New exact solutions for Hele-Shaw flow in doubly connected regions

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Radial Hele–Shaw flows are treated analytically using conformal mapping techniques. The geometry of interest has a doubly-connected annular region of viscous fluid surrounding an inviscid bubble that is either expanding or contracting due to a pressure difference caused by injection or suction of the inviscid fluid. The zero-surface-tension problem is ill-posed for both bubble expansion and contraction, as both scenarios involve viscous fluid displacing inviscid fluid. Exact solutions are derived by tracking the location of singularities and critical points in the analytic continuation of the mapping function. We show that by treating the critical points, it is easy to observe finite-time blow-up, and the evolution equations may be written in exact form using complex residues. We present solutions that start with cusps on one interface and end with cusps on the other, as well as solutions that have the bubble contracting to a point. For the latter solutions, the bubble approaches an ellipse in shape at extinction.

Keywords: Doubly connected Hele–Shaw flow, Saffman–Taylor instability, finite-time blow-up, bubble extinction, viscous fingering

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FIG. 1. Three experimental setups of the Hele–Shaw problem: (a) a finite bubble surrounded by an infinite field of viscous fluid. The interface is driven by injection or extraction of the inviscid fluid into the bubble. (b) A blob of viscous fluid in an empty cell. The interface is driven by the injection or extraction of viscous fluid. (c) a doubly-connected ring of fluid, driven by a pressure difference between the two interfaces, which is the topic of this paper.

I. INTRODUCTION

Hele–Shaw flows are of primary interest to physicists and applied mathematicians due to the Saffman–Taylor instability\(^1\), which applies whenever a less viscous fluid displaces a more viscous fluid. This mechanism is widely viewed as a paradigm for interfacial pattern formation (in dendritic crystal growth, for example), leading to extensive body of research since the mid 1980s\(^2\)–\(^4\).

A heavily studied version of the problem concerns a radial geometry for which a bubble of the less viscous fluid (an inviscid fluid, say) is completely surrounded by the more viscous fluid, and the motion driven by injection of the inviscid fluid into the bubble (see Figure 1a). The traditional experimental set-up shows that, as the inviscid fluid is injected at a constant rate, an initially near-circular interface first expands and then begins to develop fingers which grow outwards\(^5\)–\(^9\). Stability theory which includes the effects of surface tension successfully predicts the wavelengths of growing perturbations\(^5,10\)–\(^12\).

The standard mathematical model of the above physical processes ignores the effects of the inviscid fluid, leading to Laplace’s equation for the fluid pressure in the viscous fluid subject to dynamic and kinematic boundary conditions on the interface between the two fluids. The zero-surface-tension version of the problem is well known to be ill-posed; however, for the idealised case in which there is an expanding bubble in an infinite body of fluid, conformal mapping techniques can be used to derive families of exact solutions. For some of
these families, the exact solutions persist for all time, either by clearing out all the viscous fluid (the bubble must necessarily be elliptic in shape in this case)\textsuperscript{13,14}, or by growing fingers that leave behind ‘fjords’ or ‘tongues’\textsuperscript{15}. For other families, the exact solutions blow up in finite time, with the interface forming one or more cusps\textsuperscript{16}. The inclusion of surface tension in the model regularises this sort of blow-up, but prevents equivalent progress using techniques from complex variable theory. As such, much effort has been applied to numerical studies of viscous fingering in Hele–Shaw flows\textsuperscript{17,18}, and in recent times numerical and analytical studies of radial fingering with surface tension are still prevalent\textsuperscript{19–21}.

A closely-related problem results if the inviscid fluid is withdrawn from the bubble, with the fluid interface contracting in time. In this case the interface is stable, as it is the more viscous fluid displacing the inviscid one. Provided the bubble boundary is initially convex, it will shrink to a point approaching an ellipse in shape in the extinction limit\textsuperscript{22–25}. Otherwise, the bubble may undergo a change in topology by pinching off and breaking into two or more bubbles\textsuperscript{26–28}. The zero-surface-tension version of this (well-posed) problem is the time-reversal of the ill-posed bubble growth problem discussed above, meaning that by reversing time, a solution for one application can be used for the other. This property does persist once surface tension is included in the model.

Considerable attention has also been given to the moving boundary problem that arises by injection or suction of a blob of viscous fluid in a Hele–Shaw cell (see Figure 1b). Here the suction case is of most interest, as it is this regime that is unstable with the zero-surface-tension problem being ill-posed. Complex variable methods have been applied by many authors for the zero-surface-tension problem, including Refs. 29–31. Many of the known exact solutions to the zero-surface-tension suction problem blow up in finite time, accompanied by the formation of one or more zero-angle cusps on the interface (other solutions with finite-angle cusps can be constructed in principle by initiating an injection problem with a finite-angle cusp and evolving in time; time-reversibility means that the reverse of these blowing solutions are suction solutions with finite-angle cusps\textsuperscript{32,33}).

In the present study we are interested in a radial geometry in which there is a bubble of inviscid fluid in a \textit{finite} blob of viscous fluid (see Figure 1c). This geometry has previously been analysed in the context of competing instabilities in a rotating Hele–Shaw cell\textsuperscript{34}, as well as the effects of injection of viscous fluid\textsuperscript{35,36}.

