Tuning of a Piezoelectric Vibration Absorber Attached to a Damped Structure

P. Soltani[1], G. Kerschen[2], G. Tondreau[3], A. Deraemaeker[3]

1. Dynamic Research Group
Department of Mechanical Engineering
University of Sheffield, United Kingdom
E-mails: payam.soltani@sheffield.ac.uk,@gmial.com

2. Space Structures and Systems Laboratory
Department of Aerospace and Mechanical Engineering
University of Liège, Liège, Belgium
E-mail: g.kerschen@ulg.ac.be

3. Building Architecture and Town Planning (BATir)
Université Libre de Bruxelles, Brussels, Belgium
E-mails: gilles.tondreau@ulb.ac.be, aderaema@ulb.ac.be

Abstract

The objective of this paper is to derive an approximate closed-form solution to the H_∞ optimization of piezoelectric materials shunted with inductive-resistive passive electrical circuits in the presence of damping in the primary structure. To this end, the homotopy perturbation method is utilized in which the zero-order solution is the recently-developed exact solution for an undamped primary system. Simplified, though accurate, expressions for the optimum frequency and damping ratios are also provided.

Keywords: *vibration mitigation, piezoelectric tuned vibration absorber, equal-peak method, homotopy perturbation, damped structure.*

Corresponding author:
Payam Soltani
Dynamic Research Group
Department of Mechanical Engineering
University of Sheffield, United Kingdom
Emails: payam.soltani@sheffield.ac.uk,@gmail.com

1 Introduction

There exists a large body of literature regarding the design of mechanical tuned vibration absorbers (MTVAs). Considering a MTVA attached to an undamped structure, Den Hartog [1] and Brock [2] were the first to develop approximate analytical formulas based on the existence of fixed points in the receptance curve. However, it is only recently that an exact closed-form solution to this classical problem could be found. Instead of imposing two fixed points of equal amplitude, Nishihara and co-workers [3] tackled the direct minimization of the norm of the frequency response of the controlled structure. In the presence of damping, they proposed an approximate analytical solution in the form of power series of the damping in the primary system [4].

For piezoelectric tuned vibration absorbers (PTVAs) shunted with RL circuits, Hagood and von Flotow developed two strategies based on Den Hartog's fixed point method and on pole placement [5]. Hogsberg and Krenk proposed a tuning rule that is a balanced compromise between these two design criteria [6]. This balanced calibration was recently improved to account for the contributions of non-resonant vibration modes [7]. Through the development of an equivalent mechanical model of a piezoelectric element, Yamada et al. [8] introduced a new approximate analytical expression for the damping parameter that improves the PTVA performance compared to the formulas in [5]. Closed-form expressions related to shunt performance were also derived in [9]. Because all the aforementioned tuning rules are approximate, an exact closed-form solution for a PTVA attached to an undamped system was established in [10] through the extension to piezoelectric absorbers of the design philosophy proposed in [3]. We note that Den Hartog's equal peak method was extended to nonlinear PTVAs in [11].

There have been very few attempts to derive analytical formulas for PTVAs attached to damped systems, see, e.g., [12]. In this paper, a new approach that relies on the homotopy perturbation method carried out up to second order is proposed. For the zero-order solution, the recently-developed exact solution for an undamped primary system is exploited [10], which leads to an accurate closed-form solution.

The paper is organized as follows. For completeness, Section 2 briefly reviews the exact solution derived for an undamped primary system. Section 3 presents the analytical developments that lead to accurate approximations of the frequency and damping ratios of the PTVA in the damped case. To provide the reader with tractable and useful formulas, the analytical expressions are simplified using polynomial regressions in Section 4. Finally, the conclusions of the present study are drawn in Section 5.

2 Piezoelectric tuned vibration absorber attached to an undamped system: review of the exact solution

We consider a one-degree-of-freedom modal model of the primary structure coupled to a PTVA, i.e., a piezoelectric transducer (PZT) attached to a resonating resistive-inductive

(RL) shunt. The coupled system is depicted in Figure 1 and is subjected to harmonic forcing.

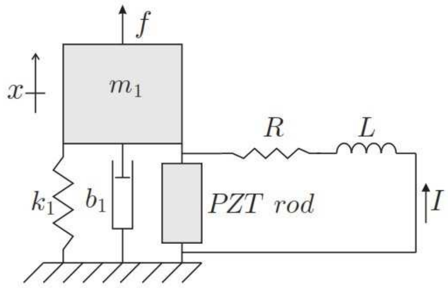


Figure 1: PTVA attached to a damped system.

