The Binomial Spin Glass

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To establish a unified framework for studying both discrete and continuous coupling distributions, we introduce the *binomial* spin glass, a class of models where the couplings are sums of m identically distributed Bernoulli random variables. In the continuum limit $m \to \infty$, the class reduces to one with Gaussian couplings, while $m = 1$ corresponds to the $\pm J$ spin glass. We demonstrate that for short-range Ising models on d-dimensional hypercubic lattices the ground-state entropy density for N spins is bounded from above by $(\sqrt{d/2m} + 1/N)\ln 2$, and further show that the actual entropies follow the scaling behavior implied by this bound. We thus uncover a fundamental non-commutativity of the thermodynamic and continuous coupling limits that leads to the presence or absence of degeneracies depending on the precise way the limits are taken. Exact calculations of defect energies reveal a crossover length scale $L^*(m) \sim L^\kappa$ below which the binomial spin glass is indistinguishable from the Gaussian system. Since $\kappa = -1/(2\theta)$, where θ is the spin-stiffness exponent, discrete couplings become irrelevant at large scales for systems with a finite-temperature spin-glass phase.

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Spin glasses are extremely rich systems that have continued to surprise for many decades [1–13]. They represent paradigmatic realizations of complexity that are abundant in nature and numerous combinatorial optimization problems [14]. Abstractions of spin-glass physics have led to new optimization algorithms and new insight into computational complexity [15–18], shed light on protein folding [19], and provided models of neural networks [20]. Notwithstanding this success, several fundamental questions still linger. These include [21] the character of the low-lying states and whether there are many incongruent [22] ground states. It has long been known that spin-glass systems with discrete couplings may rigorously exhibit an extensive degeneracy [23, 24], but these results do not extend to continuous coupling distributions [25–29]. The possibility of vanishing spectral gaps mandates the distinction of localized and extended excitations, and only the latter can give rise to a multitude of states.

In this paper, we connect the $\pm J$ and the Gaussian spin glass models by interpolating them via the *binomial* spin glass that has a tunable control parameter m. We establish bounds of the spectral degeneracy of the Ising system on bipartite graphs, which includes the usual Edwards-Anderson (EA) model with $\pm J$ ($m = 1$) and Gaussian ($m \to \infty$) couplings [10, 30–43]. We thus show that discrete (finite m) spin-glass samples exhibit an extensive ground-state degeneracy, while continuous ones ($m \to \infty$) become two-fold degenerate, while more generally the degeneracy depends on the precise way the non-commuting limits $N \to \infty$ and $m \to \infty$ are taken.

We define the *binomial Ising spin glass* on a graph of N sites [44] by the Hamiltonian

$$H_m = -\sum_{\langle \mathsf{xy} \rangle} \mathcal{J}^m_{\mathsf{xy}} s_{\mathsf{x}} s_{\mathsf{y}} \equiv -\sum_{\alpha=1}^{\mathcal{L}} \mathcal{J}^m_\alpha z_\alpha. \qquad (1)$$

Here, the sum is over sites x and y, defining a link $\alpha = \langle \mathsf{xy} \rangle$, $\mathcal{L}$ denotes the total number of links, and $s_{\mathsf{x}} = \pm 1$. The *binomial coupling* for each link α, $\mathcal{J}^m_\alpha \equiv \frac{1}{\sqrt{m}} \sum_{k=1}^m J^{(k)}_\alpha$, is a sum of m copies (or "layers") of binary couplings $J^{(k)}_\alpha = \pm 1$, each with probability p of being $+1$. The probability distribution of $\mathcal{J}^m_\alpha$,

$$\tilde{P}(\mathcal{J}^m_\alpha) = \sum_{j=0}^{m} \binom{m}{j} p^{m-j} (1-p)^j \,\delta\!\left(\mathcal{J}^m_\alpha - \frac{m-2j}{\sqrt{m}}\right), \qquad (2)$$

is a binomial. In the large-m limit, the distribution (2) approaches a Gaussian of mean $\sqrt{m}(2p-1)$ and variance $\sigma^2 = 4p(1-p)$. In particular, for $p = 1/2$, the distribution $\tilde{P}(\mathcal{J}^m_\alpha)$ approaches the standard normal distribution usually considered for the EA model [10].

