) type projection and the Improved Phillips type projection are identical. This is because they are quite similar in the computation of their subspaces. Whilst the figures for Feng and Benner show better results. For all the three methods, $RT^2 = 100$.

Table 4.5: Performance criteria values of MSE, IAE, NIAE and SSE for reduced order models via Phillips (Phillips 2000) type, Feng and Benner (Feng & Benner 2007) and Improved Phillips type projections.

| Methods | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|
| Phillips | $5.4760e-07$ | 5.0134 | $5.3019e-04$ | 0.0052 |
| Fend and Benner | $1.2352e-11$ | 0.0114 | $1.2074e-06$ | $1.168e-07$ |
| Improved Phillips | $5.4760e-07$ | 5.0134 | $5.3019e-04$ | 0.0052 |

Figure 4.10 shows the output ($y$, average speed) plots of the reduced order models and that of the high order bilinear model against time ($t$) in the first row. The second row shows the input plot against time ($t$). Figure 4.11 shows the plot absolute error of the model outputs against input in ascending order in the first row and shows the plot of absolute error values in time. In Figure 4.11, it can be observed, as it was for Table 4.5 with MSE = $5.3019e-04$, SSE $= 1.168e-07$, IAE $= 1.2074e-06$ and NIAE $=1.168e-07$ that the Feng and Benner (Feng & Benner 2007) type projection produces reduced order models

of higher accuracy when compared to Phillips (Phillips 2000) and the Improved Phillips type projection with MSE = $5.4760e-07$, SSE = 0.0052, IAE = 5.0134 and NIAE = $5,3019e-04$ when the flow model is considered.

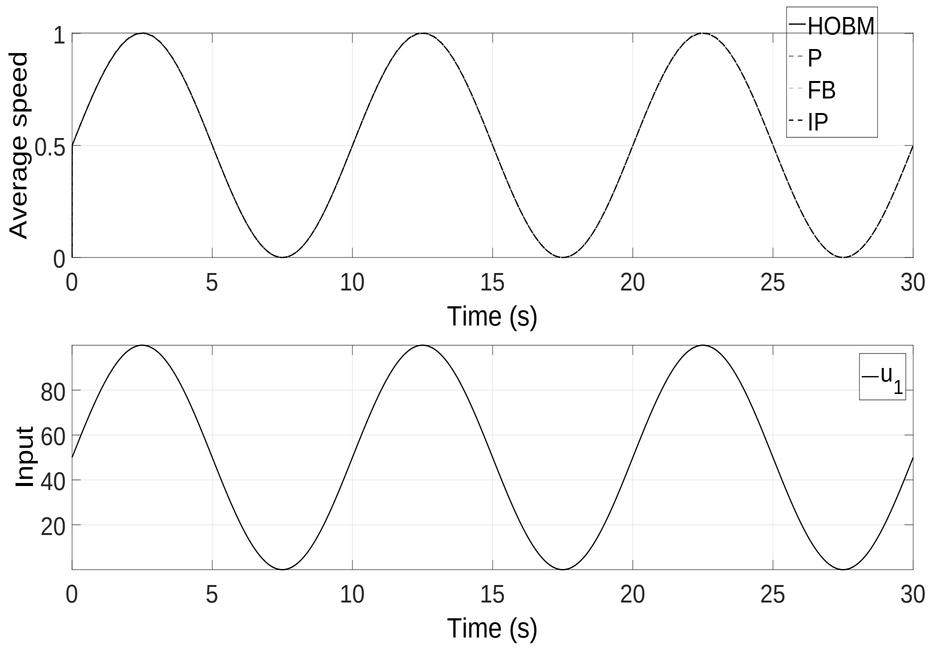



Figure 4.10: Time response $y$ (average speed) of high order bilinear model (HOBM) and reduced order models via Phillips (Phillips 2000) (P), Feng and Benner (Feng & Benner 2007) (FB) and Improved Phillips (IP) type projections for input $u$.

The application of PLA via Feng and Benner (Feng & Benner 2007) can be done by using the Krylov subspaces as defined below:

$$span\{V^{\{1\}}\} = K_{q_1}(A_\eta^{-1}, A_\eta^{-1}B_\eta) \tag{4.71}$$

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}}) \tag{4.72}$$

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^{\{k\}}\}\}. \tag{4.73}$$

Computing the reduced order model using PLA produces the following results for simulations carried out using the input,

$$u = \sin((2\pi/10) \times 50t) + 500. \tag{4.74}$$

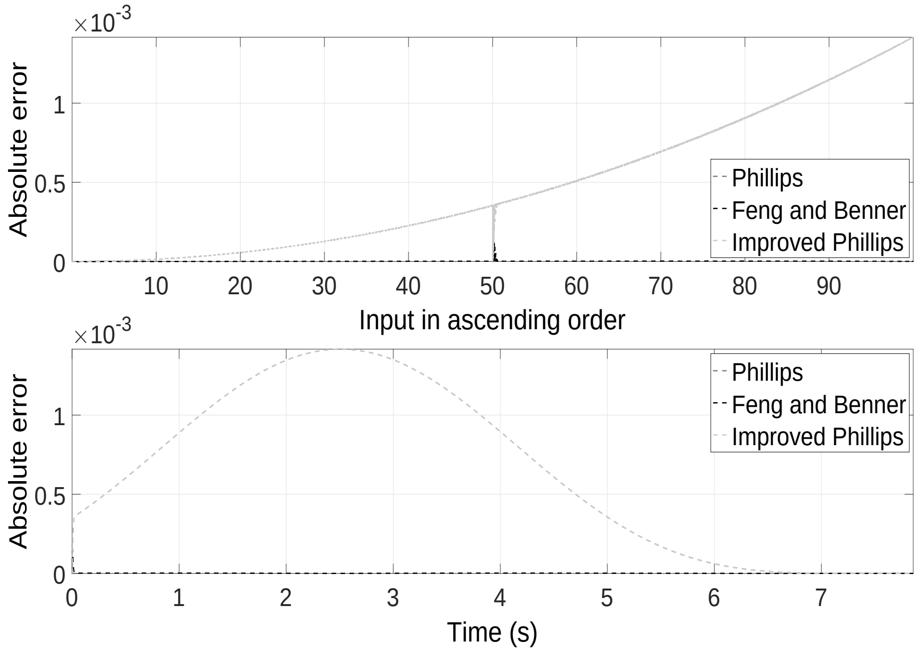



Figure 4.11: Time response $y$ of bilinear model and reduced order of different cases, input $u$ and absolute error values for all the cases using the Feng and Benner type projection.

Table 4.6 shows the performance criteria values for the reduced order models using Feng and Benner and PLA via Feng and Benner. The amplitude of the input has been increased in this case to highlight the advantage of the PLA approach. The linear approximation parameter has been chosen to be $\eta = 10$.

## 4.6 Discussion

Considering Case study 1, the proposed Improved Phillips (Phillips 2000) type approach tends to be as effective as the Feng and Benner (Feng & Benner 2007) approach for model order reduction. However, it seems to be more promising when computing models of very low order in light of the case study.

In the different cases of parameters presented for Case study 1, there is a deflation in Case 3 which results in an order of 17. Deflation does not occur

Table 4.6: Performance criteria values of IAE and SSE for reduced order models via Feng and Benner (Feng & Benner 2007) and PLA via Feng and Benner type projection.

| Methods | $IAE$ | $SSE$ |
|---|---|---|
| Fend and Benner | 73.8428 | 12.5305 |
| PLA | 73.6805 | 12.5249 |

when the operating parameters of the algorithm ($q_1 = 5$, $q_2 = 1$, $p_2 = 2$) have been reduced. This results in reduced order models of $7^{th}$ order.

In Case study 2, the Feng and Benner (Feng & Benner 2007) type projection produces a reduced order model with better performance criteria when compared to the Improved Phillips type and the Phillips (Phillips 2000) type projection. When the PLA approach is then applied via L. Feng and P. Benner (Feng & Benner 2007), the performance criteria values of the PLA were improved compared to those for the Feng and Benner (Feng & Benner 2007) approach.

Whilst it is suspected that the best reduced order model derived from all the simulated methods will depend on the type of matrices possessed by the high order system, the parametrised linear approach promises to reduce the error when applied to each of these approaches. This improved input-output preservation has been achieved at very low cost of computing $A_\eta$ and $B_\eta$. When compared to the simulation time of the other approaches, the parametrised linear approximation (PLA) is quite efficient.

The implications of using an alternate linear approximation for reducing bilinear models are not only significant for preserving the input-output behaviour of the bilinear model, Their application suggests that they can be used for reducing systems with non-invertible matrices. This is possible if the alternate linear approximate of the bilinear model is non-singular. This will be discussed further in Chapters 5 and 6.

## 4.7 Conclusion

Two novel results have been proposed in this chapter; the Improved Phillips and the parametrization of the linear model.

The present study showcases the use of different Krylov subspace projection methods which have been proposed by other authors for matching the moments and multimoments of a bilinear model. With the use of Carleman bilinearization the approximation of nonlinear models has been shown. An improved approach which matches multimoments of the multivariable transfer function of the resulting bilinear model has been implemented. In addition, it has been demonstrated that the moments of the first transfer function of a bilinear model can be improved by using an alternative linear approximation of the bilinear system. These findings have been illustrated by reducing a bilinearised nonlinear circuit model. With the use of coefficient of determination, mean square error and magnitude error and graphic plots a comparison of the different Krylov subspace reduced models has been carried out.

An experimental procedure has been done to identify the effect of the parameters of the Algorithm 3.8 on the reduced order model. The results of the experiment agree with other authors as according to (Baur et al. 2014) the input-output presevation of the reduced order model is highly dependent on the computation of $V^{\{1\}}$.

The results obtained here suggest that the use of Krylov subspaces for matching multimoments is quite subjective as the quality of the reduced or model is dependent on the nature of system matrices being used for computing the Krylov subspace. But the high dependence of the reduced order model on the linear approximation of the bilinear model can be used to improve its input-output preservation via the PLA approach.

# Chapter 5

# IP and PLA for MIMO Bilinear Models

The results presented in (Lin et al. 2007) and (Lin et al. 2009) propose Krylov subspaces for matching multimoments and moments for MIMO bilinear models. These are basically extensions of the Phillips type projection (Phillips 2000) and Bai type projection (Bai 2002) which have been proposed for SISO bilinear models.

As has been discussed in Chapter 4, the Krylov subspaces proposed in (Feng & Benner 2007) match the multimoments $m(q_1, q_2)$ and $\hat{m}(q_1, q_2)$. This has been shown to be less effective in some cases for preserving the input-output relationship of the high order models when compared to the following Krylov subspaces

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{5.1}$$

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}}) \tag{5.2}$$

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^{\{k\}}\}\}, \tag{5.3}$$

proposed in this thesis for the SISO case studies. It has also been shown that the PLA approach for MOR can be used to improve input-output preservation in all the cases discussed. While the extension of Phillips (Phillips 2000) projection to MIMO Bilinear model already exists, improvement can be achieved using other methods.

