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Abstract

In this note, we study a robust synchronization problem for multistable systems evolving on manifolds within an Input-to-State Stability (ISS) framework. Based on a recent generalization of the classical ISS theory to multistable systems, a robust synchronization protocol is designed with respect to a compact invariant set of the unperturbed system. The invariant set is assumed to admit a decomposition without cycles, that is, with neither homoclinic nor heteroclinic orbits. Numerical simulation examples illustrate our theoretical results.

I. Introduction

Over the last decades, the synchronization of complex dynamical systems and/or network of systems has attracted a great deal of attention from multidisciplinary research communities thanks to their pervasive presence in nature, technology and human society [1], [2], [3], [4]. A collective behavior occurs in the interconnection of dynamical systems and it has several potential application domains. For instance, transient stability in power network [5], cooperative multitasking and formation control [6]. The core of synchronization is the collective objective of agents in a network to reach a consensus about certain variables of interest.

The existing literature on the synchronization problem is very vast and covers many areas. In [7], the problem of formation control is investigated in swarms within the framework of output regulation in nonlinear systems. A detailed study regarding the control and synchronization of chaos can be found in [8]. The paper [9] extends optimal control and adaptive control design methods to multi-agent nonlinear systems on communication graphs. Recent advances in various aspects of cooperative control of multi-agent systems can be found in [10]. The theoretical framework for design and analysis of distributed flocking algorithms can be found in [11].

In this paper, we consider the synchronization problem for multistable systems based on the framework of Input-to-State Stability (ISS). This is a very well established method for the study of stability and robustness of nonlinear systems. The ISS property provides a natural framework of stability analysis with respect to input perturbations (see [12] and references therein). The classical definition allows to formulate and characterize stability properties with respect to arbitrary compact invariant sets (and not simply equilibria). Nevertheless, the implicit requirement that these sets should be simultaneously Lyapunov stable and globally attractive, makes the basic theory not applicable for a global analysis of many dynamical behaviors of interest, having multistability [13], [14], [15] or periodic oscillations [16], just to name a few, and only local analysis remains possible [17]. Some attempts were made to overcome such limitations by introducing the notions of almost global stability [18] and almost input-to-state stability [19], *etc.*

Recently, the authors in [20] have proposed that the most natural way of relaxing ISS condition for systems with multiple invariant sets is equivalent to relax the Lyapunov stability requirement [21] (rather than the global nature of the attractivity property). Using this relatively mild condition, they [20] have generalized the ISS theory as well as, the related literature on time invariant autonomous dynamical systems on compact spaces [22] for multistable systems. Multistability accounts for the possible coexistence of various oscillatory regimes or equilibria in the phase space of the system for the same set of parameters. Any system that exhibits multistability is called a multistable system. For a multistable system, it is frequently very difficult to predict the asymptotic regime on which this system will attain asymptotically for the given set of initial conditions and inputs [23]. In our current work, the results presented in [20], [24] are applied to provide sufficient conditions for the existence of robust synchronization for multistable systems in the presence of external inputs. The conditions obtained in this work are global.

The rest of this paper is organized as follows. Section II introduces some preliminaries about decomposable sets and notions of robustness. Our main results and the family of nonlinear systems being considered can be found in Section III. In Section IV, numerical simulation examples are given to illustrate these results. Concluding remarks in Section V close this note.

II. Preliminaries

Let M be an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without a boundary, $x \in M$ and $f : M \times \mathbb{R}^m \to T_x M$ be a map of class $\mathcal{C}^1$. In this work, we assume that all manifolds are embedded in a Euclidean space of dimension n, so they contain 0. Consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t), d(t)) \tag{1}$$

where the state $x(t) \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X(t, x; d(\cdot))$ the uniquely defined solution of (1) at time t satisfying $X(0, x; d(\cdot)) = x$. Together with (1), we will analyze its unperturbed version:

$$\dot{x}(t) = f(x(t), 0). \tag{2}$$

A set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$. For a set $S \subset M$, define the distance to S from a point $x \in M$ by $|x|_S = \inf_{a \in S} \delta(x, a)$, where the $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M. We have $|x| = |x|_{\{0\}}$ for $x \in M$, the usual Euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$, the essential supremum norm is defined as $\|d\|_\infty = \operatorname{ess\,sup}_{t \geq 0} |d(t)|$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, i.e. $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. Furthermore, $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and unbounded i.e. $\lim_{s \to \infty} \alpha(s) = \infty$. For any $x \in M$, the α- and ω- limit sets for (2) can be defined as follows:

$$\begin{aligned}
\alpha(x) &:= \left\{ y \in M \mid y = \lim_{n \to -\infty} X(x, t_n) \text{ with } t_n \searrow -\infty \right\}, \\
\omega(x) &:= \left\{ y \in M \mid y = \lim_{n \to \infty} X(x, t_n) \text{ with } t_n \nearrow \infty \right\}.
\end{aligned}$$

A. Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for (2).