We treat both cases of injection and suction of inviscid fluid in the inner bubble, so
that there is a doubly-connected region (a deformed annulus) of viscous fluid which either expands or contracts due to the pressure difference. From a mathematical point of view this geometry is interesting as it links together the bubble and blob problems described above. From a physical perspective, the geometry is relevant for experiments in which the volume of the bubble is of the same order as the volume of the viscous fluid. For example, there are recently reported experiments in which gas is injected at constant pressure into a finite volume of fluid\(^3\), and experiments on bubbles of inviscid fluid close to the edge of a viscous blob\(^38,39\). In the former, the authors observe that the inner interface develops fingers that ultimately meet the outer interface which then ruptures, while in the latter, surface tension maintains a thin film of fluid between the inner and outer inviscid regions until a time at which it suddenly breaks. This “bursting” effect is very interesting, and there are as yet no exact solutions in the literature which reproduce this behaviour.

We derive families of exact solutions to the zero-surface-tension problem by applying a conformal mapping from the fluid region to an annulus and formulating the problem in terms of a Polubarinova-Galin equation. A known class of solutions exists, whose evolution can be found by tracking the poles and zeros of the mapping function in the analytically continued plane\(^3\). Our main contribution is to present an alternate approach in which we consider the derivative of the mapping function instead of the function itself, and explicitly track the critical points in the mapping function (points at which the derivative is zero). These points, which initially lie outside the physical region, evolve in time and lead to the finite-time blow-up of the solution, by cusp formation on either inner or outer boundaries, when they intersect the boundary of the physical region. Another advantage of tracking the critical points is that the integral in the time evolution equations for the poles and zeros can be calculated exactly, using the residues at each of the critical points. Thus our method provides the exact time evolution equations, without the need to compute the integrals numerically.

Our approach follows and extends upon that of Crowdy\(^3\), who derived solutions to the doubly-connected problem in a rotating Hele–Shaw cell. This method has also been adapted\(^4\) to treat the time-dependent Saffman–Taylor finger problem with a finite amount of fluid, with the finiteness of the fluid domain again leading to an annulus in the conformally mapped plane. In Section III we reiterate the loxodromic function theory needed to construct the class of exact solutions, and in Sections IV and V we derive some examples of exact
solutions in terms of the mapping function and its derivative, respectively. We end with a discussion of future work in Section VI.

II. COMPLEX FORMULATION

A. The boundary value problem

Let \( \Omega(t) \) be the doubly-connected fluid region in the physical \((z = x + iy)\) plane. The inner and outer boundaries are denoted \( \partial \Omega_i(t) \) and \( \partial \Omega_o(t) \) respectively. According to the standard Hele–Shaw equations, there exists a velocity potential \( \phi(z, t) \), negatively proportional to the pressure field, which is harmonic (i.e., satisfies Laplace’s equation) in \( \Omega \). Assuming Bernoulli (constant pressure) conditions on each interface, we may scale variables such that

\[
\phi = \begin{cases} 
0 & z \in \partial \Omega_o, \\
1 & z \in \partial \Omega_i.
\end{cases}
\] (1)

Here we have taken the higher pressure to be on the outer interface, so that the region will contract as time increases, and expand as time decreases (recall the problem is fully time-reversible). The evolution of the fluid region is dictated by the kinematic conditions connecting the fluid velocity to the velocity of each boundary. In vector form this is

\[
v \cdot n = \nabla \phi \cdot n, \quad z \in \partial \Omega_o, \partial \Omega_i,
\] (2)

where \( n \) is a normal and \( v = (\dot{x}, \dot{y}) \) is the velocity of a point that stays on the interface.

B. Polubarinova–Galin equation

By the doubly-connected analogue of Riemann’s mapping theorem, \( \Omega \) is the image of the annulus \( \mathcal{A} = \{ \zeta \mid \rho < |\zeta| < 1 \} \) in the auxiliary \( \zeta \) plane, under some time-dependent mapping function \( z = f(\zeta, t) \) (see Figure 2). The inner radius \( \rho \), also known as the conformal modulus, is not arbitrary; it depends on the fluid region and, in general, will change over time.

The velocity potential \( \phi \) is the real part of a complex-valued function \( \Phi \) which is analytic in \( z \in \Omega \) (and therefore in \( \zeta \in \mathcal{A} \)). In the \( \zeta \)-plane, the exact form for \( \Phi \) satisfying both
Bernoulli conditions (1) is, by inspection,
\[ \Phi = \frac{\log \zeta}{\ln \rho}. \]

The remaining kinematic conditions can be written as boundary equations which specify the mapping function \( f \). This is the Polubarinova–Galin equation approach\(^{41,42}\). In complex variables, \( n = \zeta f_\zeta \) is normal to each interface and
\[ v = \frac{dz}{dt} = \begin{cases} f_t(\zeta, t) + \frac{\dot{\rho}}{\rho} \zeta f_\zeta(\zeta, t) & |\zeta| = \rho \\ f_t(\zeta, t) & |\zeta| = 1 \end{cases}. \]