In this chapter, the Improved Phillips, Feng and Benner (Feng & Benner 2007) and the parametrized linear approximation approaches for model order reduction are extended to MIMO bilinear models. An analysis and multimoment matching for bilinear models are also illustrated. The newly proposed approaches are compared to the work done in (Lin et al. 2007, Lin et al. 2009).

## 5.1 IP type projection for MIMO bilinear models

In order to extend the Improved Phillips type projection to MIMO bilinear models, the multiple bilinear state matrices will have to be taken into consideration. $V^{\{1\}}$ remains the same with the SISO case. The Krylov subspace which contains the bilinear state matrices is $V^{\{2\}}$. The result of this for MIMO bilinear projection is that there are multiple matrices which are members of $V^{\{2\}}$. This can be represented by utilising indices, i.e $V_i^{\{2\}} \in V^{\{2\}}$, where $i = 1, 2, \ldots, m$. Therefore the Krylov subspace $V^{\{2\}}$ is defined below:

$$span\{V^{\{2\}}\} = span\{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\} \tag{5.4}$$

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, N_i V^{\{1\}}),\ i = 1, 2. \ldots, m. \tag{5.5}$$

Utilizing $V^{\{1\}}$ and $V_i^{\{2\}}$, the projection matrix $V$ is then computed as given

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}. \tag{5.6}$$

The extension of Krylov subspace techniques for MOR of MIMO bilinear models follows this pattern. For completeness, the Krylov subspaces for higher $k^{th}$

subsystem of the bilinear model is given as

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{5.7}$$

$$span\{V_i^{\{k\}}\} = K_{q_k}(A^{-1}, N_i V^{\{k-1\}}), \; i = 1, 2, \ldots, m \tag{5.8}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}. \tag{5.9}$$

## 5.2 Feng and Benner type projection for MIMO bilinear models

The Feng and Benner (Feng & Benner 2007) type projection for the model order reduction of MIMO bilinear models, to the best of the author's knowledge, is being proposed first in this thesis.

As in the case with the other methods, the Krylov subspaces proposed in (Feng & Benner 2007) can be extended to MIMO bilinear models. This can be achieved for the second subsystem of the bilinear model by using the Krylov subspaces given below

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B) \tag{5.10}$$

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}N_i V^{\{1\}}), \; i = 1, 2, \ldots, m \tag{5.11}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}. \tag{5.12}$$

## 5.3 Multimoment matching for MIMO bilinear models

**Theorem 5.3.1** *For a MIMO bilinear model, the Krylov subspaces $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, N_i V^{\{1\}})$ match the multimoments, $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2$, of the higher order and reduced order bilinear models if the reduced* *order model is computed such that* $\hat{A} = V^T AV$, $\hat{N}_i = V^T N_i V$, $\hat{B} = V^T B$ *and* $\hat{C} = CV$, *where* $V^{\{1\}} \in V$, $V_i^{\{2\}} \in V$ *and* $V^T V = I$.

Proof. If the Krylov subspaces in (5.10) - (5.12) as proposed in (Feng & Benner 2007) for projection are considered, multimoment matching for MIMO bilinear systems can be established. It has been shown previously that $A^{-l_1}B \in span\{V^{\{1\}}\}$, so $A^{-l_1}B = V^{\{1\}}r_{(q_1)}$ and because $V^{\{1\}} \in V$, $A^{-l_1}B = Vp_{(q_1)}$ for $l_1 = 1, \ldots, q_1$,

$$A^{-l_1}B = Vp_{(q_1)}. \tag{5.13}$$

Therefore, for the multimoments of the reduced order model and using the definitions of the reduced order matrices

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}^{-l_1}\hat{B}) &= CV[V^T AV]^{-l_2} V^T NV(I_m \otimes [V^T AV]^{-l_1} V^T B)\\ &= CV[V^T AV]^{-l_2} V^T NV(I_m \otimes V^T V p_{(q_1)})\\ &= CV[V^T AV]^{-l_2} V^T NV(I_m \otimes p_{(q_1)}).\end{aligned} \tag{5.14}$$

Note that due to the definition of $N$ and the Kronecker product, we have

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} &= CV[V^T AV]^{-l_2} V^T N_i V p_{(q_1)}, \text{ for } i = 1, \ldots, m\\ &= CV[V^T AV]^{-l_2} V^T N_i V^{\{1\}} r_{(q_1)}, \text{ for } i = 1, \ldots, m\\ &= CV[V^T AV]^{-l_2} V^T A A^{-1} N_i V^{\{1\}} r_{(q_1)}, \text{ for } i = 1, \ldots, m.\end{aligned} \tag{5.15}$$

Since $A^{-1}N_i V^{\{1\}} \in V$, therefore $A^{-1}N_i V^{\{1\}} = Vp_{(q_1+1)}$ and

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} &= CV[V^T AV]^{-l_2} V^T AV p_{(q_1+1)} r_{(q_1)}\\ &= CV[V^T AV]^{-l_2+1} p_{(q_1+1)} r_{(q_1)}\\ &= CV[V^T AV]^{-l_2+1} V^T A A^{-1} V p_{(q_1+1)} r_{(q_1)}, \text{ for } i = 1, \ldots, m.\end{aligned} \tag{5.16}$$

Also, $A^{-2}N_i V^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$, then

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} &= CV[V^TAV]^{-l_2+1}V^TAVp_{(q_1+2)}r_{(q_1)} \\ &= CV[V^TAV]^{-l_2+2}p_{(q_1+2)}r_{(q_1)} \\ &= CV[V^TAV]^{-l_2+2}V^TAA^{-1}Vp_{(q_1+2)}r_{(q_1)}, \text{ for } i=1,\dots,m.\end{aligned} \tag{5.17}$$

Continuing this routine, it can be shown that

$$\begin{aligned}\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} &= CVp_{(q_1+l_2)}r_{(q_1)} \\ &= CA^{-l_2}N_iV^{\{1\}}r_{(q_1)}, \text{ for } i=1,\dots,m.\end{aligned} \tag{5.18}$$

Note that $A^{-l_2}N_iV^{\{1\}} = Vp_{q_1+l_2}$. Also, since $V^{\{1\}}r_{q_1} = A^{-q_1}B$,

$$\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} = CA^{-l_2}N_iA^{-l_1}B, \text{ for } i=1,\dots,m. \tag{5.19}$$

Considering the definition of $N$ and the use of a Kronecker product, we have

$$\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m\otimes\hat{A}^{-l_1}\hat{B}) = CA^{-l_2}N(I_m\otimes A^{-l_1}B), \tag{5.20}$$

where $r_{(q_1)}$ and $p_{(i)}$ are appropriate parameters for achieving orthgonality and $r_{(q_1)}\in\mathbb{R}^{q_1}$, $p_{(i)}\in\mathbb{R}^{q_1+q_1q_2m\times 1}$ for $i\le q_1$ and $p_{(i)}\in\mathbb{R}^{(q_1+q_1q_2m\times q_1)\times q_1}$ for $i>q_1$. $\square$

Using this analysis, it can be derived that for a one-sided projection of a MIMO bilinear model, the Krylov subspaces proposed in (Lin et al. 2007) match the multimoments, $\hat{m}(l_1,l_2) = m(l_1,l_2)$, $l_1 = 1,\dots,q_1-1$, $l_2 =,\dots,q_2-1$. Also, it can be derived that the Improved Phillips type projection matches the multimoments $\hat{m}(l_1,l_2) = m(l_1,l_2)$, $l_1 = 1,\dots,q_1$, $l_2 = 1,\dots,q_2-1$. The proof shown here differs from those done in (Lin et al. 2007, Lin et al. 2009) for MIMO bilinear models because they (Lin et al. 2007, Lin et al. 2009) have assumed that $VV^T = I$.

## 5.4 PLA for MIMO bilinear models

The parametrised linear approximation approach is also proposed for MIMO cases. Considering a MIMO bilinear model of the form (3.1)- (3.2), the multiple bilinear state matrices with an application of a constant input $u = [u_1\, u_2 \ldots u_m]^T$ over a short period of time results in the linear approximation of the MIMO bilinear model where the state matrix is

$$A_\eta = A + N_1\eta_1 + N_2\eta_2 + \ldots + N_m\eta_m, \tag{5.21}$$

given that $u_1 = \eta_1, u_2 = \eta_2, \ldots, u_m = \eta_m$ are the so called parameters for a linear approximation of the bilinear model. The input matrix $B_\eta$ can also be defined using these parameters such that

$$B_\eta = B \times \eta \tag{5.22}$$

where $\eta = [\eta_1\ \eta_2\ \ldots\ \eta_m]^T$. Therefore, the following set of equations can be used for computing projection bases for model order reduction using the parametrised linear approximation approach:

$$span\{V^{\{1\}}\} = K_{q1}(A_\eta^{-1}, A_\eta^{-1}B_\eta) \tag{5.23}$$

$$span\{V_i^{\{2\}}\} = K_{q2}(A^{-1}, N_iV^{\{1\}}),\ i = 1, 2, \ldots, m \tag{5.24}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}. \tag{5.25}$$

As can be observed, the set of equations use the Improved Phillips type projection for applying the parametrised linear approximation. However, the PLA for projection of bilinear models can be applied to the Phillips type (Phillips 2000) projection, Feng and Benner (Feng & Benner 2007) and Bai (Bai & Skoogh 2006), type projections for MIMO bilinear model reduction.

Two versions of the PLA are developed in this chapter. The first is (5.23) - (5.25). The second version is presented below

$$span\{V^{\{1\}}\} = K_{q_1}(A_\eta^{-1}, A_\eta^{-1}B_\eta) \tag{5.26}$$

$$span\{V_i^{\{2\}}\} = K_{q_2}(A_\eta^{-1}, N_iV^{\{1\}}),\ i = 1, 2, \ldots, m \tag{5.27}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}. \tag{5.28}$$

For systems which have noninvertible state transition matrices, (5.26) - (5.28) are used. This is because (5.24) does not use the alternate state transition matrix for computing $V^{\{2\}}$ and will not be useful in this case.