Definition 1. [22] A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that $\Lambda = \bigcup_{i=1}^{k} \Lambda_i$.

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\begin{aligned}
W^s(\Lambda) &= \{x \in M \mid |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\}, \\
W^u(\Lambda) &= \{x \in M \mid |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.
\end{aligned}$$

Define a relation on the set of invariant sets of M by: for $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$, we write $\mathcal{W} \prec \mathcal{D}$ if $W^s(\mathcal{W}) \cap W^u(\mathcal{D}) \neq \emptyset$.

Definition 2. [22] Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1) An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2) A 1-cycle is an index i such that $(W^u(\Lambda_i) \cap W^s(\Lambda_i)) \setminus \Lambda_i \neq \emptyset$.
3) A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 2, the existence of an r-cycle with $r \geq 2$ is equivalent to the existence of a heteroclinic cycle for (2) [25]. Moreover, the existence of a 1-cycle implies the existence of a homoclinic cycle for (2) [25].

Definition 3. Let $\mathcal{W} \subset M$ be a compact set containing all α and ω limit sets of (2). We say that $\mathcal{W}$ is decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, forming a filtration ordering of $\mathcal{W}$. This definition of the compact set $\mathcal{W}$ will be used all through the article.

B. Robustness notions

The following robustness notions for systems in (1) have been introduced in [20].

Definition 4. We say that the system (1) has the practical asymptotic gain (pAG) property if there exist $\eta \in \mathcal{K}_\infty$ and $q \in \mathbb{R}$, $q \geq 0$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the solutions are defined for all $t \geq 0$ and

$$\limsup_{t \to +\infty} |X(t, x; d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty) + q. \tag{3}$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 5. We say that the system (1) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |X(t, x; d)|_{\mathcal{W}} \leq \mu(\|d\|_\infty).$$

Definition 6. We say that the system (1) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the following holds for all $t \geq 0$:

$$|X(t, x; d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}).$$

To characterize (3) in terms of Lyapunov functions, it has been shown in [20] that the following notion suits:

Definition 7. We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is a practical ISS-Lyapunov function for (1) if there exists $\mathcal{K}_\infty$ functions $\alpha_1, [\alpha_2], \alpha$ and γ, and scalar $q \geq 0$ [and $c \geq 0$] such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq [\alpha_2(|x|_{\mathcal{W}} + c)],$$

the function V is constant on each $\mathcal{W}_i$ and the dissipation inequality below holds:

$$DV(x)f(x,d) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If this latter holds for $q = 0$, then V is said to be an ISS-Lyapunov function.

Notice that α_2 and c are in square brackets as their existence follows (without any additional assumptions) by standard continuity arguments.

The main result of [20] connecting these robust stability properties is stated below:

Theorem 8. *Consider a nonlinear system as in (1) and let a compact invariant set containing all α and ω limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 3). Then the following are equivalent:*

1) *The system admits an ISS Lyapunov function;*
2) *The system enjoys the AG property;*
3) *The system admits a practical ISS Lyapunov function;*
4) *The system enjoys the pAG property;*
5) *The system enjoys the LIM property and the pGS.*

A system in (1) that satisfies this list of equivalent properties is called ISS with respect to the set $\mathcal{W}$ [20].

III. Synchronization of multistable systems

The following family of nonlinear systems is considered in this section:

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t), d_i(t)), \; i = 1, \ldots, N, \; N > 1, \tag{4}$$

where the state $x_i(t) \in M_i$, with M_i an n_i-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without a boundary, the control $u_i(t) \in \mathbb{R}^{m_i}$ and the external disturbance $d_i(t) \in \mathbb{R}^{p_i}$ ($u_i(\cdot)$ and $d_i(\cdot)$ are locally essentially bounded and measurable signals) for $t \geq 0$ and the map $f_i : M_i \times \mathbb{R}^{m_i} \times \mathbb{R}^{p_i} \to T_{x_i} M_i$ is $\mathcal{C}^1$, $f_i(0,0,0) = 0$. Denote the common state vector of (4) as $x = [x_1^T, \ldots, x_N^T]^T \in M = \prod_{i=1}^N M_i$, so M is the corresponding Riemannian manifold of dimension $n = \sum_{i=1}^N n_i$ where the family (4) behaves and $d = [d_1^T, \ldots, d_N^T]^T \in \mathbb{R}^p$ with $p = \sum_{i=1}^N p_i$ is the total exogenous input.