The PZT is assumed to be a one-dimensional rod in which both the expansion and polarization directions coincide with the central axis of the rod (conventionally called '3-direction'). The stiffness of the PZT rod with short-circuited electrodes, k_{PZT}, and the capacitance with no external forces, c_{PZT}, are defined as:

$$k_{PZT} = \frac{1}{s_{33}^E}\frac{s_0}{l_0}, \quad c_{PZT} = \epsilon_3^T\frac{s_0}{l_0}. \tag{1}$$

The cross section area and length of the PZT rod are s_0 and l_0, respectively. The permittivity under constant strain and the compliance under constant electric field of the PZT rod in 3-direction are ϵ_3^T and s_{33}^E, respectively [13]. The governing equations of the coupled system are written as:

$$\begin{aligned} m_1\ddot{x} + b_1\dot{x} + \left(k_1 + \overline{k}_{PZT}\right)x - \overline{\theta}q &= f\sin\omega t, \\ L\ddot{q} + R\dot{q} + \frac{1}{\overline{c}_{PZT}}q - \overline{\theta}x &= 0. \end{aligned} \tag{2}$$

where $x(t)$ and $q(t)$ are the displacement of the primary system and the charge in the electrical circuit, respectively. The parameters

$$\overline{k}_{PZT} = \frac{k_{PZT}}{1-k_0^2}, \quad \overline{\theta} = \frac{k_0}{1-k_0^2}\sqrt{\frac{k_{PZT}}{c_{PZT}}}, \quad \overline{c}_{PZT} = c_{PZT}(1-k_0^2), \tag{3}$$

are the stiffness of the PZT rod with open electrodes, the generalized electromechanical coupling factor and the capacitance of the PZT rod under constant strain. k_0 is defined as the electromechanical coupling coefficient in d_{33}-mode:

$$k_0 = d_{33}\sqrt{\frac{k_{PZT}}{c_{PZT}}} = d_{33}\frac{1}{\sqrt{s_{33}^E\epsilon_3^T}}. \tag{4}$$

As in [10], Equations (2) are recast into:

$$\begin{aligned} \widetilde{x}'' + 2\zeta\widetilde{x}' + \widetilde{x} - \delta\alpha\widetilde{q} &= f_0\sin\gamma\tau, \\ \widetilde{q}'' + r\delta^2\widetilde{q}' - \delta\alpha\widetilde{x} + \delta^2\widetilde{q} &= 0, \end{aligned} \tag{5}$$

where prime denotes differentiation with respect to the dimensionless time $\tau = \omega_1 t$ and the other parameters are:

$$
\begin{aligned}
&\omega_1 = \sqrt{\frac{k_1 + \bar{k}_{PZT}}{m_1}}, \quad && \omega_e = \frac{1}{\sqrt{L\,\bar{c}_{PZT}}}, \quad && \gamma = \frac{\omega}{\omega_1}, \quad && \delta = \frac{\omega_e}{\omega_1}, \\
&\widetilde{x} = \sqrt{m_1}\,x, && \widetilde{q} = \sqrt{L}\,q, && r = R\,\bar{c}_{PZT}\,\omega_1, && \\
&\kappa = k_1/\bar{k}_{PZT}, && f_0 = \frac{f}{\omega_1\sqrt{\bar{k}_{PZT} + k_1}}. && &&
\end{aligned}
\tag{6}
$$

We note that the parameter

$$
\alpha = \theta\sqrt{\frac{\bar{c}_{PZT}}{\bar{k}_{PZT} + k_1}} = k_0\sqrt{\frac{\bar{k}_{PZT}}{\bar{k}_{PZT} + k_1}} = \frac{k_0}{\sqrt{1+\kappa}}, \tag{7}
$$

depends only on the stiffness ratio $\kappa = k_1/\bar{k}_{PZT}$ and the electromechanical coupling coefficient k_0. A key parameter in this paper is the damping in the primary structure:

$$
\mu = 2\,\zeta = \frac{b_1}{2\,m_1\,\omega_1} = \frac{b_1}{2\,\sqrt{m_1\,(k_1 + \bar{k}_{PZT})}}. \tag{8}
$$