**Remark 5.4.1** *Using the same approach as in Theorem 5.3.1, it can be shown that the Krylov subspaces* (5.23) - (5.24) *match the multimoments*

$$\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}_\eta^{-l_1}\hat{B}_\eta) = CA^{-l_2}N(I_m \otimes A_\eta^{-l_1}B_\eta) \tag{5.29}$$

*It can also be shown that the Krylov subspaces defined in* (5.26) - (5.27) *match the multimoments*

$$\hat{C}\hat{A}_\eta^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}_\eta^{-l_1}\hat{B}_\eta) = CA_\eta^{-l_2}N(I_m \otimes A_\eta^{-l_1}B_\eta) \tag{5.30}$$

*Note that when as $\eta_1, \eta_2, \ldots, \eta_m$ becomes negligible,*

$$\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}_\eta^{-l_1}\hat{B}_\eta) \approxeq \hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}^{-l_1}\hat{B}) \tag{5.31}$$

$$CA^{-l_2}N(I_m \otimes A_\eta^{-l_1}B_\eta) \approxeq CA^{-l_2}N(I_m \otimes A^{-l_1}B) \tag{5.32}$$

*and*

$$\hat{C}\hat{A}_\eta^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}_\eta^{-l_1}\hat{B}_\eta) \approxeq \hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}^{-l_1}\hat{B}) \tag{5.33}$$

$$CA_\eta^{-l_2}N(I_m \otimes A_\eta^{-l_1}B_\eta) \approxeq CA^{-l_2}N(I_m \otimes A^{-l_1}B). \tag{5.34}$$

The following algorithm can be used for computing Krylov subspace projection matrix, $V$ for PLA as presented in (5.23) - (5.27).

**Algorithm 5.1 (Computation of $V$ for MIMO models using PLA)**

1. ***Input:*** $A, B, N_1, \ldots, N_m, m, q_1, q_2, p_2, \eta_1, \eta_2, \ldots, \eta_m$

2. ***Compute linear approximation:*** $A_\eta = A + N_1\eta_1 + N_2\eta_2 + \ldots + N_m\eta_m$***,*** $B_\eta = B\eta$
3. ***Compute an orthonormal basis,*** $V^{\{1\}}$***, for the Krylov subspace:*** $K_{q_1}(A_\eta^{-1}, A_\eta^{-1}B_\eta)$***, using Algorithm 3.2***
4. ***for*** $i = 1 : m$***, Compute an orthonormal basis,*** $V_i^{\{2\}}$***, for the Krylov subspace:*** $K_{q_2}(A^{-1}, A^{-1}N_iV_{[p_2]}^{\{1\}})$***, using Algorithm 3.2***
5. ***end***
6. $V =$ ***orth***$([V^{\{1\}}, V_1^{\{2\}}, \ldots, V_m^{\{2\}}])$
7. ***Return*** $V$

The algorithm is implemented with the condition that $p_2 \leq q_1$ where $p_2$ is as defined in Section 3.8. Algorithm 5.1 computes a projection matrix for implementing the PLA for MIMO bilinear models. This differs from Algorithm 3.3 in steps 1 and 2. In step 1, there are more input parameters as $\eta_1, \eta_2, \ldots, \eta_m$ are added to the algorithm and in step 2, these parameters are used for computing an alternate linear approximation of the bilinear model. The new linear approximation is then used to form $V^{\{1\}}$.

Implementing PLA for singular system matrices, step 4 should be computed using the alternate system matrix $A_\eta$ as implemented in the following algorithm:

**Algorithm 5.2 (Computation of $V$ for MIMO models using PLA)**

1. ***Input:*** $A, B, N_1, \ldots, N_m, m, q_1, q_2, p_2, \eta_1, \eta_2, \ldots, \eta_m$

2. ***Compute linear approximation:*** $A_\eta = A + N_1\eta_1 + N_2\eta_2 + \ldots + N_m\eta_m$, $B_\eta = B\eta$
3. ***Compute an orthonormal basis,*** $V^{\{1\}}$, for ***the Krylov subspace:*** $K_{q_1}(A_\eta^{-1}, A_\eta^{-1}B_\eta)$, ***using Algorithm 3.2***
4. ***for*** $i = 1 : m$, ***compute an orthonormal basis,*** $V_i^{\{2\}}$, ***for the Krylov subspace:*** $K_{q_2}(A_\eta^{-1}, A_\eta^{-1}N_iV_{[p_2]}^{\{1\}})$, ***using Algorithm 3.2***
5. ***end***
6. $V = \textbf{\textit{orth}}([V^{\{1\}}, V_1^{\{2\}}, \ldots, V_m^{\{2\}}])$
7. ***Return*** $V$

Algorithm 5.2 differs from Algorithm 5.1 in step 4. Here the system matrix $A$ has been replaced with $A_\eta$ the parametrised linear approximation as defined in Equation (5.21) to compensate the singularity of $A$ at the expansion point of zero.

## 5.5 Numerical examples

In this section, two numerical examples of arbitrary bilinear models are used to compare the Phillips (Phillips 2000) type, Bai (Bai & Skoogh 2006) type, Feng and Benner (Feng & Benner 2007) type, the IP and PLA type projection techniques for MIMO bilinear models. The performance criteria used are $RT^2$, MSE, IAE, NIAE and SSE.

### 5.5.1 Example 1

Consider a time invariant MIMO bilinear model of state space dimension, $n = 1400$, number of inputs, $m = 2$, number of outputs, $p = 3$. The state matrices $A$ and $N_1$ are given as

$$A = \begin{bmatrix} -10 & 2 & 0 & \cdots & 0 \\ 2 & -10 & 2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & -10 & 2 \\ 0 & \dots & 0 & 2 & -10 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 0 & -1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}. \quad (5.37)$$

Also, $N_2 = -N_1$. $B$ is an $n \times m$ matrix while $C$ is a $p \times n$ matrix and are in the form given below

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0.8 & 0.8 & \cdots & 0.8 & 0.8 \\ 0.5 & 0.5 & \cdots & 0.5 & 0.5 \end{bmatrix}, \quad (5.38)$$

where $A \in \mathbb{R}^{n\times n}$, $N_1 \in \mathbb{R}^{n\times n}$, $N_2 \in \mathbb{R}^{n\times n}$ and an initial state, $x(0) = 0$. The system is simulated with inputs, $u_1$ (See Table 5.1.) and $u_2$ which form the simulation input, $u$, where,

Table 5.1: Table of input values $u_1$ for Example 1.

| | | | | | |
|---|---|---|---|---|---|
| $u_1$ | 5.1449 | 1.6550 | 14.4810 | 14.4897 | 19.6743 |
| Time range (s): | $t \in [0 : 0.9]$ | $t \in [1 : 1.9]$ | $t \in [2 : 2.9]$ | $t \in [3 : 3.9]$ | $t \in [4 : 4.9]$ |
| $u_1$ | 8.4851 | 11.2442 | 17.5686 | 1.4018 | 1.7853 |
| Time range (s): | $t \in [5 : 5.9]$ | $t \in [6 : 6.9]$ | $t \in [7 : 7.9]$ | $t \in [8 : 8.9]$ | $t \in [9 : 9.9]$ |
| $u_1$ | 11.74529 | | | | |
| Time range (s): | $t \in [10]$ | | | | |

$$u_2 = (\sin(t) + 1)/10 \tag{5.39}$$

$$u = [u_1 \quad u_2]^T \tag{5.40}$$

Using Algorithm 3.3 and Algorithm 3.2 discussed in Chapter 3, four of the methods proposed in (Lin et al. 2007, Lin et al. 2009) and the methods proposed in this thesis were applied to reduce the order of the bilinear model (5.37)-(5.38) by utilising the parameters, $q_1 = 5$, $q_2 = 5$, $p_2 = 4$. The outputs of the system are $y_1$, $y_2$, and $y_3$. In this numerical example, the Phillips, Bai, MIMO Fend and Benner and Improved Phillips approaches to Krylov subspace projection are compared. This comparison is done using the performance criteria discussed in Chapter 4 and visual output plots.

**Results:**

Table 5.2 shows the $RT^2$, MSE, IAE, NIAE and SSE values of the reduced order models produced by the methods implemented. As can be observed, the values

Table 5.2: Table of performance criteria values for Phillips type, Feng and Benner, Bai and Improved Phillips projection.

| | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|
| Phillips | 0.0037 | 5.2415 | 0.0336 | 0.5746 |
| Bai | 0.0015 | 3.2032 | 0.0205 | 0.2353 |
| Benner | 5.9975e-04 | 2.0073 | 0.0129 | 0.0936 |
| Improved Phillips | 5.4007e-05 | 0.5221 | 0.0033 | 0.0084 |

of each performance criteria for the different methods are quite similar. This can also be observed in Figure 5.1 which shows the inputs $u_1$, $u_2$ and the simulated outputs of the reduced order models and the high order bilinear model. Only one of the outputs from these models has been plotted which has been denoted $y_1$.

Figure 5.2 shows a zoomed in view of all the reduced order models and the original higher order bilinear model at the 8 second mark of Figure 5.1. It can

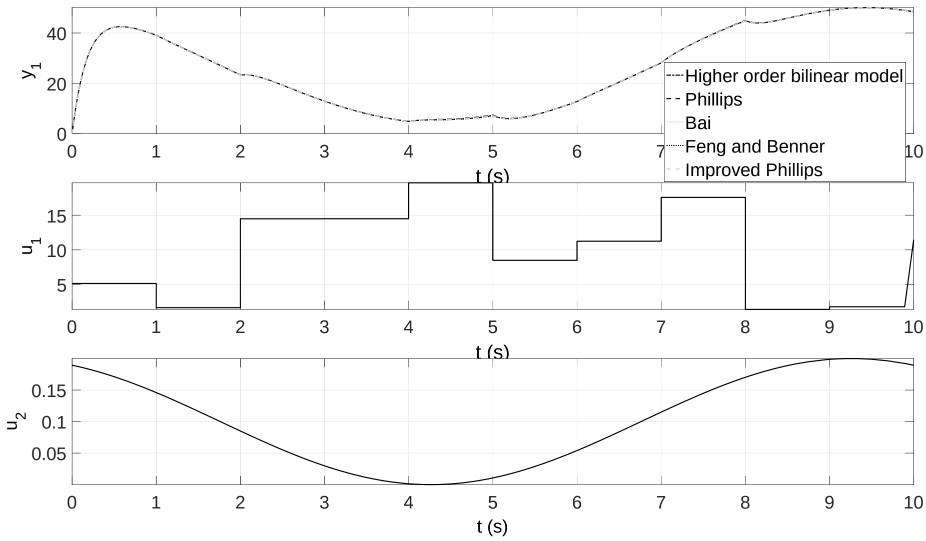



Figure 5.1: Time response $y_1$ of high order bilinear model and reduced order models using inputs $u_1$ and $u_2$.

be seen from this zoomed-in plot that the Improved Phillips produces the closest result to the original model. This result is consistent also at all other points of time.