1. *For all $i = 1, \ldots, N$, each system in (4) has a compact invariant set $\mathcal{W}_i$ containing all α and ω limit sets of $\dot{x}_i(t) = f_i(x_i(t), 0, 0)$, $\mathcal{W}_i$ is decomposable in the sense of Definition 3, and the system enjoys the AG property with respect to inputs u_i and d_i as in Definition 4.*

Under this assumption, from Theorem 8, there exist $\mathcal{C}^1$ ISS-Lyapunov functions $V_i : M_i \to \mathbb{R}$ with $\mathcal{K}_\infty$ functions $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}, \gamma_{ui}$ and γ_{di} such that

$$\alpha_{1i}(|x_i|_{\mathcal{W}_i}) \leq V_i(x_i) \leq \alpha_{2i}(|x_i|_{\mathcal{W}_i} + c_i), \; c_i \geq 0, \tag{5}$$
$$DV_i(x_i)f_i(x_i, u_i, d_i) \leq -\alpha_{3i}(|x_i|_{\mathcal{W}_i}) + \gamma_{ui}(|u_i|) + \gamma_{di}(|d_i|)$$

for all $i = 1, \ldots, N$. Define also the invariant set of disconnected and unperturbed ($u_i = d_i = 0$) family $\mathcal{W} = \prod_{i=1}^N \mathcal{W}_i \subset M$ ($0 \in \mathcal{W}$). Then, by definition, there exist functions $\nu_1, \nu_2 \in \mathcal{K}_\infty$ such that

$$\nu_1(|x|_{\mathcal{W}}) \leq \sum_{i=1}^N |x_i|_{\mathcal{W}_i} \leq \nu_2(|x|_{\mathcal{W}}) \tag{6}$$

for all $x \in M$. Since the set $\mathcal{W}$ is compact, then there are functions $\nu_3, \nu_4 \in \mathcal{K}_\infty$ and a scalar $c_0 \geq 0$ such that for all $x \in M$,

$$|x| \leq \nu_3(|x|_{\mathcal{W}}) + c_0, \; |x|_{\mathcal{W}} \leq \nu_4(|x|). \tag{7}$$

Hence, we will consider in this work, the family (4) under Assumption 1, *i.e.* a family of robustly stable nonlinear systems. In general, the sets $\mathcal{W}_i$ include equilibrium (at the origin, for instance) and limit cycles of agents in (4). There are several works devoted to synchronization and design of consensus protocols for such a family or oscillatory network [26], [27], [28]. The goal of our study is to find a condition under which the existence of a global synchronization/consensus protocol for $d = 0$ implies robust synchronization in (4) for a bounded $d \neq 0$.

Let a $\mathcal{C}^1$ function $y(x) : M \to \mathbb{R}^q$, $y(0) = 0$ be a synchronization measure for (4). We say that the family (4) is synchronized (or reached the consensus) if $y(x(t)) \equiv 0$ for all $t \geq 0$ on the solutions of the network under properly designed control actions

$$u_i(t) = \varphi_i[y(x(t))] \tag{8}$$

($\varphi_i : \mathbb{R}^q \to \mathbb{R}^{m_i}$ is a $\mathcal{C}^1$ function, $\varphi_i(0) = 0$) for $d(t) \equiv 0$, $t \geq 0$. In this case the set $\mathcal{A} = \{x \in \mathcal{W} \mid y(x) = 0\}$ contains the synchronous solutions of the unperturbed family in (4) and the problem of synchronization of "natural" trajectories is considered since $\mathcal{A} \subset \mathcal{W}$. Due to the condition $\varphi_i(0) = 0$, the convergence of y (synchronization/consensus) implies that the solutions of the interconnection belong to $\mathcal{W}$, the conditions of convergence of the synchronizing/consensus output y can be found in [26], [27], [28].