To understand the degeneracies in the spectrum, we study the entropy density of the ℓ-th energy level,

$$\mathcal{S}_\ell \equiv \frac{\sum_{\{\mathcal{J}^m_\alpha\}} P(\{\mathcal{J}^m_\alpha\}) \ln D_\ell(\{\mathcal{J}^m_\alpha\})}{N}, \qquad (3)$$

where D_ℓ is the degeneracy of the ℓ-th energy level [23]. $P(\{\mathcal{J}^m_\alpha\}) = \prod_{\alpha=1}^{\mathcal{L}} \tilde{P}(\mathcal{J}^m_\alpha)$ is the probability of the coupling configuration.

We first embark on the derivation of an upper bound on the ground state entropy density $\mathcal{S}_0$. We restrict ourselves to bipartite graphs, where any closed loop encompasses an even number of links α. Consider two spin

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configurations $|\mathsf{s}\rangle \neq |\mathsf{s}'\rangle$ and evaluate their energy difference $\Delta E = E(\mathsf{s}) - E(\mathsf{s}')$. From Eq. (1),

$$\Delta E = -\sum_{\alpha=1}^{\mathcal{L}} \mathcal{J}_\alpha^m \Big(z_\alpha(\mathsf{s}) - z_\alpha(\mathsf{s}') \Big) = -2 \sum_{\alpha=1}^{\mathcal{L}} \mathcal{J}_\alpha^m n_\alpha, \quad (4)$$

with integers $n_\alpha = 0, \pm 1$ defined by $n_\alpha \equiv [z_\alpha(\mathsf{s}) - z_\alpha(\mathsf{s}')]/2$, where $z_\alpha(\mathsf{s}) = s_\mathsf{x} s_\mathsf{y}$. If $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ are degenerate then $\Delta E = 0$. A degeneracy only occurs for some realizations $\{\mathcal{J}_\alpha^m\}$ of the couplings, and Eq. (4) can be understood as a set of conditions for the couplings to ensure this.

Consider an arbitrary reference configuration $|\mathsf{s}\rangle$ of energy $E(\mathsf{s})$ and examine its viable degeneracy with the contending $2^N - 1$ other configurations $|\mathsf{s}'\rangle$. Each of these leads to a particular set of integers $\mathsf{C}_j = \{n_\alpha\}_j$, which form the set $\{\mathsf{C}_j\}_{j=1,2^N-1}^{|\mathsf{s}\rangle}$. A subset of those, $\mathsf{Sat}_{|\mathsf{s}\rangle} = \{\mathsf{C}_{j_1}, \mathsf{C}_{j_2}, \cdots, \mathsf{C}_{j_\mathcal{N}}\}$, will satisfy the degeneracy condition $\Delta E = 0$ in Eq. (4) for some coupling realizations. There are two types of solutions to the equation $\Delta E = 0$: (i) $n_\alpha = 0, \forall \alpha$, or (ii) $n_\alpha \neq 0$, for at least one link α. It is straightforward to demonstrate that there is a single configuration $|\mathsf{s}'\rangle (\neq |\mathsf{s}\rangle)$ for which (i) $n_\alpha = 0, \forall \alpha$ [45]. This is the degenerate configuration $|\mathsf{s}'\rangle$ obtained by inverting all of the spins in $|\mathsf{s}\rangle$. To determine whether the degeneracy may be larger than two, we need to compute the probability $\mathcal{P}$ that constraints of type (ii) may be satisfied. While we cannot exactly calculate this probability for general N and m, bounds that we will derive suggest that $\lim_{N\to\infty} \lim_{m\to\infty} \mathcal{S}_\ell = 0$. As we will emphasize, different large m and N limits may yield incompatible results.