### 5.5.2 Example 2

In this example, the best algorithm from the previous Example 1, i.e. the Improved Phillips has been applied, comparing the use of standard linearization and the parametrized linear approximation method. As in Subsection 5.5.1, a similar higher order bilinear model is used. Here, consider a time invariant 2 inputs 3 outputs bilinear model with matrices $A$ and $N_1$ shown below

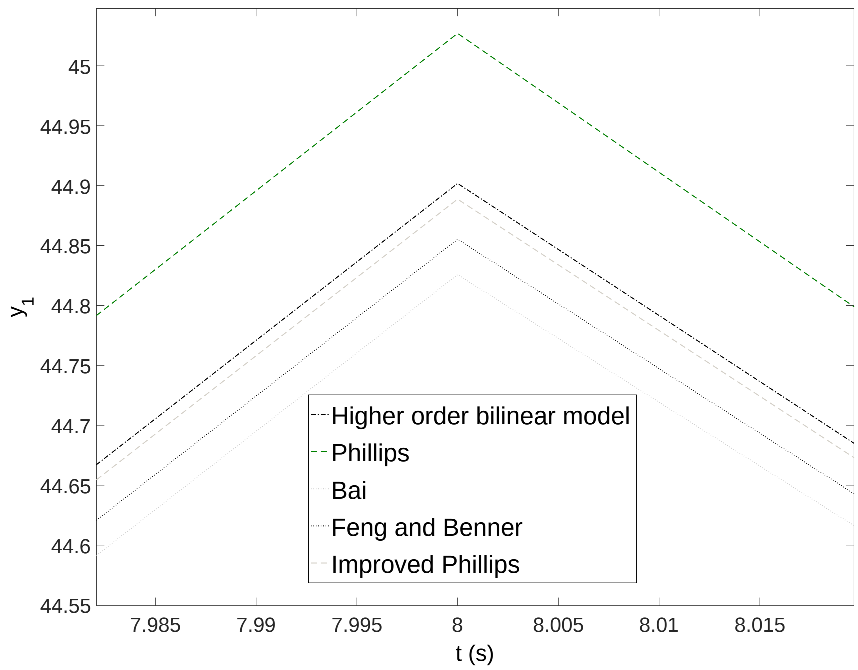



Figure 5.2: Zoomed in time response of higher order bilinear model (HOBM) and reduced order models.

$$A = \begin{bmatrix} -5 & 2 & 0 & \cdots & 0 \\ 2 & -5 & 2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & -5 & 2 \\ 0 & \dots & 0 & 2 & -5 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & -3 & 0 & \cdots & 0 \\ 3 & 0 & -3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 3 & 0 & -3 \\ 0 & \dots & 0 & 3 & 0 \end{bmatrix}. \tag{5.41}$$

The matrix $N_2 = -N_1$, where $A \in \mathbb{R}^{n\times n}$, $N_1 \in \mathbb{R}^{n\times n}$. B is an $n \times m$ matrix while C is a $p \times n$ where $n = 1400$, $p = 3$ and $m = 2$ and are given below

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0.8 & 0.8 & \cdots & 0.8 & 0.8 \\ 0.5 & 0.5 & \cdots & 0.5 & 0.5 \end{bmatrix} \tag{5.42}$$

The model order reduction is done for an initial state of zero. The same parameters as in the Subsection 5.5.1 have also been used here i.e. $q_1 = 5$, $q_2 = 5$, and $p_2 = 4$.

The parameters $\eta_1$ and $\eta_2$ are equal and have been chosen by trial and error to be equal to 0.753. The linear approximation approach for the Krylov subspaces defined in (5.23) - (5.25) have been implemented using Algorithm 5.1.

The bilinear models have been simulated using inputs $u_1$ and $u_2$.

Table 5.3: Table of input values $u_1$ for Example 2.

| | | | | | |
|---|---|---|---|---|---|
| $u_1$ | 18.3745 | 5.9107 | 51.7177 | 51.7490 | 70.2652 |
| Time range (s): | $t \in [0 : 0.9]$ | $t \in [1 : 1.9]$ | $t \in [2 : 2.9]$ | $t \in [3 : 3.9]$ | $t \in [4 : 4.9]$ |
| $u_1$ | 30.3039 | 40.1577 | 62.7450 | 5.0065 | 6.3760 |
| Time range (s): | $t \in [5 : 5.9]$ | $t \in [6 : 6.9]$ | $t \in [7 : 7.9]$ | $t \in [8 : 8.9]$ | $t \in [9 : 9.9]$ |
| $u_1$ | 40.9033 | | | | |
| Time range (s): | $t \in [10]$ | | | | |

$$u_2 = (\sin(t) + 1)/10 \tag{5.43}$$

$$u = [u_1 \quad u_2]^T \tag{5.44}$$

**Results:**

Table 5.4 shows the $RT^2$, MSE, IAE, NIAE and SSE values of the reduced order models produced by using the Improved Phillips and parametrised linear approximation approaches. Figure 5.3 shows one of the outputs of the reduced order bilinear models compared to the higher order bilinear model. In order to highlight the differences between the IP and the PLA, the input $u_1$ has been amplified as can be observed in the second row of Figure 5.3. $u_2$ remains the same and has not been plotted. In the third row of Figure 5.3, the absolute errors of both reduced order models are plotted.

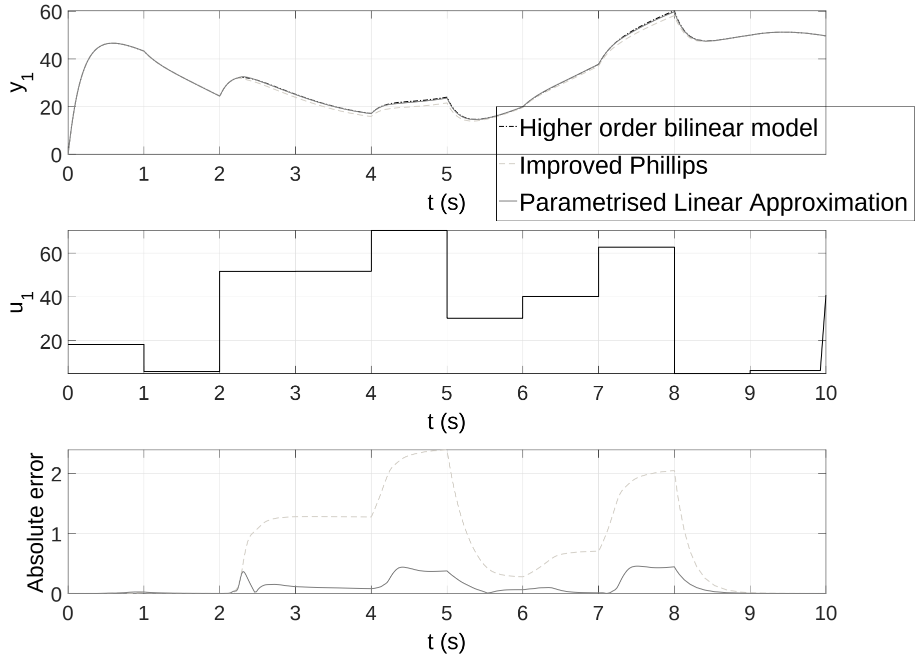



Figure 5.3: Time response $y_1$ of high order bilinear model (HOBM) and reduced order models using the Improved Phillips (IP) type projection and the Parametrised Linear Approximation (PLA). Also plotted is the input $u_1$ and the absolute error values

As can be observed there is a significant increase in the input-output preservation of the reduced order model when applying the parametrised linear approximation method. It is also expected that the parametrised linear approximation approach will yield better results when applied to the Phillips (Phillips 2000) type projection, Feng and Benner type (Feng & Benner 2007), and Bai (Bai & Skoogh 2006) type projection for the reduction of MIMO bilinear models.

## 5.6 Conclusion

In this chapter, the use of Krylov subspaces has been extended for the reduction of MIMO bilinear models and some new approaches have been proposed. The

Table 5.4: Table showing performance criteria for Improved Phillips and Parametrised Linear Approximation for MIMO bilinear models.

| | $RT^2$ | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|---|
| Improved Phillips | 99.04 | 1.8256 | 365.0072 | | 622.5193 |
| PLA | 99.86 | 0.2603 | 139.6369 | 0.4036 | 8.9180 |

Improved Phillips, Feng and Benner type (Feng & Benner 2007) and PLA approaches have been proposed. These methods have been compared by using a simulation study. These methods have also been shown to match multimoments of the bilinear model.

The simulation study done here shows that all the considered Krylov subspace methods discussed here perform very similarly, with slight improvement in the Improved Phillips, as seen in Figure 5.2. Further investigation into the linearization method shows that the introduction of a parametrised linear approximation (PLA) tends to make a significant impact on the accuracy of the reduced order model. This is even more in the case of MIMO models as multiple $N$ matrices add to the linear dynamics of the system which are not represented when a standard linear approximation $A$ is utilised for the model order reduction.

To summarize, the main contributions to literature which are proposed in this chapter are the Improved Phillips type projection proposed for MIMO bilinear systems and the Feng and Benner type projection (Feng & Benner 2007) proposed for MIMO systems and the PLA proposed for MIMO bilinear systems.

# Chapter 6

# Applications of PLA and IP Projection

## 6.1 Introduction

Reduced order models are used in many applications such as control design (Schelfhout 1996), diagnostics, hardware in the loop simulations and systems design. The need for reduced order models arises for various reasons such as cost and practicality. System level simulations, optimisation and system interaction design can be made easier in terms of time and in some cases, practically impossible without the use of reduced order models. In (Hung, Yang & Senturia 1997) a reduced order model is used to speed up the simulation time of a micromechanical device via a macromodel of reduced order. In (Nayfeh, Younis & Abdel-Rahman 2005), a state of the art literature review has been done on the development of reduced order models for micro-electromechanical systems. Reduced order models have also been used in (Filipi, Fathy, Hagena, Knafl, Ahlawat, Liu, Jung, Assanis, Peng & Stein 2006) to perform engine in the loop testing. In their paper, (Filipi et al. 2006) (Filipi et al. 2006) used an energy based model order reduction technique to reduce a high mobility multipurpose wheeled vehicle model. This was necessary because in order to carry out engine in the loop simulations, a model needs to capture all the important dynamics of the system, and at the same time the simulations must be able to run in real time.

The nature of singular systems creates further difficulties for the application of some mathematical processes. Their frequent occurrence in modelling of electric circuits and power systems has brought about an interest into finding solutions for dealing with this type of systems. This has brought about the study of singular systems (Gray & Verriest 1989) and model order reduction for singular systems as discussed in (Xu, Lam, Liu & Zhang 2003, Bender 1987). A singular system has been defined in (Weisstein 2002) as a system whose condition number is infinite.

The purpose of this chapter is to apply the PLA projection and the Improved Phillips projection techniques for the reduction of pseudo-singular bilinear models where a pseudo-singular bilinear model refers to a bilinear model with non-invertible state transition matrices. Also, an optimisation scheme will be used to estimate the parameters, $(\eta_1, \eta_2, \ldots, \eta_m, m \in \mathbb{Z})$, of a PLA MOR approach. Hybrid approaches for MOR are becoming very popular. These approaches try to combine the advantages of different MOR methods (data based and mathematical manipulation) to achieve higher accuracy or ease of implementation and practicality. For example, in (Saragih 2014), genetic algorithm has been used with balanced truncation to reduce a MIMO bilinear model.