The proposed synchronization protocol is output based, as in [26], [27], [28]. The synchronization measure y in general depends on some elements of the vectors x_i for all $i = 1, \ldots, N$. In addition, since y is a vector, then different topology of interconnection can be imposed, see examples in Section IV.

2. *The set $\mathcal{A}$ is compact, it contains all α and ω limit sets of (4), (8) for $d = 0$, and it is decomposable.*

Therefore, it is assumed that the controls $\varphi_i(y)$ ensure the network global synchronization, while decomposability in general follows from Assumption 1. We will show that in the setup as above, by selecting the shapes of φ_i, it is possible to guarantee robust synchronization of (4) for any measurable and essentially bounded input d.

By continuity arguments, there exist functions $\eta_1, \eta_2, \mu_i \in \mathcal{K}_\infty$ with a scalar $\eta_0 \geq 0$ such that for all $x \in M$:

$$|y(x)| \leq \eta_0 + \eta_1(|x|_\mathcal{W}), \; |y(x)| \leq \eta_2(|x|), \tag{9}$$
$$|\varphi_i(y)| \leq \mu_i(|y|)$$

(note that the first two inequalities are related through (7)). Then the intermediate result below can be proven under Assumption 1 for (4), (8).

Proposition 9. *Let Assumption 1 be satisfied for (4). Then there exist φ_i, $i = 1, \ldots, N$ in (8) such that the interconnection (4), (8) has pGS property with respect to the set $\mathcal{W}$.*

Proof. Consider a Lyapunov function candidate $S(x) = \sum_{i=1}^N V_i(x_i)$, where the functions V_i are given in (5). From (6), there exist two functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ and a scalar $g \geq 0$ such that for all $x \in M$:

$$\underline{\alpha}(|x|_\mathcal{W}) \leq S(x) \leq \overline{\alpha}(|x|_\mathcal{W} + g).$$

Taking the derivative of S with respect to equations in (4), (8) we obtain:

$$\dot{S} \leq \sum_{i=1}^N [-\alpha_{3i}(|x_i|_{\mathcal{W}_i}) + \gamma_{ui}(|\varphi_i(y)|) + \gamma_{di}(|d_i|)].$$

From (6) and (9), we deduce:

$$\sum_{i=1}^N \alpha_{3i}(|x_i|_{\mathcal{W}_i}) \geq 2\alpha_4(|x|_\mathcal{W}),$$

$$\sum_{i=1}^N \gamma_{ui}(|\varphi_i(y)|) \leq \sum_{i=1}^N \gamma_{ui} \circ \mu_i(|y|)$$
$$\leq \sum_{i=1}^N \gamma_{ui} \circ \mu_i(\eta_0 + \eta_1(|x|_\mathcal{W}))$$
$$\leq h + \sum_{i=1}^N \gamma_{ui} \circ \mu_i(2\eta_1(|x|_\mathcal{W})),$$

for some $\alpha_4 \in \mathcal{K}_\infty$ and where $h = \sum_{i=1}^N \gamma_{ui} \circ \mu_i(2\eta_0)$. By optimizing the shape of φ_i, it is possible to adjust the form of μ_i. In particular, providing that

$$\mu_i(s) \leq \gamma_{ui}^{-1}\left[N^{-1}\alpha_4 \circ \eta_1^{-1}(0.5s)\right]$$

for all $i = 1, \ldots, N$ we guarantee the relation $\gamma_{ui} \circ \mu_i(2\eta_1(s)) \leq \frac{1}{N}\alpha_4(s)$, then

$$\sum_{i=1}^N \gamma_{ui}(|\varphi_i(y)|) \leq h + \sum_{i=1}^N \frac{1}{N}\alpha_4(s) \leq h + \alpha_4(s).$$

Substituting the obtained terms in the inequality derived for $\dot{S}$, we obtain

$$\dot{S} \leq -\alpha_4(|x|_\mathcal{W}) + h + \gamma_d(|d|),$$

where γ_d is a function from class $\mathcal{K}_\infty$ such that $\sum_{i=1}^N \gamma_{di}(|d_i|) \leq \gamma_d(|d|)$. Finally, $\alpha_4 \circ \overline{\alpha}^{-1}[S(x)] \leq \alpha_4(2|x|_\mathcal{W}) + \alpha_4(2g)$ and

$$\dot{S} \leq -\alpha_4 \circ \overline{\alpha}^{-1}(S) + h + \alpha_4(2g) + \gamma_d(|d|),$$

which by the standard arguments [29] implies that for all $t \geq 0$

$$S(t) \leq \beta(S(0), t) + r + \gamma_d'(\|d\|_\infty)$$

for some function $\beta \in \mathcal{KL}$, $\gamma_d' \in \mathcal{K}$ and a scalar $r \geq 0$. The pGS property follows taking in mind that $\underline{\alpha}(|x(t)|_{\mathcal{W}}) \leq S(t)$, $S(0) \leq [\overline{\alpha}(|x(0)|_{\mathcal{W}} + g)]$ and the properties of a function from the class $\mathcal{KL}$. $\qquad \square$