The kinematic condition (2) becomes
\[ \Re\{v \bar{\zeta} f_\zeta\} = \Re\{\zeta \Phi\} = \frac{1}{\ln \rho} \]
and, substituting for \( v \) and rearranging, we obtain
\[ \Re \left\{ \frac{f_t}{\zeta f_\zeta} \right\} = \begin{cases} \beta_i(\zeta, t), & |\zeta| = \rho \\ \beta_o(\zeta, t), & |\zeta| = 1 \end{cases}, \]
where
\[ \beta_i(\zeta) = \frac{1}{\rho^2 \ln \rho |f_\zeta(\zeta)|^2} - \frac{\dot{\rho}}{\rho}, \quad \beta_o(\zeta) = \frac{1}{\ln \rho |f_\zeta(\zeta)|^2}. \]

The equation (5) defines the real boundary data (\( \beta_i \) and \( \beta_o \)) of a function \( f_t/\zeta f_\zeta \), analytic in the annulus \( A \). The boundary data uniquely determines the analytic function up to an imaginary constant, as we describe in the next section. In (6) and hereafter we suppress explicit dependence of the function \( f \) (and its derivatives) on \( t \), for brevity.

III. VILLAT’S INTEGRAL FORMULA AND LOXODROMIC FUNCTIONS

A. Villat’s integral formula

Since \( f_t/\zeta f_\zeta \) is analytic in \( A \), it can be constructed from its real boundary data, given in (6), up to an imaginary constant \( ic \), using Villat’s integral formula\(^{34,43}\):
\[ \frac{f_t}{\zeta f_\zeta} = I(\zeta), \quad \rho < |\zeta| < 1, \]
where
\[ I(\zeta) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \beta_o(\zeta')K(\zeta/\zeta', \rho) \frac{d\zeta'}{\zeta'} - \frac{1}{2\pi i} \oint_{|\zeta|=\rho} \beta_i(\zeta')K(\zeta/\zeta', \rho) \frac{d\zeta'}{\zeta'} - \frac{1}{2\pi i} \oint_{|\zeta|=\rho} \beta_i(\zeta') \frac{d\zeta'}{\zeta'} + ic. \]
FIG. 2. A schematic of the conformal mapping $z = f(\zeta, t)$ from the annulus $A$ to the fluid region $\Omega(t)$. The inner and outer boundaries $\partial\Omega_i$ and $\partial\Omega_o$ are the image of the circles $|\zeta| = \rho$ and $|\zeta| = 1$, respectively.

Here $K$ is the Villat kernel function, which may be defined as the infinite sum

$$K(\zeta, \rho) = 1 + \frac{2\zeta}{1 - \zeta} + 2 \sum_{j=1}^{\infty} \left( \frac{\rho^{2j}}{1 - \rho^{2j}\zeta} - \frac{\rho^{2j}/\zeta}{1 - \rho^{2j}/\zeta} \right).$$  \hspace{1cm} (9)

We take $c = 0$ without loss of generality, as a nonzero value for $c$ simply changes the parametrisation of the surfaces. Note that $\beta_o$ and $\beta_i$ are not arbitrary; by Cauchy’s integral theorem, their averages must be equal on both inner and outer circles. This immediately provides a time evolution equation for $\rho$:

$$\frac{\dot{\rho}}{\rho} = \frac{1}{2\pi \ln \rho} \int_{0}^{2\pi} \frac{1}{|f(e^{i\theta})|^2} - \frac{1}{\rho^2|f(\rho e^{i\theta})|^2} \, ds.$$  \hspace{1cm} (10)

The integral formula (7) is only valid when $\zeta$ is in $A$; to analytically continue outside, we must add in the residue in $I$ that is lost as $\zeta$ passes outside the unit circle. To continue inside, we subtract the residue that is gained. Therefore the analytic continuation of (7) is

$$\frac{f_i}{\zeta f_{\zeta}} = \begin{cases} 
I(\zeta) - 2\beta_i(\zeta) & \rho^2 < |\zeta| < \rho \\
I(\zeta) & \rho < |\zeta| < 1 \\
I(\zeta) - 2\beta_o(\zeta) & 1 < |\zeta| < \rho^{-1} 
\end{cases}.  \hspace{1cm} (11)

Here $\beta_i$ and $\beta_o$ must be analytically continued off their respective boundaries, namely,

$$\beta_i(\zeta) = \frac{1}{\rho^2 \ln \rho f_{\zeta}(\zeta) f_{\zeta}(\rho^2/\zeta)} - \frac{\dot{\rho}}{\rho}, \quad \beta_o(\zeta) = \frac{1}{\ln \rho f_{\zeta}(\zeta) f_{\zeta}(1/\zeta)}.$$  \hspace{1cm} (12)
B. Loxodromic functions and Hele–Shaw flow

A loxodromic function is a meromorphic function (except at zero) \( F \), which has the radial “periodicity” property

\[
F(\zeta) = F(m\zeta),
\]  

(13)

where \( 0 < m < 1 \) is called the multiplier\(^{44}\). Loxodromic function theory is intimately connected to elliptic function theory, since if \( F \) is loxodromic in \( \zeta \) it is elliptic in \( e^\zeta \), with periods \( \ln m \) and \( 2\pi i \). As with elliptic functions, a loxodromic function is specified up to a multiplicative constant by the location of its poles and zeros in its fundamental annulus \( m < |\zeta| < 1 \); furthermore, there must be at least two poles (counting multiplicity), and the number of poles and zeros (counting multiplicity) must be equal. A loxodromic function with given zeros and poles can thus be written explicitly as a product of simpler special functions.