In Section 6.2, a hybrid approach for model order reduction is introduced. This is followed by the use of PLA for MOR of so the called pseudo-singular systems in Section 6.3. Two case studies are provided in Section 6.4. A solar panel model with singular matrices has been reduced successfully by using PLA. Also, an optimisation scheme has been applied to the bilinear model provided in Subsection 5.5.2. This has been compared to PLA without an optimisation scheme.

## 6.2 Hybrid MOR using parameter estimation and optimisation techniques

The involvement of parameters in the computation of reduced order models means that there will be optimum parameter choices for which an optimal reduced order model can be achieved. Since the reduced order model has to reach some level of accuracy for it to be acceptable, these parameters can be said to be optimum for a given set of performance criteria.

An optimisation scheme can be taken into consideration. Tools such as genetic algorithm (Gen & Cheng 2000) and Nelder-Mead simplex (Lagarias, Reeds, Wright & Wright 1998) method for the optimisation of reduction parameters are some of the optimisation algorithms which can be used to find optimal parameters for MOR. Figure 6.1 shows a flow chart that gives a pictorial view of the proposed optimisation scheme.

The parameter initialisation stage in the flow chart (Figure 6.1) uses a best guess of the parameters, $\eta_1, \eta_2, \ldots, \eta_m$, from the user's experience with the system to be reduced. Normally, most algorithms can cope with an initialisation of zero but in this case, since the PLA achieves reduced order models with good performance criteria values without the optimisation scheme, an initialisation value can be obtained using trial and error whilst observing the accuracy of the reduced order model. The linear approximation of the bilinear model is

$$A_\eta = A + N_1\eta_1 + N_2\eta_2 + \ldots + N_m\eta_m \tag{6.1}$$

$$B_\eta = B \times \eta. \tag{6.2}$$

After which the projection matrix, $V$ is computed using Algorithm 5.1 and Algorithm 3.2. This is followed by computing the reduced order state transition

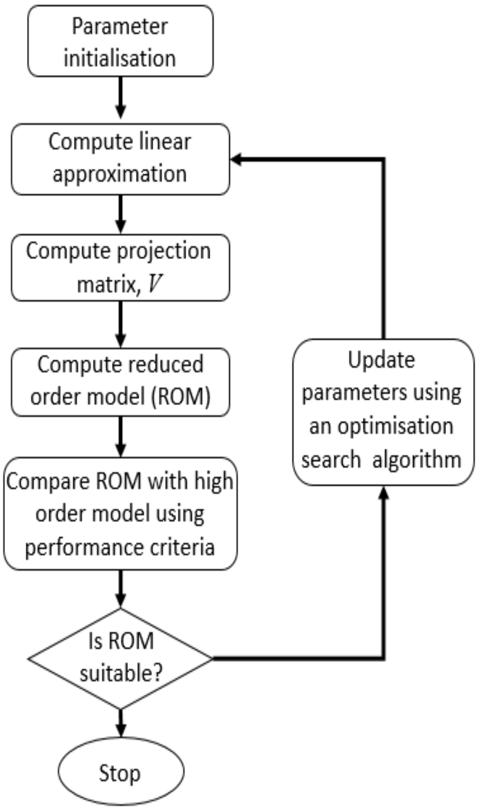



Figure 6.1: Flow chard for hybrid model order reduction using an optimisation scheme and Krylov subspace projection techniques.

matrix, bilinear state matrices, input and output matrices using Algorithm 3.4. The scheme uses a performance criteria as a cost function. Depending on the cost, the optimisation algorithm updates the parameters $(\eta_1, \eta_2, \ldots, \eta_m)$. This loop continues until a desirable cost or number of loops is achieved.

## 6.3 Parametrised linear approximation projection for pseudo-singular systems

In this section, a pseudo-singular system model is defined. It is a bilinear system that has a noninvertible state transition matrix. Consider a bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m_1} N_i x u_i + Bu \tag{6.3}$$

$$y = Cx \tag{6.4}$$

where $A \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_2}$, $C \in \mathbb{R}^{p\times n}$ and $A$ is a singular matrix. $u = [u_1, u_2, \ldots, u_{m2}]$. Note that whilst it is normal for $u$ to be composed of all $u_i$, it is not necessarily the case. In some cases, $u_i$ could be noise or some external input to the system/model. The ideal case discussed in previous chapters is for when $m_1 = m_2$.

Using an alternate linear approximation to define the linear behaviour of the pseudo-singular bilinear system model, the model can be reduced by using the Krylov subspaces as defined as follows:

$$span\{V^1\} = K_{q_1}(A_\eta^{-1}, A_\eta^{-1}B_\eta) \tag{6.5}$$

$$span\{V_i^2\} = K_{q_2}(A_\eta^{-1}, N_i V^{\{1\}}) \tag{6.6}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}. \tag{6.7}$$

Comparing (6.5)-(6.7) to (5.8)-(5.9), the state transition matrix, $A$, has been replaced by $A_\eta$, assuming that $A_\eta$ is nonsingular and forms a stable linear approximation of the bilinear model.

## 6.4 Case study on Solar Panel Model

### 6.4.1 Model description

For illustration, a solar panel model shown in Figure 6.2 which has been presented in (Tenny, Rawlings & Wright 2004, Couchman et al. 2011) will be used to demonstrate the use of parametrised linear approximation for a pseudo-singular system. The solar collector plant consists of a heat exchanger, 790 meter pipe,

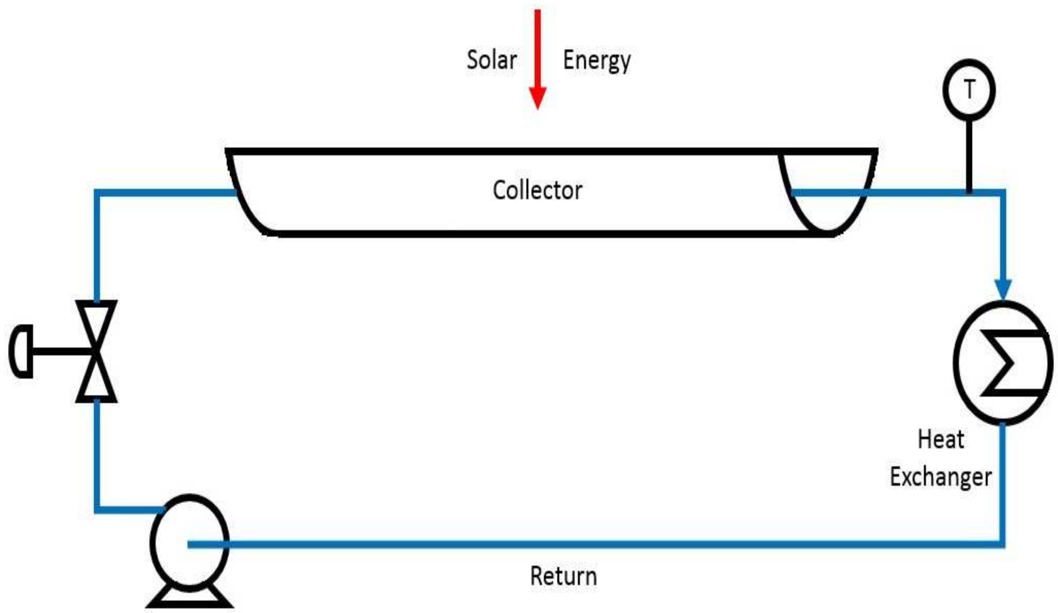



Figure 6.2: Schematic diagram of solar collector

collector and a pump. A fluid is used to collect the solar energy and is transported using the pipe to the heat exchanger for extraction of the heat energy. The fluid is then transported back to the collector. A pump is used to control the flow rate of the fluid through the pipes, therefore the fluid flow is the control variable. The operating outlet temperature of this system is 573 K.

A model of this system is derived by discretizing the return loop, heat exchanger and collector across a 1-dimensional space where the temperatures at each node are states of the model. The resulting model is of a bilinear structure (Couchman et al. 2011) with matrices of system parameters are given below

$$A = \begin{bmatrix} \mathbf{A_{11}} & 0_{12} & 0_{13} \\ 0_{21} & A_{22} & 0_{23} \\ 0_{31} & 0_{32} & 0_{33} \end{bmatrix}, \quad N = diag([N_1 \quad 0 \quad N_2]) + \begin{bmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{bmatrix}. \qquad (6.8)$$

$A_{11} = -\beta_1 * I_{(20\times20)}$, $A_{12} = 0_{(20\times1)}$, $A_{13} = 0_{(20\times20)}$, $A_{21} = 0_{(1\times20)}$, $A_{22} = -1 - \beta_2$, $A_{23} = 0_{(1\times20)}$ and $N_1 = -\alpha \times I_{(20\times1)}$, $N_2 = -\alpha \times I_{(20\times1)}$, $N_{11} = 0_{(1\times40)}$, $N_{21} = diag([\alpha \times I_{(1\times19)} \quad 0 \quad \alpha \times I_{(1\times20)}])$, $N_{22} = 0_{(40\times1)}$, where $\alpha = 8.22 \times 10^{-3}$, $\beta_1 = 1.19 \times 10^{-3}$, $\beta_2 = 5$. The matrices $B$ and $C$ are

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 1 & C_2 \end{bmatrix}, \qquad (6.9)$$

$B_{11} = \beta_1 \times T_1 \times I_{(20\times1)}$, $B_{12} = \gamma_1 \times I_{(20\times1)}$, $B_{21} = \beta_2 T_2$, $B_{31} = 0_{(20\times1)}$, $B_{32} = 0_{(20\times1)}$, $C_1 = 0_{(1\times19)}$, $C_2 = 0_{(1\times21)}$, with $\gamma = 0.541$, $T_1 = 303.15$ and $T_2 = 375.15$.

In (Stuetzle, Blair, Mitchell & Beckman 2004) this plant model was used to develop a control algorithm for achieving desired system response. A linear model predictive controller was implemented on the plant at different weather conditions and in (Stuetzle, Blair, Beckman & Mitchell 2004) the gross output of this approach is analysed.

The order of the model being discussed is $n = 41$. This higher order model is obtained as a result of discretization. However, much higher order models could be obtained if the discretization points are increased for higher accuracy of state estimation. To reduce the order of the resulting model, the implementation of the standard Krylov subspace techniques discussed (Phillips 2000, Bai & Skoogh 2006, Feng & Benner 2007) would be impossible as $A$ is a singular matrix.