Constraints $\mathsf{C}_j \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ are in *a one-to-one correspondence* with zero-energy interfaces [46], whose *size* is equal to the number g_j of non-zero integers in the set $\{n_\alpha\}_j$. That is, given a fixed reference configuration $|\mathsf{s}\rangle$ and a degenerate one $|\mathsf{s}'\rangle$, all type (ii) solutions to Eq. (4) are associated with configurations where the product $s_\mathsf{x} s'_\mathsf{x}$ is equal to -1 in a non-empty set of sites $\mathsf{x} \in R$. To avoid the trivial redundancy due to global spin inversion, consider the states $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ for which the spin at an arbitrarily chosen "origin" of the lattice assumes the value $+1$. These states are related via $|\mathsf{s}'\rangle = U_{\mathsf{s}'\mathsf{s}}|\mathsf{s}\rangle$, where the *domain-wall operator* $U_{\mathsf{s}'\mathsf{s}}$ is the product of Pauli matrices that flip the sign of the spins s'_x at the sites x where $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ differ. Regions R are bounded by zero-energy domain walls that are interfaces dual to the links with $n_\alpha = \pm 1$, i.e., surrounding the areas R where the spins in $|\mathsf{s}\rangle$ an $|\mathsf{s}'\rangle$ have opposite orientation. Each satisfied constraint $\mathsf{C}_j \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ is associated with a state $|\mathsf{s}'\rangle = U_{\mathsf{s}'\mathsf{s}}|\mathsf{s}\rangle$ that is degenerate with $|\mathsf{s}\rangle$ for some coupling realization(s).

We next formalize the counting of *independent domain walls* or clusters of free spins to arrive at an asymptotic bound on their number [Eq. (9)]. This will, in turn, provide a bound on the degeneracy. We define a *complete* set of *independent* constraints $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle} \subset \mathsf{Sat}_{|\mathsf{s}\rangle}$, of cardinality $\mathcal{M}$, to be composed of *all* constraints $\mathsf{C}_{\bar{\jmath}} \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ that lead to *linearly independent equations* of the form of Eq. (4), $\Delta E = E(\mathsf{s}) - E(\mathsf{s}_{\bar{\jmath}}) = 0$, on the coupling constants $\{J_\alpha^m\}$ [46]. All constraints in $\mathsf{Sat}_{|\mathsf{s}\rangle}$ are a consequence of the linearly independent subset of constraints $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$. Each constraint $\mathsf{C}_{\bar{\jmath}} \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ is associated with a domain wall operator $U_{\mathsf{s}_{\bar{\jmath}}\mathsf{s}}$ that generates a degenerate state $|\mathsf{s}_{\bar{\jmath}}\rangle = U_{\mathsf{s}_{\bar{\jmath}}\mathsf{s}}|\mathsf{s}\rangle$. If for a given coupling realization $\{\mathcal{J}_\alpha^m\}$ there are $M(\{\mathcal{J}_\alpha^m\}) \leq \mathcal{M}$ such independently satisfied constraints, then the states

$$|\bar{n}_1 \bar{n}_2 \cdots \bar{n}_M\rangle \equiv U_{\mathsf{s}_{\bar{1}}\mathsf{s}}^{\bar{n}_1} U_{\mathsf{s}_{\bar{2}}\mathsf{s}}^{\bar{n}_2} \cdots U_{\mathsf{s}_{\bar{M}}\mathsf{s}}^{\bar{n}_M} |\mathsf{s}\rangle, \quad (5)$$

($\bar{n}_i = 0, 1$) will include all of the spin configurations degenerate with $|\mathsf{s}\rangle$. Taking global spin inversion into account, the degeneracy of $|\mathsf{s}\rangle$ is

$$D_{\ell(|\mathsf{s}\rangle, \{\mathcal{J}_\alpha^m\})} \leq 2^{M(\{\mathcal{J}_\alpha^m\})+1}, \quad (6)$$

where, for a system defined by the coupling constants $\{\mathcal{J}_\alpha^m\}$, the index $\ell(|\mathsf{s}\rangle, \{\mathcal{J}_\alpha^m\}))$ denotes the level ℓ the state $|\mathsf{s}\rangle$ belongs to. The set $\{|\bar{n}_1 \bar{n}_2 \cdots \bar{n}_M\rangle\}$ may contain additional states not degenerate with $|\mathsf{s}\rangle$ [47].