Assuming $\gamma = \delta\,\hat{\gamma}$ and $j = \sqrt{-1}$, the receptance transfer function of the primary structure

$$
G\left(\hat{\gamma}\right) = \left|\frac{\widetilde{x}}{f_0}\right| = \left|-\frac{j\delta\,r\hat{\gamma} - \hat{\gamma}^2 + 1}{\left(j\delta\,\hat{\gamma}^3 + \delta^2 r\hat{\gamma}^2 - j\delta\,\hat{\gamma}\right)\mu + j\delta^3 r\hat{\gamma}^3 - \delta^2\hat{\gamma}^4 - j\delta\,r\hat{\gamma} + \delta^2\hat{\gamma}^2 + \alpha^2 + \hat{\gamma}^2 - 1}\right|, \tag{9}
$$

features two resonant points M and N.

For an undamped primary structure, i.e., $\mu = 0$, the optimum values of δ and r corresponding to H_∞ optimization were obtained in [10]:

$$
\delta_{opt} = 2\sqrt{\frac{\tilde{a}}{4\,S\tilde{a} - \tilde{b}}}, \quad r_{opt} = \sqrt{\frac{2\,\left[1 + \left({\delta_{opt}}^2 + 1\right)\left(\alpha^2 + \chi - 1\right){h_0}^2\right]}{\left[1 + \left(\chi\,{\delta_{opt}}^2 - 1\right){h_0}^2\right]{\delta_{opt}}^2}}. \tag{10}
$$

The amplitude h_0 of the resonant peaks is:

$$
h_0 = \frac{8}{\alpha\sqrt{2\sqrt{54\,\alpha^4 - 144\,\alpha^2 + 64} + 9\,\alpha^2 + 16}}, \tag{11}
$$

and the other auxiliary parameters are given in Appendix A. As illustrated in Figure 2, the receptance curve for $\mu = 0$ possesses two resonance peaks with exactly the same amplitude.

However, for a damped primary system, Figure 2 shows that the resonance peaks have no longer the same amplitude. The normalized difference between the amplitudes of the resonant peaks M and N

$$
\triangle\hat{G}_{M-N} \equiv \left|\frac{G(\hat{\gamma})|_M - G(\hat{\gamma})|_N}{h_0}\right|, \tag{12}
$$

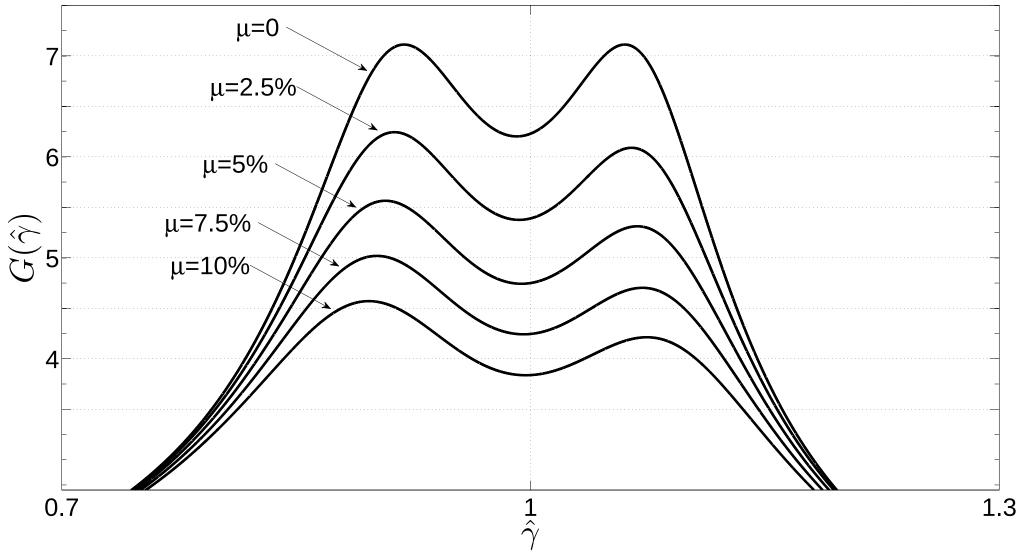


Figure 2: Frequency response of a damped primary structure coupled to a PTVA tuned using Equations (10) ($\alpha = 0.2$).

is an appropriate detuning indicator. It is presented in Figure 3 for different values of the dimensionless electromechanical coupling factor α. For instance, this figure depicts that the detuning can amount to 5% for a practical value of $\alpha = 0.2$ and for 2% damping in the primary system.

