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A Simplified Approach to Modelling the Co-Movement of Asset Returns

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Abstract

This paper proposes a simplified multivariate GARCH model (the S-GARCH model) that involves the estimation of only univariate GARCH models, both for the individual return series and for the sum and difference of each pair of series. The covariance between each pair of return series is then imputed from these variance estimates. The model that we propose is considerably easier to estimate than existing multivariate GARCH models and does not suffer from the convergence problems that characterize many of these models. Moreover, the model can be easily extended to include more complex dynamics or alternative forms of the GARCH specification. We use the S-GARCH model to estimate the minimum-variance hedge ratio for the FTSE 100 index portfolio, hedged using index futures, and compare it to four of the most widely used multivariate GARCH models. Using both statistical and economic evaluation criteria, we find that the S-GARCH model performs at least as well as the other models that we consider, and in some cases better than them.

Keywords: Multivariate GARCH; Hedging; Minimum-variance hedge ratio; FTSE 100 index.

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1. Introduction

There are many applications in finance that rely on an estimate of the multivariate conditional covariance matrix of returns. Such applications include conditional asset pricing models, portfolio optimization, minimum-variance hedging, value at risk and the pricing of options that depend on more than one underlying asset. Perhaps the most widely used approach to modeling the conditional covariance matrix of returns is the multivariate GARCH class of models.[1] A number of different multivariate GARCH models have been proposed, each imposing a different set of restrictions on the dynamic process that governs the covariance matrix of returns. These models include the Vech and Diagonal Vech models of Bollerslev, Engle and Woolridge (1988), the BEKK model and Diagonal BEKK models of Engle and Kroner (1995), the Constant Correlation model of Bollerslev (1990), the Factor ARCH model of Engle, Ng and Rothschild (1990) and the Dynamic Conditional Correlation model of Engle and Sheppard (2001) and Engle (2002).

While commonly employed in the academic literature, multivariate GARCH models suffer from a number of problems in practice. First, they tend to be computationally burdensome, typically involving the simultaneous estimation of a large number of parameters. This is particularly true of the Vech and BEKK models, both of which impose relatively few restrictions on the dynamic process that governs the evolution of the covariance matrix. Despite recent advances in technology, there are many situations where computational cost is important such as when estimating out-of-sample forecasts of the conditional covariance matrix using a rolling window over a large sample, or where forecasts of the conditional covariance matrix of returns must be computed for a large number of assets in a short period of time (such as when estimating intra-day VaR for a derivatives trading desk). In these instances, multivariate GARCH models are often eschewed by practitioners in favour of simpler alternatives such as exponentially weighted moving average (EWMA) estimators of the covariance matrix (see, for example, Engle, 2002). Second, owing to the large number of parameters that must be

[1] Other approaches to estimating the conditional covariance matrix include rolling estimators of the sample covariance matrix, exponentially weighted estimators (JP Morgan, 1994) and multivariate stochastic volatility models (Harvey, Ruiz and Shephard, 1994).

estimated simultaneously, and the non-concavity of the likelihood function, maximum likelihood estimation of multivariate GARCH models is often problematic. Establishing that estimation has properly converged (i.e. to parameter values that represent a global maximum of the likelihood function rather than a local maximum) involves a potentially computationally intensive grid-search over all of the parameters in the model. Third, compared with their univariate counterparts, it is relatively difficult to construct multi-period forecasts of the covariance matrix using multivariate GARCH models (see, for example, Kroner and Ng, 1998). Fourth, owing to their computational complexity, it is often difficult to extend multivariate GARCH models to include more complex dynamics such as longer lag specifications, the asymmetric response of volatility to return shocks, and dummy variables to capture seasonality, outliers and structural breaks.[2]

In an attempt to overcome these computational issues, a number of simpler specifications of the multivariate GARCH model have been proposed. However, the simplifications that these models entail generally come at the cost of imposing severe, and often implausible, cross-equation restrictions on the elements of the conditional covariance matrix. For example, in the Diagonal Vech and Diagonal BEKK models, each element of the conditional covariance matrix is assumed to evolve independently, meaning that shocks to the variances of individual assets have no impact on the future covariance between them. Moreover, the Diagonal BEKK model imposes cross-equation restrictions on the parameters of the variance and covariance equations which are unlikely to hold in practice. In the Constant Correlation model, time-variation in the covariance between individual assets is determined solely by time-variation in their individual variances, with the correlation coefficient assumed to be time-invariant. None of these models is able to capture, for example, the well documented feature that correlation among financial assets tends to increase as their volatility increases (see, for example, Longin and Solnik, 1995). The Factor ARCH model, which is a special case

[2] For some multivariate GARCH models, asymmetric versions have been developed. For example, de Goeij and Marquering (2004) extend the Diagonal Vech model to include asymmetric terms that capture the response to negative return shocks to either or both of the assets. Kroner and Ng (1998) show that the Vech, BEKK and Constant Correlation models can be nested within a General Dynamic Covariance model, and develop an asymmetric version of this model – the Asymmetric Dynamic Covariance model.

of the BEKK model, assumes that the covariance between any two assets derives solely from a common covariance with one or more underlying factors.

In this paper, we propose an alternative, simplified multivariate GARCH model (the S-GARCH model) that involves the estimation of only univariate GARCH models, both for the individual return series and for the sum and difference of each pair of series. The conditional covariance between each pair of return series is then imputed from these conditional variance estimates. The S-GARCH model is considerably more straightforward to estimate than the Vech and BEKK models, and because the estimation involves only univariate GARCH models – and hence only a small number of parameters in any single estimation – it does not suffer from the convergence problems that typically characterize other multivariate GARCH models. Moreover, it is easily extended to include the more complex dynamics that are commonly found in the univariate GARCH literature, or to use alternative forms of the GARCH specification. The S-GARCH model is less restrictive than the Diagonal Vech, Diagonal BEKK, Constant Correlation and Factor ARCH models, allowing the covariance between two assets to depend on the history of both their covariance and their individual variances, without imposing the restriction that the correlation coefficient between them is constant over time or that their covariance derives solely from a common covariance with an underlying factor.

We illustrate the S-GARCH model by estimating the minimum-variance hedge ratio for the FTSE 100 index portfolio, hedged using index futures. We also estimate an asymmetric specification of the S-GARCH model (the AS-GARCH model), which allows for asymmetry in the response of the conditional variances of the two assets, and the conditional covariance between them, to negative lagged return shocks. We compare the S-GARCH and AS-GARCH models to four of the most widely used multivariate GARCH models, namely the Diagonal Vech, Constant Correlation, Diagonal BEKK and Dynamic Conditional Correlation models. We evaluate the performance of each model both statistically, using a regression of each element of the realized covariance matrix on the corresponding element of the estimated covariance matrix, and economically, by considering the performance of the hedged portfolio. We find that the S-GARCH and AS-GARCH models perform at least as well as the other models that we consider, and in some cases better than them. Moreover, the computation time for the S-GARCH and AS-GARCH models is considerably lower than for the other models.