Loxodromic functions have previously been used to construct exact solutions in doubly-connected Hele–Shaw flows\(^{34,36,40}\). By taking \( \zeta \) such that \( 1 < |\zeta| < \rho^{-1} \) and using the continuation formulae (11), it can be shown that

\[
\frac{\partial}{\partial t} \left[ f(\zeta, t) - f(\rho^2\zeta, t) \right] = \zeta I(\zeta, t) \frac{\partial}{\partial \zeta} \left[ f(\zeta, t) - f(\rho^2\zeta, t) \right].
\]  

(14)

Consequently, if at an initial time, \( f(\zeta, 0) - f(\rho(0)\zeta, 0) = C \), where \( C \) is a constant, then

\[
f(\zeta, t) - f(\rho\zeta, t) = C
\]  

(15)

holds for subsequent time, as long as critical points in \( f \) are not in \( \mathcal{A} \). For \( C = 0 \), this implies that a mapping function that is initially loxodromic with multiplier \( \rho(0)^2 \) remains loxodromic, with an evolving multiplier \( \rho(t)^2 \). This property allows us to write an explicit form for \( f \) based on its poles and zeros; however, it becomes difficult to locate the critical points of the mapping (where \( f_\zeta = 0 \)), which are important in the breakdown of the problem. We consider one solution of this type, similar to that in Ref. 34, in Section IV.

Alternatively, a mapping function where \( \zeta f_\zeta \) is loxodromic, with multiplier \( \rho^2 \), will also satisfy (15), with \( C \) not necessarily zero. Thus we can write an explicit form for \( f_\zeta \), and the locations of the critical points are easier to track. We consider solutions of this type in Section V.
C. The special functions $P$ and $K$

Following Refs. 34 and 36, we define the function $P(\zeta, \rho)$ by the infinite product

$$P(\zeta, \rho) = (1 - \zeta) \prod_{j=1}^{\infty} (1 - \rho^{2j} \zeta)(1 - \rho^{2j}/\zeta).$$

(16)

Many properties of $P$ can be found in the earlier references$^{34,36}$ and are easily verified from the definition. Most importantly is

$$P(\rho^2 \zeta, \rho) = P(1/\zeta, \rho) = -\frac{P(\zeta, \rho)}{\zeta},$$

(17)

which makes $P$ useful for defining loxodromic functions. The function is also related to the Villat Kernel function (9) by

$$K(\zeta, \rho) = 1 - 2\zeta \frac{P'(\zeta, \rho)}{P(\zeta, \rho)},$$

which is useful for computing the derivative of $P$ from (9).

IV. EXACT SOLUTIONS FOR THE MAPPING FUNCTION

Suppose $f$ is initially a loxodromic function of multiplier $\rho(0)^2$ which has $N$ poles and zeros at $p_k(0)$ and $q_k(0)$, respectively (we assume all $p_k$ and $q_k$ are distinct, so that all poles and zeros are simple). We take the fundamental annulus to be $\rho < |\zeta| < \rho^{-1}$ and, since $\rho < |\zeta| < 1$ corresponds to the fluid region, all poles and zeros must be in the outer region $1 < |\zeta| < \rho^{-1}$ for the mapping function to be physically meaningful.

From the results of the previous section, $f$ remains loxodromic over time and so it has the exact form

$$f(\zeta, t) = R \zeta \prod_{k=1}^{N} \frac{P(\zeta/q_k, \rho)}{P(\zeta/p_k, \rho)}, \quad \rho^2 \prod_{k=1}^{N} \frac{q_k}{p_k} = 1,$$

(18)

where $R$ is an arbitrary multiplicative constant, and the latter condition enforces loxodromy. The poles $p_k$, zeros $q_k$, conformal modulus $\rho$ and scaling factor $R$ are all functions of time. To determine the evolution of the fluid region, we need only derive the time evolution equations for these parameters. This is achieved using the analytically continued evolution equation (11) near each zero and pole$^{40}$. 

9
A. Time evolution of poles and zeros

Considering (11) in the outer region \(1 < |\zeta| < \rho^{-1}\), we obtain a partial differential equation for \(f\):

\[
\frac{\partial f}{\partial t}(\zeta) = \zeta I(\zeta) f(\zeta) - \frac{2\zeta}{\ln \rho f(1/\zeta)}.
\]

Near a pole, \(f \sim \alpha(\zeta - p_k)^{-1}\) (for some constant \(\alpha\)) and on matching the most singular term we obtain

\[
\frac{\dot{p}_k}{p_k} = -I(p_k).
\]

Similarly, near a zero, \(f \sim f(\zeta)(\zeta - q_k)\) and matching the constant term results in

\[
\frac{\dot{q}_k}{q_k} = -I(q_k) - \frac{2}{\ln \rho f(1/q_k)}.
\]

The derivative of \(f\) is easily expressed in terms of the special functions \(P\) and \(K\) (see appendix A for details).