Note that by definition of the set $\mathcal{A}$, $|x(t)|_{\mathcal{W}} \leq |x(t)|_{\mathcal{A}} \leq |x(t)|_{\mathcal{W}} + z$ for a scalar $z \geq 0$ for all $x \in M$, then the pGS property with respect to the set $\mathcal{A}$ has also been proven.

Therefore, in the setup used in this work the boundedness of trajectories (boundedness of $|x(t)|_{\mathcal{W}}$ implies the same property for $|x(t)|$ according to (7)) follows by a proper selection of the interconnection gain in (8), *i.e.* by decreasing the control gain a certain robustness of (4), (8) is inherited after individual systems as it is stated in Assumption 1.

Theorem 10. *Let assumptions 1 and 2 be satisfied for (4), (8). Then there exist φ_i, $i = 1, \ldots, N$ in (8) such that the interconnection (4), (8) has AG property with respect to $\mathcal{A}$.*

Proof. Since all conditions of Proposition 9 are satisfied, by a proper selection of φ_i, the Lyapunov function S has the properties as in the proof above. From (9) $\alpha_4 \circ \eta_1^{-1}(0.5|y(x)|) \leq \alpha_4 \circ \eta_1^{-1}(\eta_0) + \alpha_4(|x|_{\mathcal{W}})$. Then

$$\dot{S} \leq -0.5\alpha_4(|x|_{\mathcal{W}}) - 0.5\alpha_5(|y(x)|) + h' + \gamma_d(|d|),$$

where $h' = h + 0.5\alpha_4 \circ \eta_1^{-1}(\eta_0)$ and $\alpha_5(s) = \alpha_4 \circ \eta_1^{-1}(0.5s)$. By the definition of the set $\mathcal{A}$, there exists $\theta \in \mathcal{K}_\infty$ such that $\alpha_4(|x|_{\mathcal{W}}) + \alpha_5(|y(x)|) \geq 2\theta(|x|_{\mathcal{A}})$ for all $x \in M$, then

$$\dot{S} \leq -\theta(|x|_{\mathcal{A}}) + h' + \gamma_d(|d|).$$

According to Proposition 9, the solutions are bounded. Hence, the system (4), (8) is forward complete. Following [30], for any forward complete system, there exists a smooth function $Q : M \to \mathbb{R}$ (the proof in [30] deals with Euclidean spaces, but similar arguments can be adopted here) such that for all $x \in M$ and $d \in \mathbb{R}^p$

$$\psi_1(|x|) \leq Q(x) \leq \psi_2(|x|), \quad \dot{Q} \leq 1 + \rho(|d|)$$

for some $\psi_1, \psi_2, \rho \in \mathcal{K}_\infty$. Note that there exists $\nu_5 \in \mathcal{K}_\infty$ such that $|x|_{\mathcal{A}} \leq \nu_5(|x|)$ for all $x \in M$ similarly to (7). Let us introduce a practical ISS Lyapunov function $U(x) = Q(x) + S(x)$ for (4), (8), then for all $x \in M$ and $d \in \mathbb{R}^p$ we have

$$\underline{\alpha}'(|x|_{\mathcal{A}}) \leq U(x) \leq \overline{\alpha}'(|x|_{\mathcal{A}} + g'),$$
$$\dot{U} \leq -\theta(|x|_A) + h' + 1 + \gamma_d(|d|) + \rho(|d|)$$

for properly defined $\underline{\alpha}', \overline{\alpha}' \in \mathcal{K}_\infty$ and a scalar $g' \geq 0$. Thus, U admits all requirements imposed on practical ISS Lyapunov functions, and under Assumption 2 the system (4), (8) possesses all properties in Theorem 8 and it is ISS with respect to $\mathcal{A}$. $\qquad \square$

result states that if the synchronized output y is related with $|x|_{\mathcal{W}}$ as in (9) and each system in the network is robustly stable as in Assumption 1, then the system can be robustly synchronized by a sufficiently small feedback proportional to y.