The rest of the paper is organised as follows. The following section introduces the S-GARCH model. Section 3 discusses the statistical properties of the S-GARCH model. In Section 4, we describe an extension of the S-GARCH model to allow for asymmetry in the response of the conditional covariance matrix to lagged return shocks. Section 5 presents the empirical application. Section 6 concludes.

2. The Simplified Multivariate GARCH Model

Consider two assets whose per-period abnormal returns are given by $\varepsilon_{1,t} = r_{1,t} - \mu_{1,t}$ and $\varepsilon_{2,t} = r_{2,t} - \mu_{2,t}$, where $r_{1,t}$ and $r_{2,t}$ are actual returns at time *t*, $\mu_{1,t} = E[r_{1,t} \mid \Omega_{t-1}]$ and $\mu_{2,t} = E[r_{2,t} \mid \Omega_{t-1}]$ are conditional mean returns at time *t*, and Ω_{t-1} is the time $t-1$ information set. The conditional covariance matrix of $r_{1,t}$ and $r_{2,t}$ is given by

$$\Sigma_t = \begin{bmatrix} \sigma_{1,t}^2 & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{2,t}^2 \end{bmatrix} \tag{1}$$

where $\sigma_{1,t}^2 = \mathrm{var}(\varepsilon_{1,t} \mid \Omega_{t-1})$, $\sigma_{2,t}^2 = \mathrm{var}(\varepsilon_{2,t} \mid \Omega_{t-1})$ and $\sigma_{12,t} = \sigma_{21,t} = \mathrm{cov}(\varepsilon_{1,t}, \varepsilon_{2,t} \mid \Omega_{t-1})$. Here, we propose a simple procedure to estimate the elements of the conditional covariance matrix, Σ_t, that involves the estimation of only univariate GARCH models. The procedure has two steps. In the first step, we estimate the conditional variances, $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$, with two univariate GARCH models. Each of these is estimated simultaneously with a model for the conditional mean returns, $\mu_{1,t}$ and $\mu_{2,t}$. For example, using the simplest GARCH(1, 1) specification, the conditional variances, $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$, are given by

$$\sigma_{1,t}^2 = \beta_{1,0} + \beta_{1,1}\sigma_{1,t-1}^2 + \beta_{1,2}\varepsilon_{1,t-1}^2 \tag{2}$$

$$\sigma^2_{2,t} = \beta_{2,0} + \beta_{2,1}\sigma^2_{2,t-1} + \beta_{2,2}\varepsilon^2_{2,t-1} \tag{3}$$

The models (2) and (3) can be estimated by Quasi-Maximum Likelihood to yield consistent estimates of the model parameters and corresponding estimates of the conditional variances, $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$. In the second step, we construct the series $r_{+,t} = r_{1,t} + r_{2,t}$ and $r_{-,t} = r_{1,t} - r_{2,t}$. We estimate the conditional variances of these two series, again using two univariate GARCH models with equations for the conditional means, $\mu_{+,t} = E[r_{+,t} \mid \Omega_{t-1}]$ and $\mu_{-,t} = E[r_{-,t} \mid \Omega_{t-1}]$. For example, using a GARCH(1, 1) specification, the conditional variances of $r_{+,t}$ and $r_{-,t}$ are given by

$$\sigma^2_{+,t} = \beta_{+,0} + \beta_{+,1}\sigma^2_{+,t-1} + \beta_{+,2}\varepsilon^2_{+,t-1} \tag{4}$$

$$\sigma^2_{-,t} = \beta_{-,0} + \beta_{-,1}\sigma^2_{-,t-1} + \beta_{-,2}\varepsilon^2_{-,t-1} \tag{5}$$

where $\varepsilon_{+,t} = r_{+,t} - \mu_{+,t}$, $\varepsilon_{-,t} = r_{-,t} - \mu_{-,t}$, $\sigma^2_{+,t} = \mathrm{var}(\varepsilon_{+,t} \mid \Omega_{t-1})$ and $\sigma^2_{-,t} = \mathrm{var}(\varepsilon_{-,t} \mid \Omega_{t-1})$. The auxiliary models (4) and (5) can again be estimated by Quasi-Maximum Likelihood in order to yield consistent estimates of the model parameters and corresponding estimates of the conditional variances, $\sigma^2_{+,t}$ and $\sigma^2_{-,t}$. Noting that $\mathrm{var}(r_{1,t} \mid \Omega_{t-1}) = \mathrm{var}(\varepsilon_{1,t} \mid \Omega_{t-1}) = \sigma^2_{1,t}$, $\mathrm{var}(r_{2,t} \mid \Omega_{t-1}) = \mathrm{var}(\varepsilon_{2,t} \mid \Omega_{t-1}) = \sigma^2_{2,t}$ and $\mathrm{cov}(r_{1,t}, r_{1,t} \mid \Omega_{t-1}) = \mathrm{cov}(\varepsilon_{1,t}, \varepsilon_{2,t} \mid \Omega_{t-1}) = \sigma_{12,t}$, an estimate of the conditional covariance can then be based on the following identities.

$$\sigma^2_{+,t} \equiv \sigma^2_{1,t} + \sigma^2_{2,t} + 2\sigma_{12,t} \tag{6}$$

$$\sigma^2_{-,t} \equiv \sigma^2_{1,t} + \sigma^2_{2,t} - 2\sigma_{12,t} \tag{7}$$

In particular, combining (6) and (7), we have

$$\sigma_{12,t} \equiv (1/4)(\sigma^2_{+,t} - \sigma^2_{-,t}) \tag{8}$$

The identity given by (8) is commonly used in the statistics literature in order to derive a covariance estimator based on an established variance estimator (in an unconditional setting) when no obvious multivariate extension of the variance estimator exists, such as in the case of robust estimation of the covariance matrix (see, for example, Huber, 1981). In the context of conditional volatility, Harris and Shen (2003) employ this identity to generalise a univariate robust EWMA estimator to the multivariate case. In this paper, we extend the application of this identity to the more general multivariate GARCH model.

While the S-GARCH model above is exposited using the simplest GARCH(1, 1) formulation, it would be straightforward to extend the model to allow for more complex dynamics using a general GARCH(r, s) specification, or one of the many alternative specifications from the class of univariate GARCH models. In particular, it is straightforward to include terms that capture the asymmetric response of volatility to return shocks due to changes in financial leverage (as we show below), or dummy variables that capture seasonality, outliers and structural breaks, in either the mean or the volatility of returns. Moreover, the S-GARCH model involves the estimation of only univariate GARCH models and is therefore considerably easier to implement than the Vech and BEKK models. In particular, because only a few parameters are estimated in each model, it is more likely that maximum likelihood estimation will converge properly and hence much less experimentation is required with different starting values for the model parameters.