The conformal modulus \(\rho\) is determined from the condition in (18), and following the suggestion in Ref. 40, we can find an evolution equation for \(R\) by evaluating the partial differential equation (19) at an arbitrary point (see Appendix B). Alternatively, it may be simpler to find \(R\) for certain solutions, as in the next example.

B. An example

Let there be \(N \geq 3\) poles \(p_k\) and zeros \(q_k\), evenly spaced around the circle of radius \(p\) and \(q\), respectively. In this case, the condition in (18) implies \(q = \rho^{-2/N}p\) and (18) becomes

\[
f(\zeta, t) = R\zeta \frac{P(\rho^2\zeta^N/p^N, \rho N)}{P(\zeta^N/p^N, \rho^N)}.
\]

(22)

The unknown parameters are \(p\), \(\rho\) and \(R\). Evolution equations for \(p\) and \(\rho\) are (20) and (10), respectively. To find an equation for \(R\), we note that, due to the properties of \(P\),

\[
f(\zeta_R) = R\zeta_R, \quad \zeta_R = \rho^{1-1/N}p.
\]

(23)

Now \(1 < p < \rho^{-1}\) implies \(\zeta_R \in \mathcal{A}\), thus on differentiating (23) and substituting \(f_t\) from (11) we obtain

\[
\frac{\dot{R}}{R} = (\zeta_R I(\zeta_R)) \frac{f}{f'}(\zeta_R) - \frac{\dot{\zeta}_R}{\zeta_R}.
\]
FIG. 3. The structure of zeros and poles in \( f_\zeta \) (or critical points and singularities in \( f \)) for two different solution forms. In (a), the mapping function \( f \) is of the form (22) and the critical points are free to intersect both inner and outer boundaries in finite time, leading to cusp formation (see Figure 4). In (b), \( f_\zeta \) is of the form (32), where the critical points lie on the same angle as the poles and are shielded from intersecting the unit circle. Solutions of this type always end in bubble extinction (see Figure 5) when contracting.

It is easily verified\(^{45}\) that \( f \) has \( 2N \) critical points (where \( f_\zeta = 0 \)) that lie in pairs at the angles \((2k + 1)\pi/N, k = 1, \ldots, N\), between the angles on which poles lie (see Figure 3a). For a contracting fluid ring (\( t \) increasing), the inner radius \( \rho \) may go to zero before the critical points pass into the physical region, or the critical points may reach \( |\zeta| = 1 \) first; this corresponds to the extinction of the inner bubble and the formation of cusps, respectively.

As an example, we solve the system (20), (10), (24) for \( N = 6 \), and initial conditions \( p(0) = 1.4, \rho(0) = 0.44, \) and \( R(0) = 1 \), in \texttt{Matlab}, using a Runge–Kutta scheme to advance in time (see Figure 4). The derivative of \( f \) in each equation is easily expressed in terms of the special function \( K \) (see Appendix A), while the integral \( I \) is computed using the trapezoid rule. We plot the interfaces for this solution in Figure 4. At the initial time, the outermost critical points are close to the circle of radius \( \rho^{-1} \), meaning the inner boundary is close to having cusps. As time increases, these cusps smooth off as the critical points move inward, until the innermost critical points intersect the unit circle, leading to cusp formation on the outer boundary. The solution exists only for finite time, for both expansion and contraction. Since the critical points are simple (that is \( f_\zeta(q_k) = 0 \) but \( f_{\zeta\zeta}(q_k) \neq 0 \)), a Taylor series expansion near a critical point shows that these cusps are generically of 3/2 power type, the
FIG. 4. The evolution of the fluid region given by the exact solution (22). Increasing time is indicated by lighter shades of colour. The solution blows up in finite time for both a contracting (increasing time) and contracting (decreasing time) fluid region by the formation of cusps on the outer and inner boundaries, respectively.

same as with the simply-connected case\textsuperscript{16}.

V. EXACT SOLUTIONS FOR THE DERIVATIVE OF THE MAPPING FUNCTION

A. Evolution of zeros and poles

Suppose $\zeta f_\zeta$ is initially a loxodromic function of multiplier $\rho(0)^2$ which has $N$ poles and zeros at $p_k(0)$ and $q_k(0)$, respectively. Now $\zeta f_\zeta$ remains loxodromic over time and so $f_\zeta$ has the exact form

$$f_\zeta(\zeta, t) = R \prod_{k=1}^{N} \frac{P(\zeta/q_k, \rho)}{P(\zeta/p_k, \rho)}, \quad \rho^2 \prod_{k=1}^{N} \frac{q_k}{p_k} = 1. \quad (25)$$

The evolution of poles and zeros is now found from differentiating (19):

$$f_{\zeta\zeta}(\zeta) = \zeta I(\zeta) f_{\zeta\zeta}(\zeta) + (I(\zeta) + \zeta I(\zeta)) f_\zeta(\zeta) - \frac{2}{\ln \rho f_\zeta(1/\zeta)} \left(1 + \frac{1}{\zeta} \frac{f_{\zeta\zeta}}{f_\zeta}(1/\zeta)\right). \quad (26)$$

Near a pole $p_k$, and a zero $q_k$, we have

$$\frac{\dot{p}_k}{p_k} = -I(p_k, t), \quad (27)$$
and
\[
\frac{\dot{q}_k}{q_k} = -I(q_k) + \frac{2}{\ln \rho q_k f_\zeta (1/q_k) f_\zeta (q_k)} \left( 1 + \frac{1}{q_k} \frac{1}{f_\zeta (1/q_k)} \right),
\]
respectively.