IV. EXAMPLES AND SIMULATIONS

A. Application to nonlinear pendulums without friction

Consider a network of nonlinear identical pendulums for $i = 1, \ldots, N$, $N > 1$:

$$\begin{aligned} \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= -\omega \sin(x_{1i}) + v_i + d_i, \end{aligned} \tag{10}$$

where the state $x_i = [x_{1i}, x_{2i}]$ takes values on the cylinder $M_i := \mathbb{S} \times \mathbb{R}$, the exogenous disturbance $d_i(t) \in \mathbb{R}$, the regulation input $u_i(t) \in \mathbb{R}$, and ω is a constant positive parameter. The unperturbed system is conservative with Hamiltonian $H(x_i) = 0.5x_{2i}^2 + \omega(1 - \cos(x_{1i}))$ and $\dot{H} = x_{2i}(v_i + d_i)$. The control v_i will have two parts, one to force controlled oscillations in (10) and one for the synchronization u_i:

$$v_i = -x_{2i}[H(x_i) - H^*] + u_i,$$

where $0 < H^* < 2\omega$ is the desired level of $H(x_i)$ that defines the attracting limit cycle $\Gamma_i = \{x \in M_i : H(x_i) = H^*\}$ in

$$\begin{aligned} \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= -\omega \sin(x_{1i}) - x_{2i}[H(x_i) - H^*] + u_i + d_i. \end{aligned} \tag{11}$$

Despite of the limit cycle Γ_i, each unperturbed system admits also two equilibria $[0, 0]$ and $[\pi, 0]$, the latter being a saddle point. Thus $\mathcal{W}_i = \{[0, 0] \cup [\pi, 0] \cup \Gamma_i\}$. Clearly, $\mathcal{W}_i$ is compact and contains all α and ω limit sets of (11) for $u_i = d_i = 0$. Moreover, it is straightforward to check that $\mathcal{W}_i$ is decomposable in the sense of Definition 3.

Lemma 11. *For each $i = 1, \ldots, N$, the systems in (11) have AG property.*

Proof. The conditions of Theorem 8 are satisfied for the system (11) and $\mathcal{W}_i$, thus it is enough to check a practical AG in this case. First, $|x_{1i}(t)| \leq \pi$ for all $t \geq 0$ by definition, and it is necessary to show a pAG for the coordinate x_{2i} only. For this purpose, we consider $W(x_{2i}) = 0.5x_{2i}^2$. Hence:

$$
\begin{aligned}
\dot{W} &= x_{2i}[-\omega\sin(x_{1i}) - x_{2i}[H(x_i) - H^*] + u_i + d_i] \\
&= x_{2i}[-\omega\sin(x_{1i}) - x_{2i}[0.5x_{2i}^2 + \omega(1 - \cos(x_{1i})) \\
&\quad -H^*] + u_i + d_i] \\
&\leq \omega|x_{2i}| - x_{2i}^2[0.5x_{2i}^2 + \omega(1 - \cos(x_{1i})) \\
&\quad -H^*] + 0.5x_{2i}^2 + 0.5(u_i + d_i)^2 \\
&\leq -0.5x_{2i}^4 + (0.5 + H^* + 2\omega)x_{2i}^2 + \omega|x_{2i}| \\
&\quad +0.5(u_i + d_i)^2.
\end{aligned}
$$

Since $0.5 + H^* + 2\omega > 0$ and $\omega > 0$, there exists $f_{max} > 0$ such that $-0.25x_{2i}^4 + (0.5 + H^* + 2\omega)x_{2i}^2 + \omega|x_{2i}| \leq f_{max}$ for all $x_{2i} \in \mathbb{R}$, then

$$
\begin{aligned}
\dot{W} &\leq -0.25x_{2i}^4 + f_{max} + 0.5(u_i + d_i)^2 \\
&\leq -W^2 + f_{max} + 0.5(u_i + d_i)^2.
\end{aligned}
$$

Next, applying standard arguments:

$$\limsup_{t \to +\infty} W(t) \leq \sqrt{f_{max} + 0.5(\|u_i\|_\infty + \|d_i\|_\infty)^2} \text{ and}$$

$$\limsup_{t \to +\infty} |x_{2i}(t)|^2 \leq 2\sqrt{f_{max} + 0.5(\|u_i\|_\infty + \|d_i\|_\infty)^2}.$$