After averaging over disorder, the expected number of the linearly independent satisfied constraints $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$ is

$$\langle M \rangle_m \equiv \sum_{\{\mathcal{J}_\alpha^m\}} \sum_{\mathsf{C}_{\bar{\jmath}} \in \overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}} P(\{\mathcal{J}_\alpha^m\}) \delta^{\{\mathcal{J}_\alpha^m\}}(\mathsf{C}_{\bar{\jmath}}) \equiv \sum_{\mathsf{C}_{\bar{\jmath}} \in \overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}} \mathcal{P}(\mathsf{C}_{\bar{\jmath}}). \quad (7)$$

Here, $\mathcal{P}(\mathsf{C}_{\bar{\jmath}})$ is the probability that a linearly independent constraint $\mathsf{C}_{\bar{\jmath}}$ is satisfied. The Kronecker $\delta^{\{\mathcal{J}_\alpha^m\}}(\mathsf{C}_{\bar{\jmath}})$ equals 1 if $\mathsf{C}_{\bar{\jmath}}$ is satisfied for the couplings $\{\mathcal{J}_\alpha^m\}$ and is zero otherwise. Let us bound the probability $\mathcal{P}(\mathsf{C}_{\bar{\jmath}})$ by taking the form (2) of the coupling distribution into account. From the definition of the couplings $\{\mathcal{J}_\alpha^m\}$, the sum in Eq. (4) can effectively be read as including a sum over layers $k = 1, \ldots, m$, which hence includes $g_{\bar{\jmath}} m$ nonzero terms. For general $m \geq 1$, and even $g_{\bar{\jmath}} m$, the probability that half of the nonzero integers $n_\alpha J_\alpha^{(k)}$ in Eq. (4) are $+1$ and the remainder are -1 is

$$\mathcal{P}(\mathsf{C}_{\bar{\jmath}}) = \binom{g_{\bar{\jmath}} m}{\frac{g_{\bar{\jmath}} m}{2}} \frac{1}{2^{g_{\bar{\jmath}} m}} < \frac{1}{\sqrt{g_{\bar{\jmath}} m}}. \quad (8)$$

(Eq. (4) cannot be satisfied for odd $g_{\bar{\jmath}} m$.) From asymptotic analysis [48] and Eq. (8), the probability $\mathcal{P}(\mathsf{C}_{\bar{\jmath}})$ scales (for large m) as (and, for any m, is bounded by) $1/\sqrt{g_{\bar{\jmath}} m}$. Denoting by $g_{\min}$ the smallest possible value of $g_{\bar{\jmath}}$ for the graph/lattice at hand,

$$\langle M \rangle_m \leq \frac{\mathcal{M}}{\sqrt{g_{\min} m}}. \quad (9)$$

On a general graph, the number $\mathcal{M}$ of linearly *independent* constraints $\mathsf{C}_{\bar{\jmath}}$ on the coupling constants $\{\mathcal{J}_\alpha^m\}$ cannot be larger than their total number, $\mathcal{M} \leq \mathcal{L}$, i.e., the number of links $\mathcal{L}$ on this graph. Putting all of the pieces together, Eqs. (6) and (9) imply

$$\sum_{\{\mathcal{J}_\alpha^m\}} P(\{\mathcal{J}_\alpha^m\}) \ln D_{\ell(|\mathsf{s}\rangle, \{\mathcal{J}_\alpha^m\})} \leq (1 + \frac{\mathcal{L}}{\sqrt{g_{\min} m}}) \ln 2. \quad (10)$$

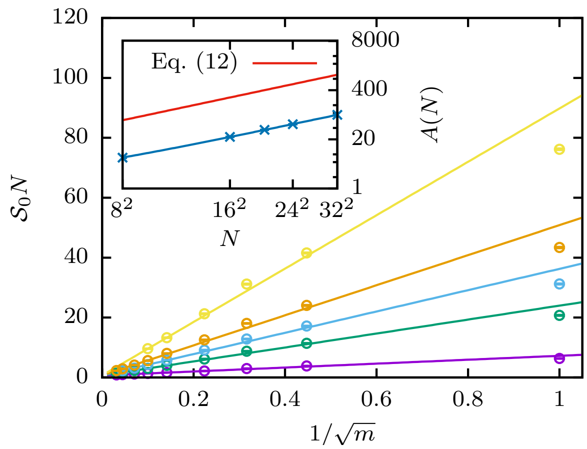

FIG. 1. Ground-state entropy $\mathcal{S}_0 N$ of the binomial Ising spin glass with m layers, cf. Eq. (1), on square lattices of $N = L^2$ spins from exact ground-state calculations (from the bottom: $L = 8$, 16, 20, 24, and 32). Lines are fits of the form of (13) to the data for sufficiently large m. The inset shows the linear scaling of the amplitude $A(N)$. The top line indicates the constraint imposed by the upper bound (11).