Since \( q_k \) are now the critical points, and the integrands in (7) are singular at these points, the integral term \( I \) can be expressed exactly by computing residues. However, the integrands in (8) have an accumulation point at zero, so the residue theorem cannot be applied directly. Instead, we use the properties of \( f \) and \( K \) to write (8) in terms of integrals around paths bounding annuli, in which their integrands will be meromorphic.

Firstly, we substitute into the Villat integral formula (8) the given boundary data (12):
\[
I(\zeta, t) = \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{K(\zeta/\zeta', \rho)}{\ln \rho f_\zeta (\zeta') f_\zeta (1/\zeta')} \frac{d\zeta'}{\zeta'} - \frac{1}{2\pi i} \oint_{|\zeta'|=\rho} \frac{1}{\rho^2 \ln \rho f_\zeta (\zeta') f_\zeta (\rho^2/\zeta')} \frac{d\zeta'}{\zeta'}.
\]
Note that the terms involving \( \dot{\rho}/\rho \) have cancelled out, since \( \oint_{|\zeta'|=\rho} K(\zeta/\zeta', \rho) \frac{d\zeta'}{\zeta'} = -1 \). Now as a consequence of the assumed loxodromy, \( \rho^2 f_\zeta (\rho^2/\zeta) = \overline{f_\zeta (1/\zeta)} \), which allows us to write the first two integrals as one whose path encloses the annulus \( \rho < |\zeta'| < 1 \). To handle the third integral, we use the property\(^{40}\)
\[
1 = \frac{1}{2} (K(\zeta, \rho) - K(\rho^{-2} \zeta, \rho)).
\]
This allows us to split and recombine the last integral in a form whose path encloses the annulus \( \rho < |\zeta'| < \rho^{-1} \). We obtain
\[
I(\zeta) = \frac{1}{2\pi i \ln \rho} \left[ \oint_{|\zeta'|=1} - \oint_{|\zeta'|=\rho} \right] \frac{K(\zeta/\zeta', \rho)}{\zeta' f_\zeta (\zeta') f_\zeta (1/\zeta')} d\zeta' + \frac{1}{4\pi i \ln \rho} \left[ \oint_{|\zeta'|=\rho^{-1}} - \oint_{|\zeta'|=\rho} \right] \frac{K(\zeta', \rho)}{\zeta' f_\zeta (\zeta') f_\zeta (1/\zeta')} d\zeta'.
\]
Both paths of integration enclose annuli in which their integrands are meromorphic, with a finite number of poles. In (27) and (28) we are interested in computing \( I \) at \( p_k \) and \( q_k \), which all lie in the region \( 1 < |\zeta| < \rho^{-1} \); we therefore consider (30) in that region. In their respective regions, the first integrand has simple poles at \( \zeta' = 1/q_k \), and the second has simple poles at \( \zeta' = q_k, \zeta' = 1/q_k \), and \( \zeta' = 1 \) (due to the pole in \( K \)). Summing the residues at these poles results in
\[
I(\zeta) = -\frac{1}{\ln \rho |f_\zeta (1)|^2} - \frac{1}{\ln \rho} \sum_{j=1}^{N} \frac{K(\zeta, \rho) - K(\zeta, \rho)}{q_j f_\zeta (1/q_j) f_\zeta (q_j)}.
\]
\[\tag{31}\]
This formula is used to replace $I$ in the evolution equations (27) and (28), so that the integral (7) need never be computed numerically.

B. An example

If the poles and zeros of $f_\zeta$ are evenly radially spaced, one can easily obtain a radially symmetric exact solution of a form analogous to (22), where the outer boundary forms fingers instead of cusps. However, we wish to emphasise the possible extinction behaviour of the inner bubble, and so choose a non radially symmetric solution for which the inner bubble tends to an ellipse of nonzero eccentricity, rather than a circle.

Let $f_\zeta$ have $N = 4$ poles and zeros at $\pm p_1, \pm i p_2$ and $\pm q_1, \pm i q_2$, respectively; thus

$$f_\zeta(\zeta) = R \frac{P(\zeta^2/q_1^2, \rho^2)}{P(\zeta^2/q_2^2, \rho^2)} P(-\zeta^2/q_2^2, \rho^2).$$

(32)

In this case we use (27)-(28) for the evolution of the poles and zeros, and determine $\rho$ from the condition in (25). It follows from this condition that $p_1 > q_1$ and $p_2 > q_2$, so that the pole and zero structure of $f_\zeta$ is as in Figure 3b. An evolution equation for the scaling factor $R$ is found by evaluating (26) at a point (see Appendix B). With this structure, the poles act to block the zeros from intersecting the unit circle, preventing the formation of cusps on the outer boundary when the region is contracting (time increasing). The solution therefore continues until bubble extinction. When the region is expanding (time decreasing), the critical points will intersect the circle $|\zeta| = \rho^{-1}$ after a certain time, so that finite-time blow-up occurs with cusps (also of $3/2$ power type) forming on the inner boundary.