3. Properties of the Simplified Multivariate GARCH model

Different specifications of the multivariate GARCH model impose different restrictions on the dynamic process that governs the covariance matrix of returns, reflecting a balance between flexibility in the dynamic structure of the model and ease of computation. In an attempt to overcome the computational difficulty of more general unrestricted Vech and BEKK models, a number of simpler specifications of the multivariate GARCH model have been proposed, such as the Diagonal Vech, Diagonal BEKK, Constant Correlation and Factor ARCH models. However, the simplifications that these models entail come at the cost of imposing severe, and often implausible, cross-equation restrictions on the elements of the covariance matrix. For example, in the Diagonal Vech and Diagonal BEKK models, each element of the covariance matrix is assumed to evolve independently, implying that shocks to the variances of individual assets have no impact on the future covariance between them. In contrast, with the Constant Correlation model, the covariance between individual assets is specified as the product of the conditional variances and a constant correlation coefficient, so that time-variation in the conditional covariance is determined solely by time-variation in their individual variances. The Factor ARCH model assumes that the covariance between any two assets derives solely from a common covariance with one or more underlying factors.

The precise dynamic structure of the S-GARCH model depends on the specification of the univariate GARCH models used at each of the two estimation stages. Here, we illustrate the properties of the S-GARCH model under the univariate GARCH(1,1) specification used above. In particular, substituting (6) into (4), and noting that $\varepsilon_{+,t} = \varepsilon_{1,t} + \varepsilon_{2,t}$ and $\varepsilon_{-,t} = \varepsilon_{1,t} - \varepsilon_{2,t}$, yields

$$\begin{aligned} \sigma_{1,t}^2 + \sigma_{2,t}^2 + 2\sigma_{12,t} &= \beta_{+,0} + \beta_{+,1}(\sigma_{1,t-1}^2 + \sigma_{2,t-1}^2 + 2\sigma_{12,t-1}) \\ &\quad + \beta_{+,2}(\varepsilon_{1,t-1}^2 + \varepsilon_{2,t-1}^2 + 2\varepsilon_{1,t-1}\varepsilon_{2,t-1}) \end{aligned} \tag{9}$$

while substituting (7) into (5) yields

$$\begin{aligned} \sigma_{1,t}^2 + \sigma_{2,t}^2 - 2\sigma_{12,t} &= \beta_{-,0} + \beta_{-,1}(\sigma_{1,t-1}^2 + \sigma_{2,t-1}^2 - 2\sigma_{12,t-1}) \\ &\quad + \beta_{-,2}(\varepsilon_{1,t-1}^2 + \varepsilon_{2,t-1}^2 - 2\varepsilon_{1,t-1}\varepsilon_{2,t-1}) \end{aligned} \tag{10}$$

Subtracting (10) from (9) and rearranging yields

$$\sigma_{12,t} = \theta_0 + \theta_1\sigma_{12,t-1} + \theta_2\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \theta_3(\sigma_{1,t-1}^2 + \sigma_{2,t-1}^2) + \theta_4(\varepsilon_{1,t-1}^2 + \varepsilon_{2,t-1}^2) \tag{11}$$

where $\theta_0 = (\beta_{+,0} - \beta_{-,0})/4$, $\theta_1 = (\beta_{+,1} + \beta_{-,1})/2$, $\theta_2 = (\beta_{+,2} + \beta_{-,2})/2$, $\theta_3 = (\beta_{+,1} - \beta_{-,1})/4$ and $\theta_4 = (\beta_{+,2} - \beta_{-,2})/4$. Thus the S-GARCH model with a univariate GARCH(1, 1) specification is equivalent to a first order diagonal Vech model but with a covariance equation augmented by the average lagged variance and the average lagged squared return. In this respect, the simplified model is more flexible than both the Diagonal Vech model (which assumes that the covariance between $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is determined solely by their lagged covariance, i.e. $\theta_3 = \theta_4 = 0$), and the Constant Correlation model (which assumes that the covariance between $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is determined solely by their lagged variances). Moreover, unlike the Factor ARCH model, it does not impose a factor representation of returns in order to estimate their conditional covariance. If $\beta_{+,1} = \beta_{-,1}$ and $\beta_{+,2} = \beta_{-,2}$, the model reduces to the Diagonal Vech model. The S-GARCH model is also more general than the FlexM-GARCH model of Ledoit, Santa-Clara and Wolf (2003), who propose a simplified approach to the estimation of the Diagonal Vech model that also involves two stages. In the first stage, they estimate the conditional variance equations for individual assets using a univariate GARCH model. In the second stage, they estimate the conditional covariance for each pair of assets using a Diagonal VECH model, but with the conditional variances set to their estimated values from the first stage, which reduces the number of parameters to be estimated. The S-GARCH model therefore nests the FlexM-GARCH model. Moreover, the S-GARCH model employs only univariate GARCH models in its estimation and is therefore more easily implemented in standard econometric packages.

We can also establish the relationship between the parameters of the S-GARCH model and the unconditional covariance matrix. For the variances of $r_{1,t}$ and $r_{2,t}$, we have the standard result that the long run variances implied by the GARCH(1, 1) model are given by

$$\sigma_1^2 = \frac{\beta_{1,0}}{1 - \beta_{1,1} - \beta_{1,2}} \tag{12}$$

$$\sigma_2^2 = \frac{\beta_{2,0}}{1 - \beta_{2,1} - \beta_{2,2}} \tag{13}$$

By taking the unconditional expectation of (4) and (5) and substituting into the unconditional expectation of (8), it is straightforward to show that the long run covariance of $r_{1,t}$ and $r_{2,t}$ implied by the S-GARCH model is given by

$$\sigma_{12} = \frac{\beta_{+,0}}{4(1-\beta_{+,1}-\beta_{+,2})} - \frac{\beta_{-,0}}{4(1-\beta_{-,1}-\beta_{-,2})} \tag{14}$$

In common with the Diagonal Vech model, and the FlexM-GARCH model of Ledoit, Santa-Clara and Wolf (2003), a potential problem with the S-GARCH model is that the resulting estimate of the conditional correlation matrix, $\hat{\Gamma}_t$, is not necessarily positive semi-definite (PSD). Moreover, in contrast with the Diagonal Vech model, it is not possible to impose any restrictions on the estimated parameters to ensure that the resulting estimate of the correlation matrix is PSD, since the parameters to be restricted are estimated in separate equations. There are several approaches to this problem. The first – and simplest – is to extract the eigenvalues and eigenvectors of the estimated correlation matrix, $\hat{\Gamma}_t$, and truncate any negative eigenvalues at zero (or some small number, to ensure invertibility of the covariance matrix). The original eigenvectors and truncated eigenvalues are then used to construct a pseudo-correlation matrix, which is rescaled so that it has a unit diagonal. The resulting correlation matrix will be PSD by construction, although it will not generally be the 'nearest' PSD matrix to the original correlation matrix. The second is to use a numerical algorithm to choose a PSD correlation matrix, $\tilde{\Gamma}_t$, such that the distance $\left\|\tilde{\Gamma}_t - \hat{\Gamma}_t\right\|$ is minimised according to some metric such as the Frobenius or Hadamard norm (see, for example, Higham, 2002). The third is to transform the matrices of the parameter estimates of the multivariate GARCH model in such a way that the correlation matrix that is implied by these parameter estimates is PSD (see, for example, Ledoit, Santa-Clara and Wolf, 2003). Any of these approaches could be applied to the S-GARCH model to ensure that the estimated correlation matrix is PSD.