Trying to evaluate the l.h.s. of Eq. (10) we must take into account that whatever $|\mathsf{s}\rangle$ we pick might be a ground state for some coupling configurations, but will be an excited state for others. Hence we cannot directly infer a bound to the average entropy $\mathcal{S}_\ell$ from (10). Since the inverse temperature $1/(k_B T) = \partial \ln D/\partial E$, however, the system's ground-state degeneracy for couplings $\{\mathcal{J}_\alpha^m\}$ is typically lower than (or equal to) that of any other level ℓ [49], i.e., $D_0 \leq D_\ell$. This monotonicity of $D(E)$ implies that, typically, $\mathcal{S}_0 N = \sum_{\{\mathcal{J}_\alpha^m\}} P(\{\mathcal{J}_\alpha^m\}) \ln D_0(\{\mathcal{J}_\alpha^m\}) \leq \sum_{\{\mathcal{J}_\alpha^m\}} P(\{\mathcal{J}_\alpha^m\}) \ln D_{\ell(|\mathsf{s}\rangle,\{\mathcal{J}_\alpha^m\})}$. Then, Eq. (10) yields

$$\mathcal{S}_0 \leq \left(\frac{\mathcal{L}}{N\sqrt{g_{\min} m}} + \frac{1}{N}\right) \ln 2. \tag{11}$$

This is the promised rigorous bound. For $p \neq 1/2$ one has a lower entropy density than that of $p = 1/2$. Thus, Eq. (11) constitutes a generous upper bound on $\mathcal{S}_0$ for general p. To study higher energy levels, consider the average of Eq. (10) over all possible 2^N reference spin configurations $|\mathsf{s}\rangle$. Performing this average and invoking the monotonicity of $D(E)$ suggests that the entropy density $\mathcal{S}_\ell$ of Eq. (3) of low-lying excited levels $\ell > 0$ is, typically, also bounded by the r.h.s of Eq. (11). For d-dimensional hypercubic lattices with periodic boundary conditions, the ratio $\mathcal{L}/N = d$ while $g_{\min} = 2d$. Thus, $\mathcal{S}_0 \leq (\sqrt{d/2m} + 1/N) \ln 2$. Eq. (11) further suggests that, in the thermodynamic ($N \to \infty$) limit [50],

$$\mathcal{S}_0(m') \sim \sqrt{\frac{m}{m'}}\, \mathcal{S}_0(m) \quad \text{for finite} \quad m, m' \gg 1. \tag{12}$$

We now study the exact m dependence of the ground state entropies of the binomial model on the square lattice with periodic boundaries and $N = L^2$. To this end, we employed an implementation of the Pfaffian technique of counting dimer coverings of the lattice as discussed in Ref. [51], which is a generalization of earlier methods [52, 53] to fully periodic lattices. In Fig. 1, we present the results for the ground-state entropy, averaged over 1000 coupling realizations for each lattice size. The data are well described by

$$\mathcal{S}_0 N = \left(\frac{A(N)}{\sqrt{m}} + 1\right) \ln 2. \tag{13}$$

Linear fits in $1/\sqrt{m}$ for fixed N work well for sufficiently large m, as is illustrated by the straight lines in Fig. 1. Thus, for any finite N, as $m \to \infty$ the ground-state entropy is equal to $\ln 2$, implying a single degenerate ground-state pair. The slope $A(N)$ shown in the inset follows a linear behavior, $A(N) = aN + b$, and we find $a = 0.0858(4)$ and $b = 1.09(12)$. For not too small m, our data are hence fully consistent with

$$\mathcal{S}_0 = \left(\frac{a}{\sqrt{m}} + \frac{1}{N} + \frac{b}{N\sqrt{m}}\right) \ln 2. \tag{14}$$

When $N \gg \sqrt{m} \gg 1$, Eq. (14) is consistent with the physically inspired [50] scaling of Eq. (12). For large N, the bound of Eq. (11) would have been asymptotically saturated if $a \simeq 1$, far larger than the actual value of a. The behavior in the double limit $m, N \to \infty$ is subtle: (1) for $m \to \infty$, N finite, we have a single ground-state pair; (2) for $N \to \infty$, m finite, there is a finite ground-state entropy $\sim \ln 2/\sqrt{m}$; (3) for $N \to \infty$, $m \to \infty$, $\kappa = N/\sqrt{m}$ fixed, there is a finite number $2^{a\kappa}$ of ground-state pairs. Thus clearly the continuum and thermodynamic limits are not commutative in general. Note further that according to the bound $\mathcal{S}_0 \leq (\sqrt{d/2m} + 1/N) \ln 2$ for hypercubic lattices additional rich behavior is expected if the limit of high dimensions is correlated with that of large m.