### 6.4.2 MOR procedure

The model order reduction procedure uses the Krylov subspaces defined in (6.5) - (6.7). The projection matrix has been computed using Algorithm 5.2 whilst the reduced order models have been computed using Algorithm 3.4. The results presented are for $7^{th}$, $9^{th}$ and $11^{th}$ order models using the simulation inputs

$$u_1 = 0.6 \quad \text{for:} t \in [0 s : 200 s] \tag{6.10}$$

$$u_1 = 8.0 \quad \text{for:} t \in [200 s : 1750 s] \tag{6.11}$$

$$u_2 = 1.4 \quad \text{for:} t \in [0 s : 1000 s] \tag{6.12}$$

$$u_2 = 0.6 \quad \text{for:} t \in [1000 s : 1750 s] \tag{6.13}$$

$$u = [u_1 \quad u_2]. \tag{6.14}$$

The higher order model and the reduced order models have been simulated with a nonzero initial condition of $x(0) = 465$. The results are presented in the next subsection.

### 6.4.3 Results

The first row of Figure 6.3 shows the output plots of the $7^{th}$ and the $9^{th}$ order models compared to the solar panel model output. The second row of Figure 6.3

Table 6.1: $RT^2$, MSE, IAE, NIAE and SSE values for model order reduction of solar panel model using the Parametrised Linear Approximation

| | $RT^2$ | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|---|
| $7^{th}$ order model | 99.84 | 5.0398e+05 | 1.7284+06 | 542.1694 | 1.6067e+09 |
| $9^{th}$ order model | 99.91 | 3.0202e+05 | 0.9560+06 | 299.8695 | 0.9628e+09 |
| $11^{th}$ order model | 99.98 | 0.4977e+05 | 0.3512+06 | 110.1527 | 0.1587e+09 |

shows the input values $u_1$ and $u_2$ as defined in (6.10) - (6.14) over a time length of 1700 seconds.

Figure 6.4 shows the output of the solar panel model compared to a reduced order model of $11^{th}$ order. The second row of Figure 6.4 shows the absolute error

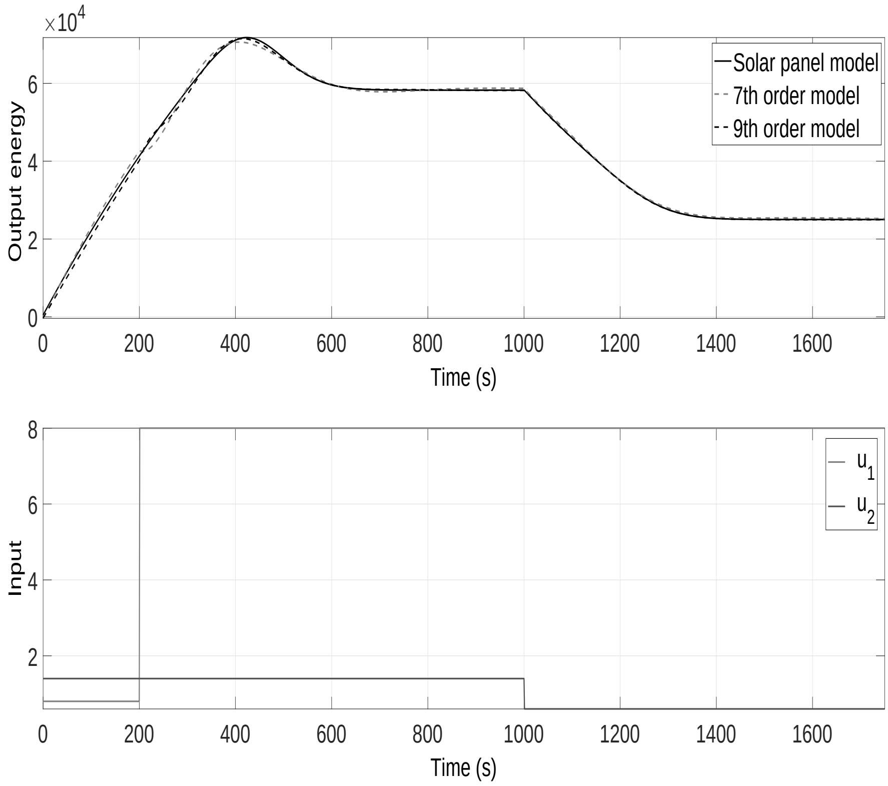



Figure 6.3: Input and output plot of solar panel model (SPM) compared to $7^{th}$ and $9^{th}$ order models using PLA type projection

values divided by 100 of the $7^{th}$, $9^{th}$, and $11^{th}$ reduced order models. As can be observed in this figure, the accuracy of the reduced order model increases as the order increases. This trend can also be observed in Table 6.1 which shows the coefficient of determination ($RT^2$), integral of absolute error (IAE), integral of absolute error divided by number of samples (NIAE), mean square error (MSE), and the sum of square of error (SSE) values for the reduced order models.

In (Couchman et al. 2011), the order of this same model has been reduced using an input constraint with balanced truncation to 3rd, 7th and 10th order. As is the case here, reducing the model to orders below 7 reduces its accuracy considerably.

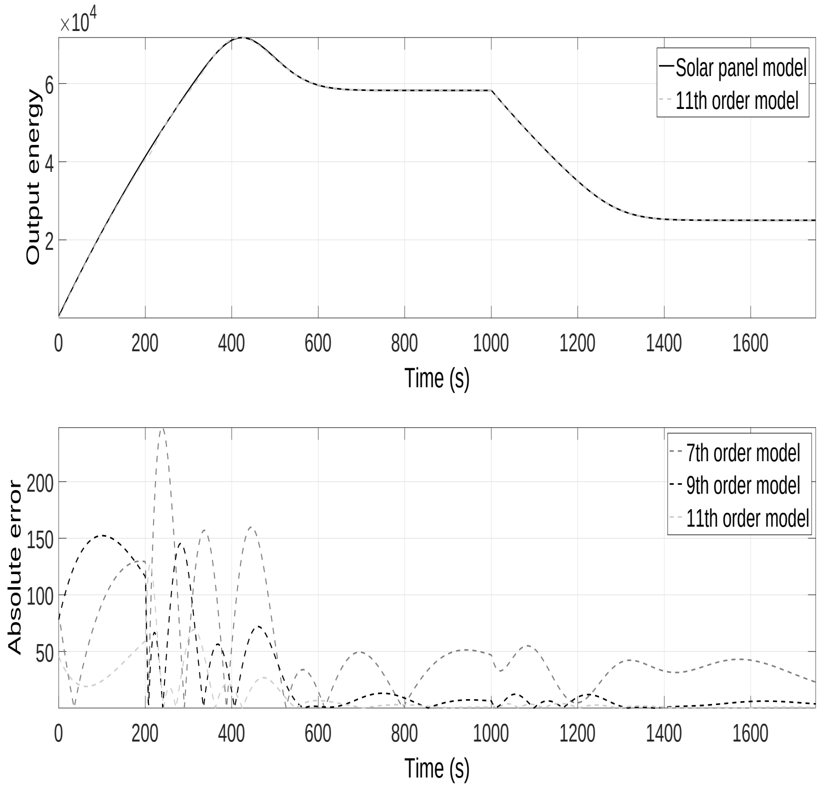



Figure 6.4: Plot of $11^{th}$ order model compared to solar panel model (SPM) and absolute error plots for $7^{th}$, $9^{th}$ and $11^{th}$ order models

## 6.5 MOR with optimisation

In order to demonstrate the use of optimisation algorithms for model order reduction, consider an arbitrary biliniear system of the form presented in (Lin et al. 2007, Lin et al. 2009) where the number of inputs is 2 and the number of outputs is 3. The bilinear model structure considered here has 2 bilinear state matrices i.e. $m_1 = 2$.

$$A = \begin{bmatrix} -10 & 2 & 0 & \cdots & 0 \\ 2 & -10 & 2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & -10 & 2 \\ 0 & \dots & 0 & 2 & -10 \end{bmatrix} \tag{6.15}$$

$$N_2 = \begin{bmatrix} -10 & 2 & 0 & \cdots & 0 \\ 2 & -10 & 2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & -10 & 2 \\ 0 & \dots & 0 & 2 & -10 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 0 & -1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \tag{6.16}$$

Also, $N_1, N_2, \in \mathbb{R}^{n \times n}$. $B$ is an $n \times m$ matrix while C is a $p \times n$ matrix and are in the form given below

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0.8 & 0.8 & \cdots & 0.8 & 0.8 \\ 0.5 & 0.5 & \cdots & 0.5 & 0.5 \end{bmatrix}. \tag{6.17}$$

Using Algorithm 5.1, an optimisation scheme called the Nelder-Mead simplex method which is inbuilt in MATLAB has been used to estimate the parameters $\eta_1$ and $\eta_2$. As an initial guess, the parameter values $\eta_1 = 0.79$ and $\eta_2 = 0.79$ has been used. The reduced order model to be optimised is of dimension 25 where $q_1 = 5$, $q_2 = 5$ and $p_2 = 4$.

At each iteration step, the scheme returns a cost for the estimated reduced order model. In this case, the squared error (SE) has been used as the cost function:

$$\text{SE} = \sum_{i=1}^{n_s} (\hat{y}_i - y_i)^2 \tag{6.18}$$

where $y_i$ is the output of the higher order bilinear model at time $i$, $\hat{y}_i$ is the output of the reduced order bilinear model at time $i$, and $n_s$ is the number of samples collected for each output. The input which has been used for simulation in the optimisation scheme is given. This is called the estimation input.

Table 6.2: Input values $u_1$ for parameter estimation.

| | | | | | |
|---|---|---|---|---|---|
| $u_1$ | 18.3745 | 5.9107 | 51.7177 | 51.7490 | 70.2652 |
| Time range (s): | $t \in [0 : 0.9]$ | $t \in [1 : 1.9]$ | $t \in [2 : 2.9]$ | $t \in [3 : 3.9]$ | $t \in [4 : 4.9]$ |
| $u_1$ | 30.3039 | 40.1577 | 62.7450 | 5.0065 | 6.3760 |
| Time range (s): | $t \in [5 : 5.9]$ | $t \in [6 : 6.9]$ | $t \in [7 : 7.9]$ | $t \in [8 : 8.9]$ | $t \in [9 : 9.9]$ |
| $u_1$ | 40.9033 | | | | |
| Time range (s): | $t \in [10]$ | | | | |

$$u_2 = (\sin(2\pi \times t) + 90)/10 + 0.1 \tag{6.19}$$

$$u = [u_1 \quad u_2]^T \tag{6.20}$$

The results to be shown are for simulations using the estimation data set and then the validation data set i.e. input and output values. The validation input is given as follows.

$$u_2 = (\sin(2\pi \times t) + 90)/10 + 0.2 \tag{6.21}$$

$$u = [u_1 \quad u_2]^T \tag{6.22}$$

### 6.5.1 Results

Figure 6.5 displays the outputs $y_1$ of the reduced order models derived from using the PLA and the optimised PLA and the higher order bilinear model (HOBM). The second row of this figure shows the absolute error values of the reduced order models. In these plots, the optimised PLA shows a better fit for the higher order bilinear model.