Another potential problem with the S-GARCH model is that while standard errors of the estimated parameters of the conditional variance equations for $r_{1,t}$ and $r_{2,t}$ are readily available from the univariate estimation procedure, those of the conditional covariance equation are not because the parameters of this equation are themselves imputed from the parameters of the auxiliary conditional variance equations for $r_{+,t}$ and $r_{-,t}$. From a practical perspective, this is not likely to be a problem in many cases since the majority of applications that employ the GARCH framework (whether univariate or multivariate) are concerned with forecasting performance and tests of economic significance (in the context of the application) rather than with statistical inference. For example, when using a multivariate GARCH model to estimate a time-varying minimum variance hedge ratio, it is common to focus on the properties of the resulting hedge portfolio, rather than on the statistical properties of the conditional covariance between spot and futures returns (see, for example, Baillie and Myers, 1991; Kroner and Sultan, 1993). Of course, in some cases, it may be desirable to conduct inference about the parameters of the conditional covariance equation. In these cases, one solution would be to use the bootstrap methodology of Efron (1979) to generate simulated standard errors (see, for example, Ledoit, Santa-Clara and Wolf, 2003). While clearly detracting somewhat from the simplicity of implementation of the S-GARCH model, such an approach is nevertheless relatively straightforward and can be easily implemented in most econometric packages. To implement the bootstrap approach, we proceed as follows.

Step 1: Estimate the S-GARCH model given by (2), (3), (4), (5) and (8) using the actual data. Save the estimated parameters, $\hat{\beta} = (\hat{\beta}_{1,0}, \hat{\beta}_{1,1}, \hat{\beta}_{1,2}, \hat{\beta}_{2,0}, \hat{\beta}_{2,1}, \hat{\beta}_{2,2}, \hat{\beta}_{+,0}, \hat{\beta}_{+,1}, \hat{\beta}_{+,2}, \hat{\beta}_{-,0}, \hat{\beta}_{-,1}, \hat{\beta}_{-,2})$, the estimated residuals, $\hat{\varepsilon}_t = [\hat{\varepsilon}_{1,t} \quad \hat{\varepsilon}_{2,t}]'$, and the estimated conditional covariance matrix, $\hat{\Sigma}_t$. The parameters $\hat{\beta}$ are used to calculate the parameters, $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4)$, of the implied covariance equation given by (11).

Step 2: Construct standardised residuals, $\tilde{\varepsilon}_t = \hat{\Sigma}_t^{-1/2} \hat{\varepsilon}_t$.

Step 3: Estimate the unconditional mean ($\overline{\tilde{\varepsilon}}$) and unconditional covariance matrix ($\hat{\Sigma}_{\tilde{\varepsilon}}$) of the standardised residuals, $\tilde{\varepsilon}_t$.

Step 4: Compute zero-mean, unit-variance standardised residuals, $\tilde{\tilde{\varepsilon}}_t = \hat{\Sigma}_{\tilde{\varepsilon}}^{-1/2} (\tilde{\varepsilon}_t - \overline{\tilde{\varepsilon}})$.

Step 5: Resample with replacement from $\tilde{\tilde{\varepsilon}}_t$ to yield the bootstrapped zero-mean, unit-variance standardised residuals ε_t^{**}.

Step 6: Arbitrarily set the first observation of the bootstrapped covariance matrix, Σ_1^*. Use the estimated model (the variance equations given by (2) and (3) and the implied covariance equation given by (11)) to recursively compute the rescaled bootstrapped residuals, $\varepsilon_t^* = (\Sigma_t^*)^{1/2}\varepsilon_t^{**}$, and the bootstrapped covariance matrix, Σ_t^*, $t = 2,\ldots,T$. To mitigate the influence of Σ_1^*, an additional, say, 100 initial observations can be estimated and then discarded.

Step 7: Estimate the S-GARCH model given by (2), (3), (4), (5) and (8) using the bootstrapped residuals, ε_t^*. Save the estimated parameters, $\hat{\beta}^*$ and use these to calculate the coefficients, $\hat{\theta}^*$, of the implied covariance equation given by (11).

This procedure is repeated, say, 100 times.[3] The bootstrapped standard errors of the elements of $\hat{\theta}$ are then given by the standard deviations (across the bootstrap replications) of the elements of $\hat{\theta}^*$.

4. The Asymmetric S-GARCH Model

It is now well established that in many cases, the conditional variance of returns is not a symmetric function of lagged return shocks. In particular, it is commonly found that negative return shocks have a greater impact on the conditional variance than positive return shocks (see, for example, Black, 1976). Much work has been done to extend the univariate GARCH framework to capture this asymmetry and a wide range of models have been developed, including the EGARCH model (Nelson, 1991), the GJR-GARCH model (Glosten, Jagannathan and Runkle, 1993), the quadratic GARCH model (Sentana, 1995) and the smooth transition GARCH model (Gonzalez-Rivera, 1998). These models differ in the way volatility is assumed to respond to lagged return shocks. For example, the GJR model defines a function

[3] To achieve accurate standard error estimates, at least 25 bootstrap replications are needed (see, for example, Efron, 1979).

$$w_i(\varepsilon_{i,t-1}) = \begin{cases} 1 & \text{if } \varepsilon_{i,t-1} < 0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

This function is then used to weight the lagged squared residual, $\varepsilon^2_{i,t-1}$, in the conditional variance equation. In the S-GARCH framework, this asymmetry can be easily incorporated in the conditional variances of the two assets. In particular, we can re-specify the conditional variance equations as

$$\sigma^2_{1,t} = \beta_{1,0} + \beta_{1,1}\sigma^2_{1,t-1} + \beta_{1,2}\varepsilon^2_{1,t-1} + \beta_{1,3}w_1\varepsilon^2_{1,t-1} \tag{16}$$

$$\sigma^2_{2,t} = \beta_{2,0} + \beta_{2,1}\sigma^2_{2,t-1} + \beta_{2,2}\varepsilon^2_{2,t-1} + \beta_{2,3}w_2\varepsilon^2_{2,t-1} \tag{17}$$

The parameters $\beta_{1,3}$ and $\beta_{2,3}$ measure the asymmetry in the response of $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$ to lagged shocks to $r_{1,t}$ and $r_{2,t}$, respectively. When $\beta_{1,3} = \beta_{2,3} = 0$, the conditional variance equations reduce to the symmetric univariate GARCH models given by (2) and (3). Alternative asymmetric specifications can be obtained by suitable choice of the weight function $w_i(\varepsilon_{i,t-1})$ (see, for example, Taylor, 2005).