3 Analytical solution for a piezoelectric tuned vibration absorber attached to a damped system

Given the dimensionless electromechanical coupling factor α and the damping parameter μ, the objective of this section is to correct the analytical expressions of the frequency and damping ratios in Equation (10) so as to maintain equal peaks in the receptance curve of a damped primary system.

By introducing new auxiliary parameters $\theta \equiv \delta^2$ and $g \equiv (\delta\,\hat{\gamma})^2$, the square of the receptance transfer function of the primary damped structure $G^2 \equiv |G(\hat{\gamma})|^2$ is given by:

$$G^2 = \frac{gr^2\theta + g^2 - 2\,g + 1}{\left[\begin{array}{c} g^4\theta^2 + \theta\,\left(r^2\theta^2 + \mu^2 - 2\,\theta - 2\right) g^3 + \\ \left[1 + \left(\mu^2 r^2 - 2\,r^2 + 1\right)\theta^2 + \left(-2\,\alpha^2 - 2\,\mu^2 + 4\right)\theta\right] g^2 + \\ \left\{\left[\left(2\,\mu\,r + 2\right)\alpha^2 + r^2 + \mu^2 - 2\right]\theta + 2\,\alpha^2 - 2\right\} g + \\ \alpha^4 - 2\,\alpha^2 + 1. \end{array}\right]} \tag{13}$$

To enforce two resonance peaks of equal amplitude, the optimality conditions should

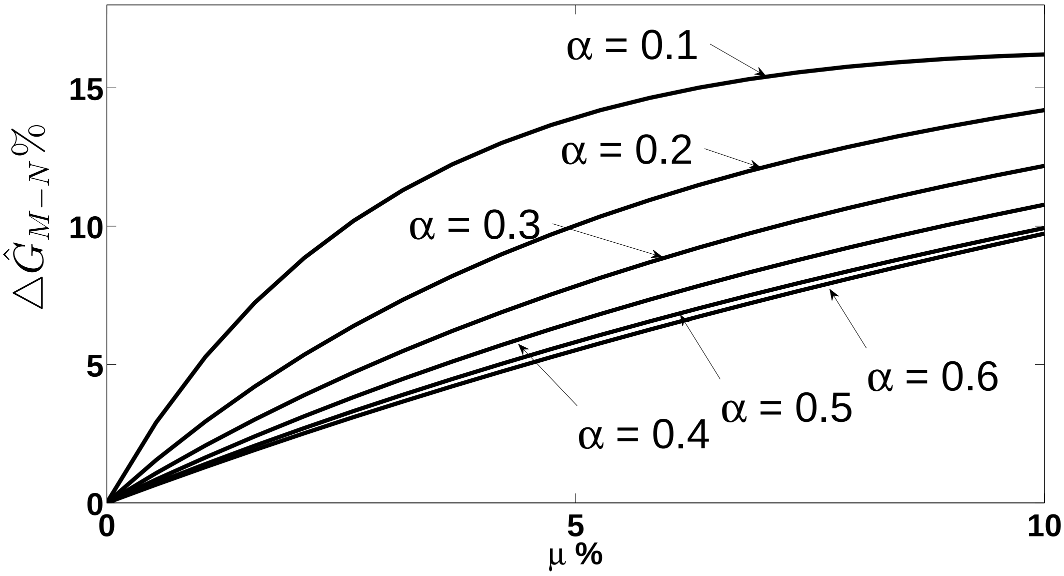


Figure 3: Normalized difference between the resonant peaks for different values of α.

be [4]:

$$\begin{aligned} C_1 &= \tfrac{\partial G^2}{\partial r}|_{g_M,g_N} = 0, \\ C_2 &= \tfrac{\partial G^2}{\partial g}|_{g_M,g_N} = 0, \\ C_3 &= G^2|_{g_M} - G^2|_{g_N} = 0. \end{aligned} \tag{14}$$

C_1 indicates that the resonant amplitudes take a minimum value at a certain combination of the dimensionless damping r and dimensionless frequency g, i.e., $(r_M,\, g_M)$ and $(r_N,\, g_N)$. C_2 represents a condition for finding the two resonance frequencies, and C_3 guarantees the equality of the resonance peaks.