Take $|x_i| = \sqrt{x_{1i}^2 + x_{2i}^2}$ then

$$\limsup_{t \to +\infty} |x_i(t)| \leq \sqrt{\pi^2 + 2\sqrt{f_{max} + 0.5(\|u_i\|_\infty + \|d_i\|_\infty)^2}}$$

and the pAG property holds since $|x_i|_{\mathcal{W}_i} \leq |x_i|$. $\qquad\square$

As a consequence, Assumption 1 is satisfied for (11) and we may select the synchronization measure y for the network. The synchronization problem for nonlinear pendulums has been widely considered previously [31], [32], [27] (usually for unperturbed systems without a limit cycle, for example, with $v_i = -\kappa x_{2i} + u_i$ for some $\kappa > 0$). In this work we will consider

$$y = Ax_2,$$

where $x_2 = [x_{21}, \ldots, x_{2N}]^T$ and $A \in \mathbb{R}^{N \times N}$ is a Metzler matrix whose off-diagonal elements $A_{i,j} \in \{0,1\}$ for all $1 \leq i \neq j \leq N$ and $\sum_{j=1}^N A_{ij} = 0$, $\sum_{j=1}^N |A_{ij}| \neq 0$ for all $i = 1, \ldots, N$ (for example, $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ for $N = 2$). It is necessary to check (9) for this y: obviously the function η_2 exists. To evaluate the constant η_0 and the function η_1 it is necessary to calculate $|x_i|_{\mathcal{W}_i}$ (and $|x_i|_{\mathcal{W}}$). Note that $|y|^2 \leq \|A\|_2 \sum_{j=1}^N x_{2i}^2$, then it is enough to estimate a relation between x_{2i} and $|x_i|_{\mathcal{W}_i}$. There exist $\delta_1, \delta_2 \in \mathcal{K}_\infty$ such that for all $x \in M_i$

$$\delta_1(|x_i|_{\mathcal{W}_i}) \leq \Delta(x_i) \leq \delta_2(|x_i|_{\mathcal{W}_i})$$

where $\Delta(x_i) = \min\{\sin^2(x_{1i}) + 0.5x_{2i}^2, |H(x_i) - H^*|\}$. Then it is enough to establish the boundedness of x_{2i} by $\Delta(x_i)$, but a direct computation shows:

$$0.5x_{2i}^2 \leq \Delta(x_i) + H^*$$

and (9) is valid for y. Take

$$\varphi_i(y) = \epsilon\tanh(y_i), \ \epsilon > 0,$$

then we may suppose that Assumption 2 is satisfied for some sufficiently small ϵ. The results of simulations confirm this conclusion, see for example Fig. 1, where for $N = 4$ and

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & 0 & -2 \end{bmatrix}, \ \omega = 2, \ H^* = 2, \ \epsilon = 0.1$$

the results for two scenarios are given: Fig. 1,a without disturbances and Fig. 1,b with disturbances as $[d_1, d_2, d_3, d_4]^T = [0.7\sin(2t), -0.25\sin($
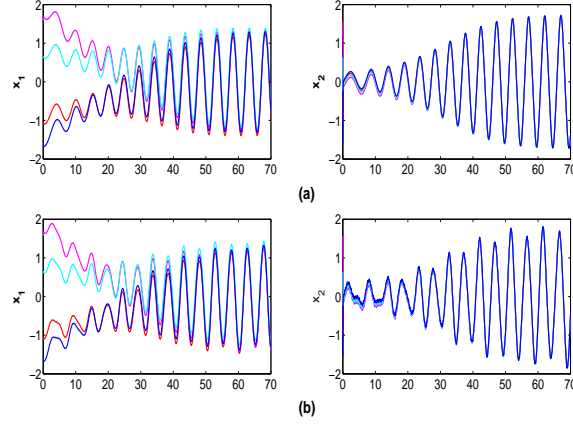
Figure 1. The results of simulation for (11)