While the literature on asymmetry in the univariate case is well established, the multivariate case is much less developed. Nevertheless, it is clear that asymmetry in the conditional variances of two assets is likely to be matched by a similar asymmetry in their conditional covariance. Kroner and Ng (1998) develop a generalised multivariate GARCH framework (which nests the Vech, BEKK and Constant Correlation models) and extend this to the asymmetric case. They find that, indeed, the covariance between small and large stocks increases more in response to negative return shocks than to positive return shocks. Similarly, de Goeij and Marquering (2004) find that there is a significant asymmetry in the covariance between stock and bond returns in the US. The simplest way to capture asymmetry in the covariance equation, in the spirit of the GJR-GARCH model, is to define the weight

$$w_3(\varepsilon_{1,t-1}, \varepsilon_{2,t-1}) = \begin{cases} 1 & \text{if } \varepsilon_{1,t-1} < 0 \text{ and } \varepsilon_{2,t-1} < 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$
$$= w_1 w_2$$

To incorporate asymmetry in the covariance equation of the S-GARCH model, we specify the auxiliary conditional variance equations for $r_{+,t}$ and $r_{-,t}$ as

$$\sigma^2_{+,t} = \beta_{+,0} + \beta_{+,1}\sigma^2_{+,t-1} + \beta_{+,2}\varepsilon^2_{+,t-1} + \beta_{+,3} w_3 \varepsilon_{1,t-1}\varepsilon_{2,t-1} \tag{19}$$

$$\sigma^2_{-,t} = \beta_{-,0} + \beta_{-,1}\sigma^2_{-,t-1} + \beta_{-,2}\varepsilon^2_{-,t-1} + \beta_{-,3} w_3 \varepsilon_{1,t-1}\varepsilon_{2,t-1} \tag{20}$$

Following the same approach as in the previous section, the implied covariance between $r_{1,t}$ and $r_{2,t}$ is given by

$$\begin{aligned}\sigma_{12,t} &= \theta_0 + \theta_1 \sigma_{12,t-1} + \theta_2 \varepsilon_{1,t-1}\varepsilon_{2,t-1} + \theta_3(\sigma^2_{1,t-1} + \sigma^2_{2,t-1}) + \theta_4(\varepsilon^2_{1,t-1} + \varepsilon^2_{2,t-1}) \\ &\quad + \theta_5 w_3 \varepsilon_{1,t-1}\varepsilon_{2,t-1}\end{aligned} \tag{21}$$

where $\theta_5 = (\beta_{+,3} - \beta_{-,3})/4$ measures the asymmetry in the response of $\sigma_{12,t}$ to a negative shock to the returns of both assets. When $\beta_{+,3} = \beta_{-,3}$, $\theta_5 = 0$ and the model reduces to the symmetric covariance equation given by (11). Again, other forms of asymmetry can be achieved by suitable choice of the weight function. Moreover, it would be straightforward to include dummy variables not just for when both lagged return shocks are negative, but also for when just one of the return shocks is negative (see, for example, de Goeij and Marquering, 2004).

5. Empirical Illustration: Estimation of the Minimum Variance Hedge Ratio

In this section, we illustrate the S-GARCH and Asymmetric S-GARCH (AS-GARCH) models by estimating the minimum-variance hedge ratio for the FTSE 100 index portfolio, hedged using the FTSE100 index futures contract. When the conditional covariance matrix of spot and futures returns is time-varying, the minimum-variance hedge ratio at time *t* is equal to

$$h_t = \frac{\sigma_{sf,t}}{\sigma^2_{f,t}} \quad (22)$$

where $\sigma_{sf,t}$ is the conditional covariance of spot and futures returns and $\sigma^2_{f,t}$ is the conditional variance of futures returns (see, for example, Kroner and Sultan, 1993).

Data

We obtained daily closing prices for the FTSE 100 index from Datastream, and for the FTSE 100 index futures contracts from LIFFE, for the period 04 May 1984 to 24 February 2006, which is the longest common sample available. At any one time, there are four futures contracts outstanding. On each day, we use the nearest contract to delivery, but rollover to the next nearest contract on the first day of the delivery month in order to avoid thin trading and expiration effects. Using the daily closing spot and futures prices, we computed continuously compounded returns. The futures return is calculated with reference to the previous day's spot price rather than the previous day's futures price (see, for example, Solnik, 2000). We removed the returns on the rollover dates from the sample in order to avoid spurious jumps in the futures price that arise from suddenly increasing the maturity of the futures contract. We also removed outliers corresponding to the five days following the stock market crash of 19 October 1987. Table 1 gives summary statistics for the spot and futures returns. The mean returns for the two series are similar, and both are very close to zero. The volatility of futures returns is somewhat higher than the volatility of spot returns, which is a common empirical finding (see, for example, Kroner and Sultan, 1993). The ARCH(4) portmanteau test for up to fourth order serial correlation in squared returns shows that both spot and futures returns display significant volatility clustering. Both spot and futures returns are highly leptokurtic (which is consistent with the existence of time-varying volatility) and negatively skewed. The Jarque-Bera statistic very strongly rejects the null hypothesis of normality. The unconditional correlation of spot and futures returns is 0.951.

[Table 1]

Methodology

In order to implement the S-GARCH model, we first estimate the conditional variances of $r_{s,t}$ and $r_{f,t}$. In principal, any conditional volatility model could be used, but to illustrate our approach, we use the simplest univariate GARCH(1, 1) specification. This permits a comparison with other multivariate GARCH models. Since our interest is in modelling the conditional covariance matrix of spot and futures returns, and not expected returns, we include only a constant in the mean equation for both spot and futures returns. The model for $r_{s,t}$ is therefore given by

$$r_{s,t} = \mu_s + \varepsilon_{s,t} \tag{23}$$

$$\sigma^2_{s,t} = \beta_{s,0} + \beta_{s,1}\sigma^2_{s,t-1} + \beta_{s,2}\varepsilon^2_{s,t-1} \tag{24}$$

and the model for $r_{f,t}$ is given by

$$r_{f,t} = \mu_f + \varepsilon_{f,t} \tag{25}$$

$$\sigma^2_{f,t} = \beta_{f,0} + \beta_{f,1}\sigma^2_{f,t-1} + \beta_{f,2}\varepsilon^2_{f,t-1} \tag{26}$$