Let us turn to the study of excitations. By construction, cf. Eq. (4), for finite m the energy is "quantized" in multiples of $1/\sqrt{m}$. It is therefore natural to expect a closing of the spectral gap as $m \to \infty$. That this is indeed the case can be shown rigorously for the one-dimensional binomial spin glass in its thermodynamic limit, with different behaviors for odd and even m, see the discussion in the Supplemental Material [54]. The closing of the gap is a consequence of the existence of (rare) local excitations, i.e., finite-size clusters of almost free spins [55]. Whether gapless *non-local* excitations exist and which form they take in the thermodynamic limit is a long-standing question [56]. One possible approach of investigating such excitations consists of subjecting individual samples to a system spanning perturbation by a change of boundary condition and studying how this affects the energy and configuration of the ground state. Such defect energy calculations [57] enable us to extract a scaling $\langle|\Delta E|\rangle \sim L^\theta$ of the defect energies with the spin stiffness exponent θ. Generalizing Peierls' argument [58–61] for the stability of the ordered phase, one should

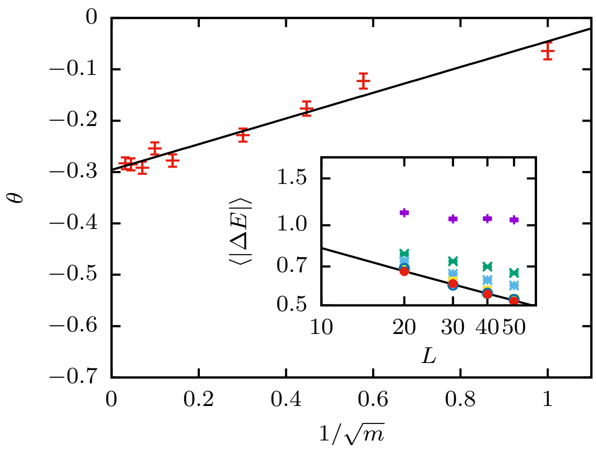

FIG. 2. Effective spin stiffness exponents $\theta = \theta(m)$ resulting from fits of the power law $\langle|\Delta E|\rangle = BL^{\theta}$ to the defect energies for the binomial model of m layers (inset, from the top: $m = 1$, 5, 11, 51, 201, and 1001), averaged over 10 000 disorder samples. The solid line of the inset corresponds to the Gaussian model.

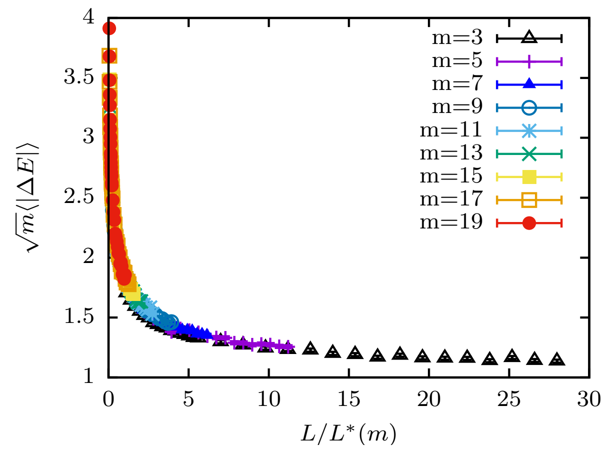

FIG. 3. Scaling collapse of the defect energies of the binomial model for system sizes rescaled with the crossover length scale $L^*(m) \sim m^{\kappa}$ with $\kappa = 1.79$.