Figure 6.6 shows the input $u_1$ as defined in Table 6.2 in the first row. The second row of Figure 6.6 shows the second input $u_2$ as defined in (6.20)

Table 6.3: Input values $u_1$ for model validation.

| | | | | | |
|---|---|---|---|---|---|
| $u_1$ | 14.8532 | 7.1170 | 35.5489 | 47.0612 | 47.0612 |
| Time range (s): | $t \in [0 : 0.9]$ | $t \in [1 : 1.9]$ | $t \in [2 : 2.9]$ | $t \in [3 : 3.9]$ | $t \in [4 : 4.9]$ |
| $u_1$ | 22.2576 | 28.3737 | 42.3934 | 6.5557 | 7.4058 |
| Time range (s): | $t \in [5 : 5.9]$ | $t \in [6 : 6.9]$ | $t \in [7 : 7.9]$ | $t \in [8 : 8.9]$ | $t \in [9 : 9.9]$ |
| $u_1$ | 28.8365 | | | | |
| Time range (s): | $t \in [10]$ | | | | |

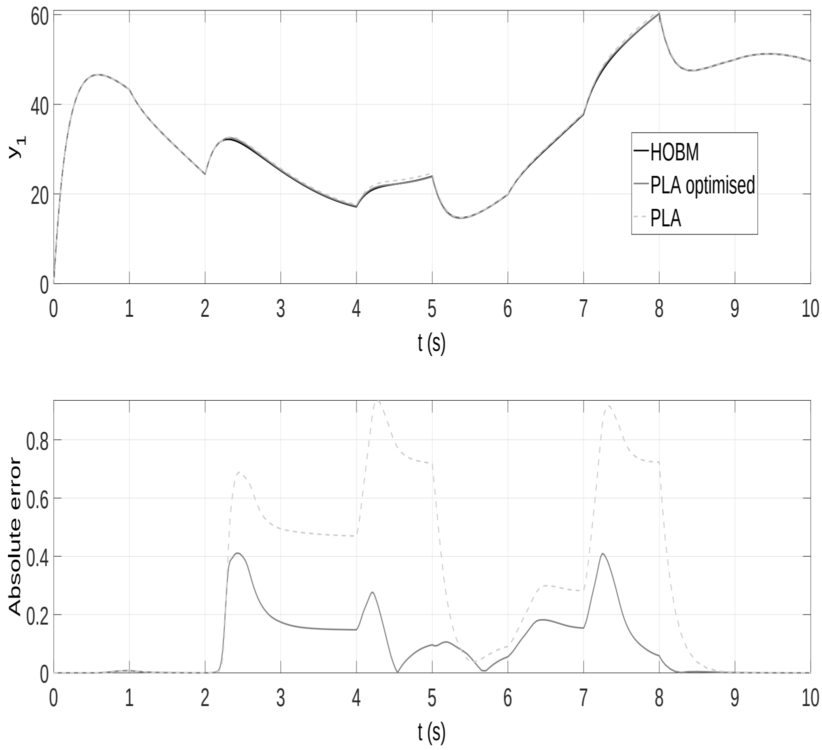



Figure 6.5: Plot of outputs $y_1$ and absolute error for PLA, optimised PLA and high order bilinear model using estimation input

In Table 6.4, the performance criteria values of the coefficient of determination ($RT^2$), the mean square error (MSE), the integral of absolute error (IAE), the integral of absolute error divided by number of samples (NIAE) and the sum of square of error (SSE) are displayed.

Table 6.4: Table of $RT^2$, MSE, IAE, NIAE and SSE values for model order reduction of solar panel model using parametrised linear approximation

| | $RT^2$ | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|---|
| PLA | 99.86 | 0.2603 | 139.6369 | 0.4036 | 8.9180 |
| PLA optimised | 99.99 | 0.0258 | 40.9249 | 0.1183 | 90.0735 |

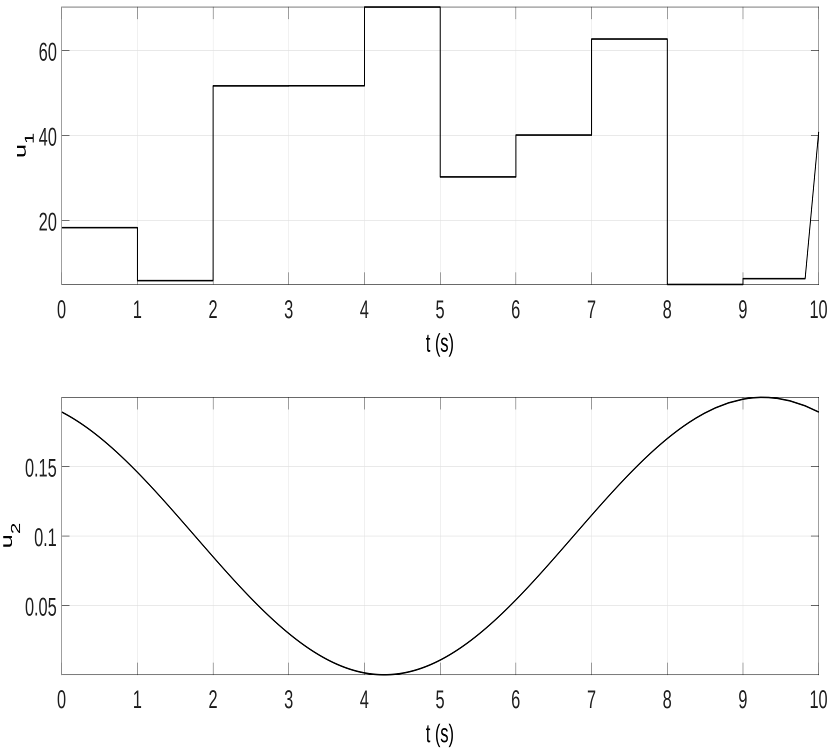



Figure 6.6: Plot of inputs $u_1$ and $u_2$ used for parameter optimisation of the PLA parameters and the simulation output given in Figure 6.5 as defined in Table 6.2 and (6.19).

The results presented in Figures 6.5 and Table 6.4 are for the input used for parameter estimation. These validation results using a new set of inputs are shown in Figures 6.7.

Table 6.5: Table of $RT^2$, MSE, IAE, NIAE and SSE values for model order reduction of solar panel model using Parametrised Linear Approximation

| | $RT^2$ | MSE | IAE | NIAE | SSE |
|---|---|---|---|---|---|
| PLA | 99.99 | 0.0165 | 43.1672 | 0.1680 | 4.2396 |
| PLA optimised | 99.98 | 0.0545 | 24.4731 | 0.0952 | 14.0130 |

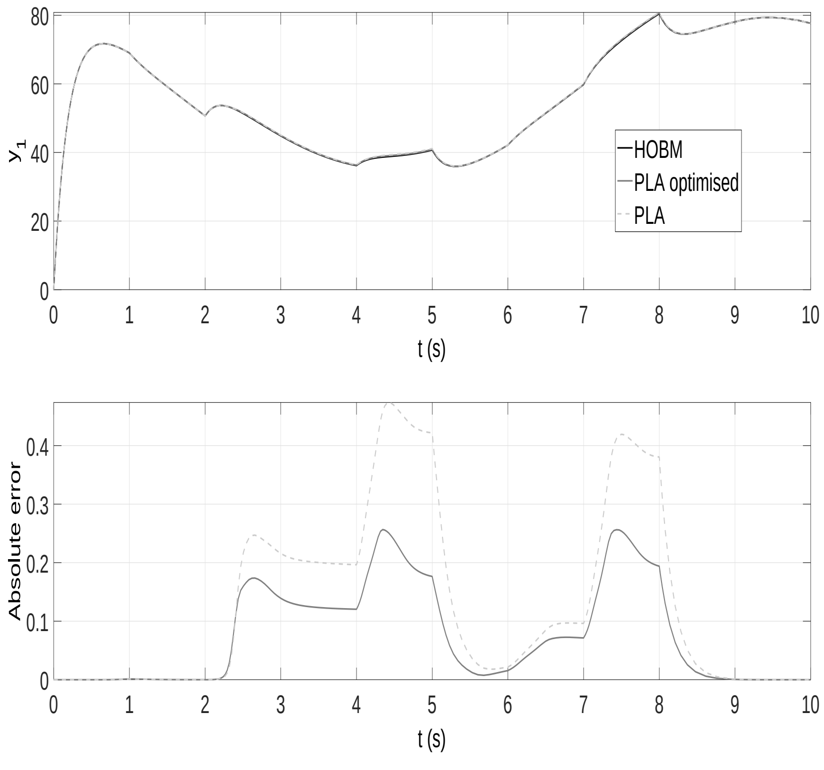



Figure 6.7: Plot of outputs $y_1$ and absolute error for PLA, optimised PLA and higher order bilinear model using validation input

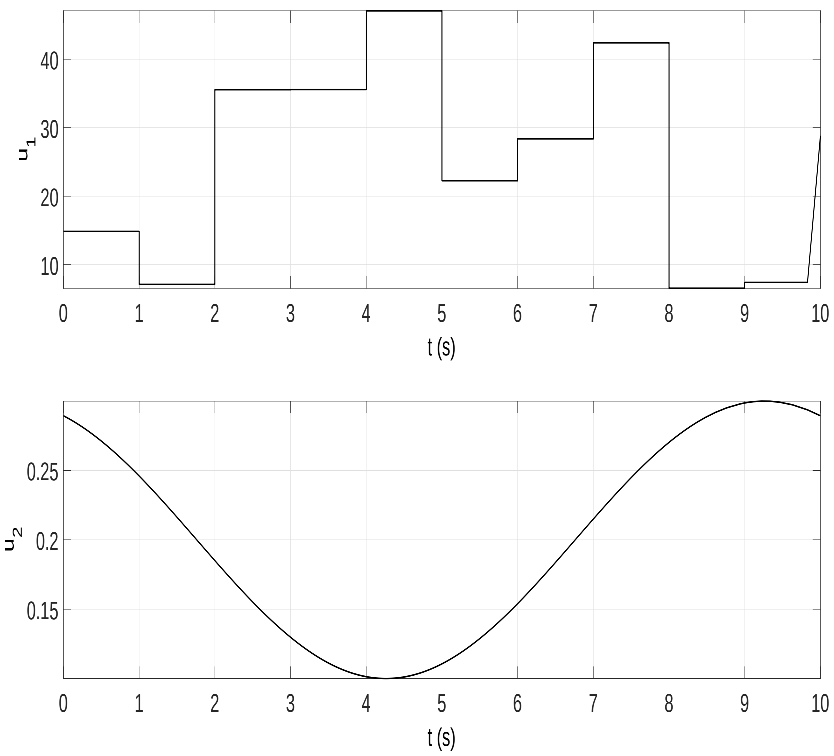



Figure 6.8: Plot of inputs $u_1$ and $u_2$ used for validating the optimised PLA parameters.