These are used to provide estimates of the conditional variances, $\hat{\sigma}^2_{s,t}$ and $\hat{\sigma}^2_{f,t}$. We then construct the new series, $r_{+,t} = r_{s,t} + r_{f,t}$ and $r_{-,t} = r_{s,t} - r_{f,t}$. We estimate the conditional variances of $r_{+,t}$ and $r_{-,t}$ also using two univariate GARCH(1, 1) models. Again, we include only a constant term in the mean equation. The model for $r_{+,t}$ is therefore given by

$$r_{+,t} = \mu_+ + \varepsilon_{+,t} \tag{27}$$

$$\sigma^2_{+,t} = \beta_{+,0} + \beta_{+,1}\sigma^2_{+,t-1} + \beta_{+,2}\varepsilon^2_{+,t-1} \tag{28}$$

and the model for $r_{-,t}$ is given by

$$r_{-,t} = \mu_{-} + \varepsilon_{-.t} \tag{29}$$

$$\sigma^2_{-,t} = \beta_{-,0} + \beta_{-,1}\sigma^2_{-,t-1} + \beta_{-,2}\varepsilon^2_{-,t-1} \tag{30}$$

These are used to provide estimates of the conditional variances, $\hat{\sigma}^2_{+,t}$ and $\hat{\sigma}^2_{-,t}$. We then estimate the conditional covariance as

$$\hat{\sigma}_{sf,t} = (1/4)(\hat{\sigma}^2_{+,t} - \hat{\sigma}^2_{-,t}) \tag{31}$$

The estimated conditional variance of futures returns, $\hat{\sigma}^2_{f,t}$, and conditional covariance of spot and futures returns, $\hat{\sigma}_{sf,t}$, are then used to compute the minimum-variance hedge ratio using equation (22). For the AS-GARCH model, we use the specifications given by (16), (17), (18), (19) and (20). We estimate each of the univariate GARCH models above by maximum likelihood with a conditional normal distribution, using the BHHH algorithm with a convergence criterion of 0.00001 applied to the coefficient values. Note that if $r_{f,t}$ and $r_{s,t}$ are both conditionally normally distributed, then $r_{+,t}$ and $r_{-,t}$ will also be conditionally normally distributed by construction. If $r_{f,t}$ and $r_{s,t}$ are conditionally non-normal then we can rely on the consistency results of Quasi-Maximum Likelihood (see Bollerslev and Woolridge, 1992).[4] For both the S-GARCH and AS-GARCH models, we compute standard errors of the estimated coefficients of the implied covariance equation using the bootstrap method described in Section 3, with 100 replications.

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[4] Although Quasi-Maximum Likelihood estimation is consistent if $r_{f,t}$ and $r_{s,t}$ are conditionally non-normal, more efficient estimators of the conditional covariance matrix can be obtained by using, for example, the APARCH model of Ding, Granger and Engle (1993) (see Nelson and Foster, 1996). This is easily accommodated in the S-GARCH framework.

Evaluation

We compare the results for the S-GARCH and AS-GARCH models with four of the most commonly used multivariate GARCH models, namely the Diagonal Vech (DVech) model of Bollerslev, Engle and Woolridge (1988), the Diagonal BEKK (DBEKK) model of Engle and Kroner (1995), the Constant Correlation (CC) model of Bollerslev (1990) and the Dynamic Conditional Correlation (DCC) model of Engle and Sheppard (2001). As with the simplified model, we estimate each of these multivariate GARCH models by maximum likelihood with a conditional normal distribution, using the BHHH algorithm with a convergence criterion of 0.00001 applied to the coefficient values. In each case, the mean equation for both spot and futures returns is specified to include only a constant. For the CC and DCC models, the variance equations are estimated simultaneously using a Diagonal Vech model, as is common in practice.

In order to evaluate the performance of the six multivariate GARCH models, we employ two approaches. The first is a statistical evaluation. If a multivariate GARCH model is correctly specified then it should generate estimates of the realized covariance matrix that are conditionally unbiased. For each element of the covariance matrix, we test this using a regression of the realized variance (or covariance) on the estimated variance (or covariance). As a measure of the realized covariance matrix, we use the squares and cross-product of $\hat{\varepsilon}_{s,t}$ and $\hat{\varepsilon}_{f,t}$. We therefore estimate the following three regressions.

$$\hat{\varepsilon}_{s,t}^2 = \delta_{s,0} + \delta_{s,1}\sigma_{s,t}^2 + v_{s,t} \tag{32}$$

$$\hat{\varepsilon}_{f,t}^2 = \delta_{f,0} + \delta_{f,1}\sigma_{f,t}^2 + v_{f,t} \tag{33}$$

$$\hat{\varepsilon}_{s,t}\hat{\varepsilon}_{f,t} = \delta_{sf,0} + \delta_{sf,1}\sigma_{sf,t} + v_{sf,t} \tag{34}$$

If the estimated covariance matrix is conditionally unbiased, the intercept in each regression should be zero and the slope coefficient should be unity (see, for example, Andersen and Bollerslev, 1998). We estimate these regressions using OLS. The null hypothesis of conditional unbiasedness is tested for each regression using an F-statistic. The second approach is an economic evaluation. In particular, we report the standard deviation of the hedged portfolio for each model. The more accurate the estimated conditional covariance matrix, the lower should be the standard deviation of the hedged portfolio. We also report the standard deviation of the hedge ratio itself, which gives some indication of the likely transaction costs associated with a dynamic hedging strategy based on each model, on the basis that if transaction costs are proportional to the change in the composition of the hedge portfolio, a more variable hedge ratio implies higher transaction costs since it involves a greater degree of rebalancing.

Results

Panel A of Table 2 reports summary statistics for $\hat{\sigma}_{sf,t}$ estimated from the six models, and the correlation matrix of these estimates across the six models. The DBEKK, S-GARCH and AS-GARCH models generate covariance estimates that are very close on average to the unconditional covariance of spot and futures returns reported in Table 1, while the other models generate covariance estimates that are on average either too high (DVech and DCC) or too low (CC). The DBEKK model has the lowest standard deviation, while the DCC model has the highest. The correlations between $\hat{\sigma}_{sf,t}$ across the six models are high, except in the case of the DBEKK model, which has a relatively low correlation with all of the remaining models. The highest correlation for $\hat{\sigma}_{sf,t}$ is between the DVech and DCC models. Panel B reports the summary statistics and correlation matrix for $\hat{\rho}_{sf,t} = \hat{\sigma}_{sf,t} / (\hat{\sigma}_{s,t}\hat{\sigma}_{f,t})$ for each of the six models. While the average values of $\hat{\rho}_{sf,t}$ are similar for all six models, there are considerable differences in their variation. The S-GARCH and AS-GARCH models generate the most variable $\hat{\rho}_{sf,t}$, closely followed by the DVech and DCC models. For the DBEKK model, the standard deviation of $\hat{\rho}_{sf,t}$ is considerably lower than that of the DVech, DCC, S-GARCH and AS-GARCH models, while for the CC model, the standard deviation of $\hat{\rho}_{sf,t}$ is zero, as expected. Note that for all six models, $-1 \le \hat{\rho}_{sf,t} \le 1$ for all *t*, implying that all six models generate a conditional correlation matrix that is positive semi-definite in this particular sample. Hence there is no need to apply any of the adjustments described in Section 3. As with $\hat{\sigma}_{sf,t}$, the correlation of $\hat{\rho}_{sf,t}$ is lowest between the DBEKK model and the remaining models. The highest correlation is between the S-GARCH and DVech models, reflecting the fact that the DVech model is nested by the S-GARCH model. The correlation is also relatively high between the DVech and DCC models, and between the S-GARCH and AS-GARCH models. To illustrate the differences between the models, Figure 1 plots the estimated correlation coefficient from the six models for a short sub-sample.