We solve for $p_1, p_2, q_1, q_2$ and $R$ as before in Matlab using a Runge-Kutta scheme, with initial conditions $p_1(0) = 1.2$, $q_1(0) = 1.7$, $p_2(0) = 1.1$, $q_2(0) = 1.4$, and $R(0) = 1$. The interfaces themselves are reconstructed by integrating (32)$^{46}$. The result is shown in Figure (5). The solution continues up to the complete extinction of the inner bubble. For our initial conditions, a numerical examination of the Laurent series of $f$,

$$f(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^n,$$

show that $\rho^{-1} a_{-1}$ and $\rho a_1$ are of order 1 as $\rho \to 0$, while $\rho^n a_n \to 0$ for $|n| \geq 2$ ($a_0$ represents a translation, so does not change the shape of the inner boundary). This implies the extinction behaviour of the inner boundary is elliptic in nature. To our knowledge, the exact solution
FIG. 5. The evolution of the fluid region of the exact solution (32). Increasing time is indicated by lighter shades of colour. The interior bubble contracts to an ellipse of nonzero eccentricity.

presented here is the first to the multiply-connected bubble extinction problem showing this behaviour.

VI. DISCUSSION

We have demonstrated a method of calculating a large class of exact solutions to the zero-surface-tension problem of a doubly-connected expanding or contracting fluid region in a Hele–Shaw cell. This method involves tracking the zeros and poles of the mapping function or its derivative. By using the derivative, the critical points of the mapping function (which lead to finite-time blow-up) are explicitly located, and the integrals involved in the evolution equations are exactly expressed as sums of residues at these critical points. This novel approach to obtaining exact solutions to the doubly-connected problem provides the main contribution in the paper.

As with the problems treated by Crowdy\textsuperscript{34} and Crowdy and Tanveer\textsuperscript{40}, the finite amount of viscous fluid in our problem is bounded by two moving interfaces, and the problem is ill-posed regardless of whether the region of fluid is expanding or contracting. This is because each option involves the inviscid fluid displacing the viscous fluid. As such, an interesting feature of some of the solutions we have presented is that they start with cusps on one interface and end with cusps on the other. These cusps are generically of $3/2$ power type. We emphasise that the addition of surface tension regularises cusp formation in Hele–Shaw flows, so these solutions would not be observed in experiments.
We have focused on the behaviour of contracting bubbles, where solutions either cease to exist with cusp formation on the outer boundary (finite-time blow-up), or continue until the inner bubble contracts to a point (bubble extinction). In the case of extinction, the inner bubble asymptotes to an ellipse whose eccentricity depends on the pole and zero locations. This behaviour is reminiscent of shrinking bubbles in an infinite body of fluid\textsuperscript{16,22}, as well as the extinction of a bubble driven by a constant pressure on a fixed outer boundary\textsuperscript{22–25}. There are no known non-trivial exact solutions to the latter problem; complex variable methods similar to those employed in this paper may offer a new way of calculating some families of solutions. We also conjecture that the elliptical extinction behaviour seen in our solution is universal, as it is in the fixed outer boundary variant, in the sense that (almost) all solutions with bubble extinction have bubbles contracting to an ellipse at extinction (the exceptional case, treated in Refs. 28 and 27, is borderline between extinction and breakup into multiple bubbles). For other initial conditions for which the bubble boundary is sufficiently non-convex, we expect that the contracting bubble may break up into two or more bubbles\textsuperscript{26–28}. The classes of solutions treated in this paper do not cover this eventuality, as it involves a change in the topology of the fluid region, which can therefore no longer be mapped from a doubly-connected annulus.

As mentioned a number of times above, a feature of the zero-surface-tension problem considered here is that it is time-reversible, so that in addition to bubble contraction, our exact solutions also describe bubble growth in a doubly-connected domain. The bubble growth solutions we have presented all exhibit finite-time blow-up with a cusp singularity forming on the inner interface. These are therefore analogous of the bubble growth solutions in an infinite body of fluid that are discussed in Example 1 of Ref. 16. Unfortunately we have not found families of solutions that exhibit the distinctive fingering behaviour observed in experiments\textsuperscript{37} analogous to those shown in Figure 5 of Ref. 15. As such, another interesting facet of this problem which we have not explored is the behaviour of expanding fluid rings. Any solutions that are characterised by viscous fingering will either continue indefinitely or end with ‘bursting’, with the inner and outer boundaries meeting at one or more points. Bursting is the normal behaviour seen in experiments\textsuperscript{37–39}, and in terms of the conformal mapping from an annulus can only occur in the limit $\rho \to 1$, which represents a significant numerical challenge. The construction of a family of solutions that demonstrate either of these behaviours would be of some considerable interest.
The authors would like to acknowledge discussions held with Prof. Darren Crowdy that proved very valuable.