B. Application to nonlinear pendulums with friction

This example is taken from [27]. Consider a network of nonlinear non-identical pendulums for $i = 1, \ldots, N$, $N > 1$:

$$
\begin{aligned}
\dot{x}_{1i} &= x_{2i}, \\
\dot{x}_{2i} &= -\Omega_i^2 \sin(x_{1i}) - \kappa x_{2i} + d_i,
\end{aligned} \tag{12}
$$

where the state $x_i = [x_{1i}, x_{2i}]$ takes values on the cylinder $M_i := \mathbb{S} \times \mathbb{R}$, the exogenous disturbance $d_i(t) \in \mathbb{R}$, κ is a constant parameter and Ω_i^2 is the angular frequency of individual pendulums. The unperturbed system has a Hamiltonian $H(x_i) = 0.5 x_{2i}^2 + \Omega_i^2 (1 - \cos(x_{1i}))$ and $\dot{H} = x_{2i} d_i - \kappa x_{2i}^2$. Each unperturbed system has two equilibria $[0, 0]$ and $[\pi, 0]$ (the former is attractive and the later one is a saddle-point), thus $\mathcal{W}_i = \{[0, 0] \cup [\pi, 0]\}$ is a compact set containing all α- and ω-limit sets of (12) for $d_i = 0$. In addition, it is easy to check that $\mathcal{W}_i$ is decomposable in the sense of Definition 3 [33].

Lemma 12. *[34] For each $i = 1, \ldots, N$, the systems in (12) is ISS with respect to the set $\mathcal{W}_i$.*

As a consequence, Assumption 1 is satisfied for (12) (remark that admitting an ISS Lyapunov function is equivalent to enjoying AG property by Theorem 8) and we may select the synchronization measure y for the network. Since in [27], the authors have considered the first coordinate as synchronization measure, we follow here the same idea:

$$
y = A \sin(x_1),
$$

where $x_1 = [x_{11}, \ldots, x_{1N}]^T$ and $A \in \mathbb{R}^{N \times N}$ is a Metzler matrix as in the first example.

Since the global boundedness of trajectories of (12) for bounded inputs is proven in Lemma 12, then a local analysis around equilibria is sufficient to show the synchronization measure convergence. It is straightforward to check that linearized around equilibria dynamics has $y = 0$ as a stable and attractive manifold. By this, the convergence of y is guaranteed locally. Then by taking,

$$
\varphi_i(y) = \beta y_i, \beta > 0,
$$

we may suppose that Assumption 2 is satisfied for some sufficiently small β. The results of simulations confirm this conclusion, see Fig. 2 where a) is the disturbance free case and b) represents the simulation result with disturbances. The simulation parameters are $N = 5$, $\Omega_i^2 = 0.02i$, $\beta = 0.1$, the disturbance inputs are $[\phi_1, \ldots, \phi_5]^T = [0.1 \sin(t), -0.15 \sin(t), -0.2 \sin(t), 0.15 \sin(t), 0.2 \sin(t)]^T$ and

$$
A = \begin{bmatrix}
-3 & 1 & 1 & 0 & 1 \\
1 & -3 & 1 & 1 & 0 \\
1 & 1 & -3 & 1 & 0 \\
0 & 1 & 1 & -3 & 1 \\
1 & 0 & 0 & 1 & -2
\end{bmatrix}.
$$

V. Conclusions

In this work, sufficient conditions for robust synchronization were derived based on an extension of the ISS framework to systems evolving on a (non-compact) manifold and with multiple invariant sets. The condition imposed on the controller ($\varphi_i(0) = 0$) ensures that the convergence of the synchronization measure implies that the interconnection belongs to the decomposable set $\mathcal{W}$. Practical global stability analysis of the interconnection was done with respect to $\mathcal{W}$. The asymptotic gain property of the interconnection with respect to the set of synchronous solutions $\mathcal{A}$ ($\mathcal{A} \subset \mathcal{W}$) was also proved.
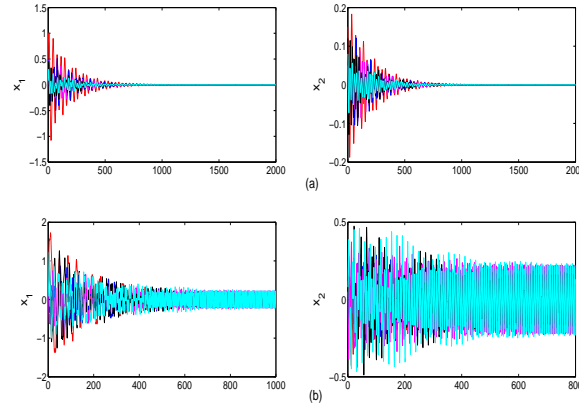
Figure 2. The result of simulation for (12)

Numerical simulations demonstrated the effectiveness of our method to network of both identical and nonidentical nodes. Remark that our results are applicable only to systems that allow decomposition without cycles.

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