[Table 2]

[Figure 1]

Table 3 reports the implied covariance equation for the S-GARCH and AS-GARCH models, given by equations (11) and (21), respectively, and the corresponding equation for the DVech model. Standard errors are reported in parentheses. The three models give similar estimates for the parameters on the lagged conditional covariance and the lagged cross-product of returns. However, the S-GARCH and AS-GARCH models also put a small weight on the lagged values of the average conditional variance and the average squared return, which are restricted to be zero in the DVech model. For the S-GARCH model, these parameters are insignificant. However, in the AS-GARCH model, the estimated parameter on the lagged average squared return is significantly negative, suggesting that the covariance between spot and futures returns decreases in response to volatility shocks. The asymmetry term in the AS-GARCH model is significantly positive, implying that the conditional covariance between the two assets shows a greater response to return shocks that are both negative, which is consistent with Kroner and Ng (1998) and de Goeij and Marquering (2004). Moreover, the size of the asymmetry coefficient is comparable with the values reported in these studies.

[Table 3]

Table 4 reports the results of the regressions to test the conditional unbiasedness of $\sigma^2_{s,t}$, $\sigma^2_{f,t}$ and $\sigma_{sf,t}$ for each of the six multivariate GARCH models. For $\sigma^2_{s,t}$, the null hypothesis of conditional unbiasedness is rejected at the five percent significance level for the CC, DBEKK and DCC models. For the DBEKK model, the rejection is very strong. For $\sigma^2_{f,t}$, the null hypothesis of conditional unbiasedness is rejected for the DVech and DCC models. For $\sigma_{sf,t}$, the null hypothesis of conditional unbiasedness is rejected for the CC, DBEKK and DCC models. Thus, for the DCC model, the null hypothesis is rejected for all three elements of the conditional covariance matrix. Only for the S-GARCH and AS-GARCH models is the null hypothesis not rejected for all three elements of the conditional covariance matrix.

[Table 4]

The first row of Panel A of Table 5 reports the standard deviation of the estimated hedge ratio for the six multivariate GARCH models. The CC model yields the lowest hedge ratio standard deviation, reflecting the fact that it imposes the restriction that the correlation coefficient between spot and futures returns is constant, while the other models allow the correlation coefficient to vary over time. The DCC model yields the highest hedge ratio standard deviation, followed by the S-GARCH, AS-GARCH, DVech and DBEKK models. The second row of Panel A of Table 5 reports the standard deviation of the hedge portfolio daily return for the five multivariate GARCH models. The DVech model yields the lowest hedge portfolio standard deviation, while the S-GARCH and AS-GARCH models yield return standard deviations that are only marginally higher. For the DBEKK model, the hedge portfolio standard deviation is considerably higher than for the remaining models. Panel B of Table 5 tests the pair-wise significance of the differences in hedge portfolio variance between the six models. The DBEKK model yields a hedge portfolio variance that is significantly higher than for all of the other models. For the remaining five models, there is no significant difference in hedge portfolio variance.

[Table 5]

6. Conclusion

While commonly employed in the academic literature, multivariate GARCH models suffer from a number of problems in practice owing to the complexity of their specification. In particular, because of the large number of parameters that must be estimated simultaneously, multivariate GARCH models tend to be computationally burdensome. Moreover, because the likelihood function of these models is not globally concave, there is no guarantee that maximum likelihood estimation will converge to the correct parameter values, particularly when the models are supplemented by more complicated dynamics, asymmetric terms or dummy variables.

In this paper, we propose a simple but effective multivariate GARCH model (the S-GARCH model) that overcomes these problems. The model that we propose involves the estimation of only univariate GARCH models, both for the individual return series and for the sum and difference of each pair of series. The covariance terms in the covariance matrix are then imputed from these variance estimates. Since the estimation involves only univariate GARCH models, it is considerably more straightforward to estimate than existing multivariate GARCH models and does not suffer from the convergence problems that typically characterise many of these models. We show that the S-GARCH model is equivalent to a first-order diagonal Vech model but with a covariance equation augmented by the average lagged variance and the average lagged squared error. In this respect, the simplified model is more flexible than both the Diagonal Vech model (which assumes that the covariance of returns is determined solely by their lagged covariance), and the Constant Correlation model (which assumes that the covariance of returns is determined solely by their lagged variances). We also estimate an asymmetric specification of the S-GARCH model (the AS-GARCH model), which allows for asymmetry in the response of the conditional variances of the two assets, and the conditional covariance between them, to negative lagged return shocks.

We illustrate the S-GARCH and AS-GARCH models, and compare them to four of the most widely used multivariate GARCH models – the Diagonal Vech model, the Constant Correlation model, the Diagonal BEKK model and the Dynamic Conditional Correlation model – by estimating the minimum-variance hedge ratio for the FTSE 100 index portfolio, hedged using index futures. We evaluate the performance of each model both statistically, using a regression of each element of the realized covariance matrix on the corresponding element of the estimated covariance matrix, and economically, by considering the performance of the hedged portfolio. We find that by both measures, the S-GARCH and AS-GARCH models perform at least as well as the other models that we consider, and in some cases better than them. Moreover, the S-GARCH and AS-GARCH models are computationally much easier to estimate. Indeed, for the empirical application in this paper, the estimation time for the S-GARCH model was less than half that of the other models. The estimation time for the asymmetric AS-GARCH model was only marginally longer.

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Table 1 Summary Statistics of FTSE 100 Spot and Futures Returns

	$r_{s,t}$	$r_{f,t}$
Mean	0.030%	0.016%
Standard Deviation	0.983%	1.071%
Skewness	-0.189	-0.167
Excess kurtosis	3.401	2.974
Jarque-Bera	2731.522	2088.932
ARCH(4)	737.432	354.220
Covariance	0.0001001	
Correlation	0.951	

Notes: The table reports summary statistics for continuously compounded spot and futures returns for the FTSE 100 for the period 04 May 1984 to 24 February 2006. The Jarque-Bera statistic tests the null hypothesis of zero skewness and excess kurtosis, and has a $\chi^2(2)$ distribution with a critical value of 5.99 at the 5% significance level. The ARCH(4) statistic tests the null hypothesis that the first four partial autocorrelations of squared returns are zero, and has a $\chi^2(4)$ distribution with a critical value of 9.49 at the 5% significance level.