Appendix A: Derivatives of \( f \)

The evolution equations (20)-(21), or (27)-(28), require the differentiation of \( f \) or \( f_\zeta \), respectively. Since \( f \) is defined in (18) as a product, it is simplest to use its logarithmic derivative. Given the relationship between \( P \) and \( K \), this is easily found to be

\[
\frac{f_\zeta}{f}(\zeta) = \frac{1}{\zeta} + \frac{1}{2\zeta} \sum_{k=1}^{N} (K(\zeta/p_k, \rho) - K(\zeta/q_k, \rho)). \tag{A1}
\]

Similarly, if \( f_\zeta \) is as in (25), its logarithmic derivative is

\[
\frac{f_{\zeta\zeta}}{f_\zeta}(\zeta) = \frac{1}{2\zeta} \sum_{k=1}^{N} (K(\zeta/p_k, \rho) - K(\zeta/q_k, \rho)). \tag{A2}
\]

This formula is not valid at \( \zeta = q_m \), since \( f_\zeta = 0 \) there. Instead we use a limit definition to obtain

\[
f_{\zeta\zeta}(q_m) = \lim_{\zeta \to q_m} \frac{f_\zeta(\zeta)}{\zeta - q_k} = - \frac{R\mu(\rho)}{q_m P(q_m/p_m, \rho)} \prod_{k=1, k \neq m}^{N} \frac{P(q_m/q_k, \rho)}{P(q_m/p_k, \rho)}, \tag{A3}
\]

where \( \mu(\rho) \) is the infinite product

\[
\mu(\rho) = \prod_{j=1}^{\infty} (1 - \rho^{2j})^2. \tag{A4}
\]

The evolution equation for the scaling factor \( R \) requires the time derivative of \( f \) or \( f_\zeta \). Again using the logarithmic derivative we find the time derivative of (18) to be

\[
\frac{f_t}{f}(\zeta) = \frac{\dot{R}}{R} + \frac{1}{2} \sum_{k=1}^{N} \left( \frac{\dot{q}_k}{q_k} K(\zeta/q_k, \rho) - \frac{\dot{p}_k}{p_k} (K(\zeta/p_k, \rho)) \right) + \dot{\rho} \sum_{k=1}^{N} \left( \frac{P^*(\zeta/q_k, \rho)}{P(\zeta/q_k, \rho)} - \frac{P^*(\zeta/p_k, \rho)}{P(\zeta/p_k, \rho)} \right) + \dot{\rho},
\]

where \( P^*/P \) is the logarithmic derivative of \( P \) with respect to its second argument (\( \rho \)). An infinite sum definition of this special function is easily derived from (16). A similar formula of course holds for \( f_{\zeta t}/f_\zeta \) when \( f_\zeta \) is as defined in (25).
Appendix B: The time evolution of $R$

Following Ref. 40, we find the evolution equation for the scaling factor $R$ by evaluating the partial differential equation (26) at any given point $x$, where $1 < |x| < \rho^{-1}$. Substitution of the time derivative of $f_\zeta$ from (B1) into (26) results in

$$
\dot{R} = -\frac{1}{2} \sum_{k=1}^{N} \left( \frac{\dot{q}_k}{q_k} K(x/q_k, \rho) - \frac{\dot{p}_k}{p_k} (K(x/p_k, \rho)) \right) - \rho \sum_{k=1}^{N} \left( \frac{P^s(x/q_k, \rho)}{P(x/q_k, \rho)} - \frac{P^s(x/p_k, \rho)}{P(x/p_k, \rho)} \right) - \frac{\dot{\rho}}{\rho}
+ xI(x) \frac{\bar{f}_\zeta}{f_\zeta}(x) + I(x) + xI_\zeta(x) - \frac{2}{\ln \rho f_\zeta(x)f_\zeta(1/x)} \left( 1 + \frac{1}{x} \frac{\bar{f}_\zeta}{f_\zeta}(1/x) \right).
$$

(B1)

The Villat integral $I$ and its derivative $I_\zeta$ are found from (31).

REFERENCES


45Since $\zeta f_\zeta$ is loxodromic with $N$ second order poles at $p_k$, $f_\zeta$ will have $2N$ zeros (possibly counting multiplicity), with the same $N$-fold symmetry as $f$, that is

$$f_\zeta(\zeta) = A \frac{P(\zeta^N/r_1^N, \rho^N)P(\zeta^N/r_2^N, \rho^N)}{P(\zeta^N/p^N, \rho^N)^2}, \quad \rho^2 r_1 r_2 = 1$$

where $r_1, r_2 \in \mathbb{C}$. While the exact values of $r_1$ and $r_2$ are not easily found, the loxodromy condition above implies $r_2 = p_2^{-2/N} r_1$. Furthermore, we let $\zeta^N = Z$, and consider

$$F(Z) = (-Z)^{1/N} \frac{P(\rho^2 Z/p^N, \rho^N)}{P(Z/p^N, \rho^N)}$$

(which is proportional to $f$), so that $F$ is real, continuous and differentiable on the negative real axis. Since $F(Z) = F(\rho^{2n} Z)$, the mean value theorem implies $F$ must have a critical point (say $Z = r_1^N$) on the negative axis, from which it follows that $r_1$ (and subsequently $r_2$) lies on an angle $(2k + 1)\pi/N$ in the $\zeta$-plane.

46In this example, symmetry allows us to determine the (time varying) constant of integration trivially. For a more general solution, we can easily write a time evolution equation of $f(1)$, say, from (19), that allows us to determine this constant.

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