Figure 6.7 displays the output plots of the reduced order models compared with that of the higher order bilinear model (HOBM). Figure 6.8 shows the input plots $u_1$ (top) and $u_2$ (bottom). These inputs are as defined in Table 6.3 and (6.21) respectively.

In Table 6.5, the performance criteria values of the coefficient of determination ($RT^2$), the mean square error (MSE), the integral of absolute error (IAE), the integral of absolute error divided by number of samples (NIAE) and the sum of square of error (SSE) of the validation output values are shown.

## 6.6 Conclusion

In this chapter, two unique applications of the parametrised linear approximation (PLA) approach to applying Krylov subspaces for model order reduction are discussed and demonstrated. The first is the use of PLA for the reduction of pseudo-singular systems. These are systems/models which have singular system matrices. This means that the Krylov subspace model order reduction methods discussed in Chapters 3, 4 and 5 i.e the Phillips type (Phillips 2000), Feng and Benner type (Feng & Benner 2007), Bai type (Bai & Skoogh 2006) and the Improved Phillips type projection methods will not be applicable for model order reduction. However using the PLA approach, a pseudo-singular model which has been used in (Couchman et al. 2011) has been reduced. In order to reduce this system using PLA, the Krylov subspaces $K_{q_1}(\mathbb{N}, \mathbb{M})$ and $K_{q_2}(\mathbb{N}, \mathbb{M})$ were defined using the parameterised linear approximation which is invertible. The case study on Solar Panel Model demonstrated the advantage of the use of PLA approach. Not only is it useful for reducing models with noninvertible matrices, it can also achieve this at accuracy of $RT^2$ values of 99.98.

In the second application and example, a hybrid model order reduction approach has been proposed. This approach suggests that the PLA parameters $\eta_1, \ldots, \eta_m$ can be optimised by using an optimisation scheme. Using the example provided in Chapter 5, the optimised reduced order model was compared to the case where the PLA parameters are equal and the optimised reduced order model shows a better input-output preservation for the two sets of test input, i.e. the input used for optimisation and the input used for validation.

# Chapter 7

# Conclusion and Further Work

This chapter will summarise the findings and contributions of this thesis. Also it will discuss future research ideas which are the result of the findings therein. In the next subsection, a brief description of the research objectives is revisited. This is followed by the research conclusion, contributions and further work.

## 7.1 Conclusion

This thesis presents novel approaches in the reduction of bilinear models. Single-input-single-output (SISO), multi-input-multi-output (MIMO) and their pseudo-singular model cases are considered.

In Chapters 2 and 3, the basic concepts of Krylov subspace model order reduction approaches are discussed. Chapter 3 discusses the state of the art of one-sided Krylov subspace projection for bilinear models (Phillips 2000, Feng & Benner 2007) and their application to MIMO bilinear models (Lin et al. 2007, Lin et al. 2009, Lin et al. 2007, Lin et al. 2009). These Krylov projection types discussed multimoments of bilinear models/systems and in order to do this, matrix inversion is necessary. This limits their application only to models which have invertible matrices. Also, the error of the derived models tend to become larger as the input increases and are not flexible in their implementation.

In Chapter 4, a new set of Krylov subspaces for matching multimoments has been proposed. The new approach is called the Improved Phillips (IP) type projection and takes advantage of the fact that model order reduction of bilinear models are dominated by the linear approximation of the bilinear model as has been discussed in (Baur et al. 2014). This new method therefore achieves a linear approximation by matching the multimoments $m(l_1, l_2), l_1 = 1, \ldots q_1, l_1 = 1, \ldots q_2 - 1$. Going further, to continue exploiting the linear approximation of the bilinear model, another new approach is proposed called the Parametrised Linear Approximation (PLA). This Parametrised Linear Approximation is shown to preserve the input-output relationship of a bilinear system using three examples which show the advantages of using the Improved Phillip type projection, the Parametrised Linear Approximation and the combination of both. The input-output preservation of the reviewed methods in Chapter 3 and the proposed methods in Chapter 4 have been analysed using input-output plots, coefficient of determination ($RT^2$), mean square error (MSE), integral of absolute error (IAE), sum of square error (SSE), integral of absolute error divided by number of samples (NIAE) and plots of absolute error against ascending input values.

Subsequently, the Feng and Benner type projection (Feng & Benner 2007), Improved Phillips type and the Parametrised Linear Approximation type projection have been extended to MIMO structures in Chapter 5. The multimoment matching for Feng and Benner type (Feng & Benner 2007) has been analysed for MIMO models and the analysis can be extented to other types of Krylov subspace projections in literature i.e projection for MIMO bilinear models as proposed in (Lin et al. 2007, Lin et al. 2009), where the Phillip type projection has been extended to MIMO cases in (Lin et al. 2007) and the Bai type projection in (Lin et al. 2009). Using the same criteria that has been used in Chapter 4, the Feng and Benner type projection for MIMO bilinear models, Improved

Phillips type projection for MIMO bilinear models and the parametrised linear approximation for MIMO bilinear models have been compared to the work done in (Lin et al. 2007, Lin et al. 2009). In the numerical simulation results, it has been found that the parametrised linear approximation (PLA) shows good input-output preservation when compared to the other types. This is expected because the PLA uses a so called better linear approximation for the SISO and MIMO bilinear model reduction.

Whilst the Feng and Benner type projection (Feng & Benner 2007) matches more multimoments when compared to the Phillips type projection (Phillips 2000, Lin et al. 2007) and the Improved Phillips type for both SISO and MIMO models, it is not always the case that the reduced order models produced give a better approximation. This is because during the Krylov subspace reduction process, the multiplication of matrices which are nonsingular produce loss of information and therefore the Feng and Benner approach (Feng & Benner 2007) cannot guarantee a better reduced order model and in some cases it will be impossible to compute the projection matrix if there is a total loss of rank in the resulting matrices. It has been said that the Improved Phillips type projection combines the advantages of the Phillips type projection (Phillips 2000) and the Feng and Benner type projection (Feng & Benner 2007) by matching more moments of the linear approximation of the bilinear model and avoiding the loss of information when computing the projection matrix.

The application of the model order reduction techniques developed so far were applied to a pseudo-singular system in Chapter 6. Therein, a pseudo-singular system has been defined as a system with system matrices that cannot be inverted. The application of the Improved Phillips type projection and the PLA approach shows the viability in the use of the proposed Krylov subspaces for reducing systems of this kind. As opposed to the other systems where the alternate linear approximation is applied to only match the linear moments, in this case the linear approximation has shown to be useful for computing subsequent Krylov subspaces for the bilinear approximation. This means that the singular system matrix can be replaced by a non-singular alternate/approximate which exhibits the same characteristics. The reduced order models derived show good accuracy. A second example which has been considered in Chapter 6 exploits the hybrid combination of optimisation techniques and the PLA approach. This has been shown to improve the reduced order model when compared to the PLA without an optimisation scheme.

In summary, the contributions of the research reported in this thesis are as follows:

1. The matching of a higher number of multimoments whilst avoiding the multiplication of nonsingular matrices. This has been called the Improved Phillip type projection (Chapter 3).

2. The proposal of a reduced order modelling approach using Kylov subspaces by applying a so called better linear approximation. This approach is called the Parametrised Linear Approximation (PLA).

3. The analysis of multimoment matching for the Feng and Benner type projection (Feng & Benner 2007), Phillip type projection (Phillips 2000) and the Improved Phillip type projection.

4. The extension of the Improved Phillip type projection, Parametrised Linear Approximation projection and the Feng and Benner type projection (Feng & Benner 2007) to MIMO cases.

5. The analysis of multimoment matching for MIMO bilinear model reduction using Krylov subspaces.

6. The use of PLA for the reduced order modelling of pseudo-singular bilinear systems to enable the reduction of systems with nonsingular system

matrices.

7. The use of an optimisation scheme for finding parameters which form an alternate linear approximation of a bilinear system/model and the use of these parameters for model order reduction.

This thesis also served as a resource for understanding and implementation of reduced order modelling using Krylov subspaces.

## 7.2 Further work

The research done in this thesis brings about various new scopes of research which are quite interesting.

1. Exploration of other nonlinear approximations: The focus of this thesis has been on bilinear and nonlinear models of a certain type, i.e. those nonlinear models which can be bilinearised. The scope can also be extended to quadratic approximations (Chen 1999) and quadratic-bilinear control systems (Benner & Breiten 2012*b*). Quadratic approximations have been reported to be less effective when compared to bilinear approximations. The application of an alternate linear approximation for computing the projection matrices will improve the input-output preservation of quadratic and quadratic-bilinear approaches to MOR.

2. Extension to other bilinear systems with singularity or ill-conditioned matrices: This work makes it possible to apply bilinear model order reduction using Krylov subspaces to systems which would have been otherwise impossible to reduce. A further application of the techniques to other systems which have singular and/or ill-conditioned matrices either due to their derivation from Carleman bilinearisation or through the discretisation process is an interesting prospect. More practical examples of systems

which result in singularity or ill-conditioned matrices will further highlight the effectiveness of the PLA approach.

3. Further exploration of optimization techniques: optimisation of model order reduction techniques are also an interesting area to look at. In this thesis, Krylov subspaces were combined with parameter estimation. Here, the only algorithm looked at is the use of the Nelder-Mead optimisation algorithm. However, it has been suggested in (Abdullah, Deris, Anwar & Arjunan 2013) that other algorithms such as the Firefly Algorithm (FA) (Yang 2009), Particle Swarm Optimization (PSO) (Kennedy 2011, Campbell 2009) and the Hybrid Firefly Evolutionary Optimization (Abdullah et al. 2013), significantly outperform the MATLAB® implementation of the Nelder-Mead algorithms. Candidate algorithms to be investigated should include the aforementioned nature inspired optimisation algorithms as well as efficient gradient based constrained optimisation algorithms such as the interior-point or sequential-quadratic programming (SQP) (Nocedal & Wright 2006) solvers implemented in the MATLAB function fmincon. Exploring other optimisation schemes and combination of other parameter estimation techniques with Krylov subspace MOR should improve the accuracy of the reduced order models. Such improvement would enable faster as well as more accurate online simulations. This should result in improved performance of model based control schemes exploiting such reduced order models.

4. Further hybrid approaches: This involves the hybrid implementation of Improved Phillip and PLA with other classical model estimation algorithms such as balanced truncation and $H_2$ model order reduction. In the future, the combination of PLA with balanced truncation and $H_2$ model order reduction are likely to make the use of these techniques more practical and less time consuming. For example, as has been discussed in litera-

ture, Gramian based approaches can achieve higher accuracy and Krylov subspace techniques can be used for models with much higher dimensions.

5. Two sided projection approaches: the natural progression after considering one-sided approaches is to consider two-sided projection. This can be achieved by using a different set of Krylov subspaces for defining the left projection matrix. This will increase the number of multimoments matched by the IP and PLA approaches.

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