Table 2 Summary Statistics and Correlation Matrices for $\hat{\sigma}_{sf,t}$ and $\hat{\rho}_{sf,t}$

Panel A: $\hat{\sigma}_{sf,t}$

	DVech	CC	DBEKK	DCC	S-GARCH	AS-GARCH
Mean	0.000104	0.000096	0.000100	0.000105	0.000100	0.000099
Stand dev	0.000095	0.000084	0.000069	0.000099	0.000096	0.000093
Skewness	3.446	3.460	3.451	3.399	3.572	3.642
Kurtosis	15.394	15.625	17.256	14.899	16.734	17.758
Minimum	0.000027	0.000028	0.000037	0.000025	0.000025	0.00002
Maximum	0.000894	0.000804	0.000782	0.000907	0.000940	0.00096

Correlation Matrix

	DVech	CC	DBEKK	DCC	S-GARCH	AS-GARCH
DVech	1.000	0.998	0.875	0.999	0.998	0.984
CC		1.000	0.856	0.987	0.981	0.985
DBEKK			1.000	0.859	0.866	0.885
DCC				1.000	0.949	0.981
S-GARCH					1.000	0.987
AS-GARCH						1.000

Panel B: $\hat{\rho}_{sf,t}$

	DVech	CC	DBEKK	DCC	S-GARCH	AS-GARCH
Mean	0.952	0.947	0.953	0.950	0.947	0.948
Stand dev	0.038	0.000	0.026	0.039	0.042	0.047
Skewness	-3.284	N/A	-1.584	-2.389	-3.128	-3.357
Kurtosis	18.356	N/A	6.316	9.136	17.492	19.338
Minimum	0.626	0.947	0.693	0.680	0.579	0.511
Maximum	0.992	0.947	0.997	0.994	0.998	0.998

Correlation Matrix

	DVech	CC	DBEKK	DCC	S-GARCH	AS-GARCH
DVech	1.000	N/A	0.568	0.937	0.990	0.983
CC		1.000	N/A	N/A	N/A	N/A
DBEKK			1.000	0.476	0.530	0.610
DCC				1.000	0.863	0.894
S-GARCH					1.000	0.978
AS-GARCH						1.000

Notes: The table reports summary statistics and correlations for $\hat{\sigma}_{sf,t}$ and $\hat{\rho}_{sf,t}$ for the six multivariate GARCH models.

Table 3 Estimated Covariance Equation for DVech, S-GARCH and AS-GARCH Models

	DVech	S-GARCH	AS-GARCH
θ_0	1.133e-06 (1.332e-07)	1.292e-06 (3.400e-07)	1.384e-06 (4.590e-07)
θ_1	0.923 (0.003)	0.915 (0.008)	0.911 (0.009)
θ_2	0.068 (0.003)	0.079 (0.007)	0.064 (0.009)
θ_3	-	-0.002 (0.004)	0.004 (0.006)
θ_4	-	-0.001 (0.004)	-0.013 (0.005)
θ_5	-	-	0.0584 (0.015)

Notes: The table reports the parameters of the estimated covariance equation for the DVech, S-GARCH and AS-GARCH models. Standard errors are reported in parentheses. For the S-GARCH and AS-GARCH models, these are estimated using a semi-parametric bootstrap with 100 replications.

Table 4 Results for Conditional Bias Regressions

Panel A: $\hat{\varepsilon}_{s,t}^2 = \delta_{s,0} + \delta_{s,1}\sigma_{s,t}^2 + v_{s,t}$

	DVech	**CC**	**DBEKK**	**DCC**	**S-GARCH**	**AS-GARCH**
$\hat{\delta}_{s,0}$	-0.000	-0.000	-0.000	0.000	0.000	-0.000
$\hat{\delta}_{s,1}$	0.974	1.083	1.272	0.933	0.981	1.047
F-statistic	1.033	4.069	21.915	3.957	0.196	1.356

Panel B: $\hat{\varepsilon}_{f,t}^2 = \delta_{f,0} + \delta_{f,1}\sigma_{f,t}^2 + v_{f,t}$

	DVech	**CC**	**DBEKK**	**DCC**	**S-GARCH**	**AS-GARCH**
$\hat{\delta}_{s,0}$	0.000	0.000	-0.000	0.000	0.000	0.000
$\hat{\delta}_{s,1}$	0.926	1.023	1.054	0.901	0.928	0.984
F-statistic	3.815	0.558	0.988	6.736	2.877	0.134

Panel C: $\hat{\varepsilon}_{s,t}\hat{\varepsilon}_{f,t} = \delta_{sf,0} + \delta_{sf,1}\sigma_{sf,t} + v_{sf,t}$

	DVech	**CC**	**DBEKK**	**DCC**	**S-GARCH**	**AS-GARCH**
$\hat{\delta}_{s,0}$	0.000	-0.000	-0.000	0.000	0.000	-0.000
$\hat{\delta}_{s,1}$	0.962	1.096	1.225	0.923	0.960	1.036
F-statistic	1.539	4.928	14.911	4.756	0.903	0.792

Notes: The table reports the results of estimating the conditional bias regressions for $\sigma_{s,t}^2$, $\sigma_{f,t}^2$ and $\sigma_{sf,t}^2$. The F-statistic tests the null hypothesis that $\hat{\delta}_{i,0} = 0$ and $\hat{\delta}_{i,1} = 1$ ($i = s, f, sf$), and has an F(2, 5590) distribution with a critical value of 2.99 at the 5% significance level.

Table 5 Results for Minimum Variance Hedge Portfolio

Panel A: The standard deviation of the estimated hedge ratio and the hedge portfolio return

	DVech	CC	DBEKK	DCC	S-GARCH	AS-GARCH
$\sigma_{\hat{h}}$	0.087	0.076	0.087	0.103	0.090	0.092
σ_p	0.295%	0.298%	0.366%	0.298%	0.296%	0.296%

Panel B: F-tests for the pairwise differences in hedge portfolio return variance

	CC	DBEKK	DCC	S-GARCH	AS-GARCH
DVech	0.449	0.000	0.568	0.827	0.755
CC		0.000	0.852	0.590	0.655
DBEKK			0.000	0.000	0.000
DCC				0.725	0.794
S-GARCH					0.925

Notes: Panel A reports the standard deviation of the estimated hedge ratio, $\sigma_{\hat{h}}$, and the standard deviation of the hedge portfolio return, σ_p, for each of the six multivariate GARCH models. Panel B reports the one-tailed probability that the variances of the hedged portfolio returns are not significantly different between each pair of the six models.

Figure 1 Estimated Correlation Coefficient