find $\theta > 0$ for cases where there is a finite-temperature spin-glass phase, and $\theta \leq 0$ otherwise. The latter case is expected for dimensions $d = 1$ and $d = 2$, whereas θ is positive for $d \geq 3$ [62, 63]. We employed techniques based on minimum-weight perfect matching [64, 65] to perform such calculations for the binomial model on the square lattice. The resulting disorder-averaged defect energies from *exact* ground-state calculations for samples with periodic and antiperiodic boundaries are shown in the inset of Fig. 2. As m increases, the decay of defect energies as a function of L becomes steeper and the data approach the behavior of the Gaussian EA model. The effective spin stiffness exponents θ extracted from fits of the type $\langle|\Delta E|\rangle = BL^{\theta}$ are shown in the main panel of Fig. 2. These exponents appear to interpolate smoothly between the limiting cases of the Gaussian model with $\theta = -0.2793(3)$ and the $\pm J$ system with $\theta = 0$ [62, 65]. Asymptotically, however, we expect that $\theta(m) = 0$ for any finite value of m when $L \gtrsim L^*(m)$. The scaling of the crossover length $L^*(m) \sim m^{\kappa}$ follows by considering the model with the unscaled couplings $\sqrt{m}\mathcal{J}_{\alpha}^{m}$, for which the energy gap Δ is independent of m. The discreteness of the spectrum becomes apparent once the corresponding defect energies $\sqrt{m}\langle|\Delta E|\rangle \sim L^{\theta}$ have decayed below the size of the gap, i.e., for

$$L \geq L^*(m) \sim m^{-1/(2\theta)},$$

such that $\kappa = -1/(2\theta)$. For the $d = 2$ system we have $\theta = -0.2793(3)$ [65], such that $\kappa = 1.790(2)$, which is in excellent agreement with the actual defect energies for our system shown in Fig. 3.

It is clear that if $\theta < 0$, as is the case for the Gaussian spin glass in two dimensions, excitations of a divergent length scale may entail a vanishing energy penalty. At zero temperature, the discreteness of the spectrum is then always seen at large scales $L \gtrsim L^*(m)$. On the other hand, for $\theta \geq 0$ (i.e., $d \geq 3$), the above arguments imply that the discreteness does not matter at large scales. Also, in this case one should inspect the full probability distribution of domain wall energies and the weight it carries in the limit $\Delta E \to 0$ [55]. In how far such excitations correspond to incongruent states, however, one might only be able to infer by inspecting the configurations themselves.

In summary, we introduced and discussed the *binomial spin glass*. This class of models affords controlled access to the enigmatic continuous ($m \to \infty$) finite dimensional EA model. Its $m = 1$ realization is the quintessential discrete spin glass, the $\pm J$ model. We derived bounds on the spectral degeneracy of the binomial Ising spin glass on general graphs and suggested an asymptotic scaling that is fully supported by exact two-dimensional calculations. The behavior of defect energies suggests the existence of a crossover length $L^*(m) \sim L^{-1/2\theta}$ below which the binomial model behaves like the Gaussian system. Our results show that the existence of degeneracies depends on the particular way of taking the thermodynamic ($N \to \infty$) and continuous coupling ($m \to \infty$) limits, and limiting states with and without degeneracies can be reached by corresponding correlated limiting processes, thus accommodating theories that postulate degeneracies as well as pictures stipulating a unique ground-state pair. An intriguing prediction regards an effectively negative crossover scaling exponent in three dimensions, where hence discreteness of the spectrum is expected not to matter at large scales.

The physics of spin-glass models and, in particular, the role of degeneracies has also recently attracted attention from another side. In the context of quantum annealing [66] as implemented in the devices by D-Wave and similar machines that are being developed by competing consortia, degeneracies are not a desired feature as the quantum annealing process does not sample such states uniformly [67]. On the other hand, continuous coupling distributions may also be undesired because of increased susceptibility to external noise implied by chaos in spin glasses [68–71]. Our binomial glasses may allow for realizations that suffer the least from these combined problems. While the present system is already a generalization of the usually considered spin-glass models, we believe that the approach of decomposing continuous couplings into discrete layers and the intriguing consequences it allowed us to uncover in terms of the general non-commutativity of the thermodynamic and continuous coupling limits is promising and we expect exciting applications to models in other fields.

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[48] Notice that one can write the asymptotic form

$$\mathcal{P}(\mathsf{C}_{\bar{\jmath}}) \sim \sqrt{\frac{2}{\pi g_{\bar{\jmath}} m}}, \tag{15}$$

for large m after applying Stirling's approximation.
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