By plugging Equation (13) into the optimality conditions (14), we obtain:

$$C_1 = \left[-g^2 - 2\,(-\frac{1}{2}\,r^2\theta - 1)g - 1\right]\mu - 2\,r\theta\,g^2 - 2\,\left(-\theta - 1\right)rg + \alpha^2 r - 2\,r, \tag{15}$$

and

$$\begin{aligned} C_2 &= 2\,g^3r^4\theta^4 - 4\,g^2\left[g^2 - 2\,g + 1 + \left(\tfrac{1}{4}\,\mu^2 - \tfrac{1}{2}\right)r^2\right]r^2\theta^3 + -2\,a_{2C_2}\,g\theta^2 + a_{1C_2}\,\theta + \\ &\quad 2\,\alpha^2\,(g-1)\,(\alpha^2 + g - 1)\,, \end{aligned} \tag{16}$$

with the following auxiliary parameters:

$$\begin{aligned} a_{2C_2} &= g^4 - 4\,g^3 + \left[6 + (\mu^2 - 2)\,r^2\right]g^2 + \left[-4 + (-\alpha^2 - 2\,\mu^2 + 4)\,r^2\right]g + 1 + (\mu^2 - 2)\,r^2, \\ a_{1C_2} &= (-\mu^2 + 2)\,g^4 + (4\,\mu^2 - 8)\,g^3 + (2\,\alpha^2\mu\,r - 2\,\alpha^2 - 6\,\mu^2 + 12)\,g^2 + (4\,\alpha^2 + 4\,\mu^2 - 8)\,g + \\ &\quad (\alpha^4 - 2\,\alpha^2)\,r^2 - 2\,\alpha^2\mu\,r - \mu^2 - 2\,\alpha^2 + 2 \end{aligned} \tag{17}$$

and

$$
\begin{array}{rl}
C_3 & = r^4 g_M \, g_N \, (g_M + g_N)\, \theta^4 + r^2 a_{3C_3}\, \theta^3 + a_{2C_3}\, \theta^2 + a_{1C_3}\, \theta + \\
& -2 \, \left[\left(\frac{1}{2}\, \alpha^2 + g_N - 1\right) g_M + \frac{1}{2}\, \alpha^2 g_N - \alpha^2 - g_N + 1\right] \alpha^2.
\end{array} \tag{18}
$$

with:

$$
a_{3C_3} = \left[{g_M}^3 g_N + (2\, {g_N}^2 - 4\, g_N + 1)\, {g_M}^2 + g_N \, ({g_N}^2 - 4\, g_N + 2 + (\mu^2 - 2)\, r^2)\, g_M + {g_N}^2\right],
$$

$$
\begin{array}{rl}
a_{2C_3} & = (g_N - 1)^2 \, {g_M}^3 + \left[-2 + {g_N}^3 - 4\, {g_N}^2 + (5 + (\mu^2 - 2)\, r^2)\, g_N\right] {g_M}^2 + \\
& \left[-2\, {g_N}^3 + (5 + (\mu^2 - 2)\, r^2)\, {g_N}^2 + (-4 + (-2\, \alpha^2 - 4\, \mu^2 + 8)\, r^2)\, g_N + 1 + (\mu^2 - 2)\, r^2\right] g_M + \\
& g_N \, \left[{g_N}^2 - 2\, g_N + 1 + (\mu^2 - 2)\, r^2\right],
\end{array}
$$

$$
\begin{array}{l}
a_{1C_3} = (g_N - 1)^2 \, (\mu^2 - 2)\, {g_M}^2 + \\
\left[(-2\, \mu^2 + 4)\, {g_N}^2 + (-2\, \alpha^2 \mu\, r + 2\, \alpha^2 + 4\, \mu^2 - 8)\, g_N - 2\, \mu^2 - 2\, \alpha^2 + 4\right] g_M + \\
(\mu^2 - 2)\, {g_N}^2 + (-2\, \alpha^2 - 2\, \mu^2 + 4)\, g_N + (-\alpha^4 + 2\, \alpha^2)\, r^2 + 2\, \alpha^2 \mu\, r + \mu^2 + 2\, \alpha^2 - 2.
\end{array} \tag{19}
$$

Equations (15), (16), and (18) should be solved simultaneously to obtain the solution

$$
\mathbf{X} = \left\{ \begin{array}{c} \theta \\ r \\ g_M \\ g_N \end{array} \right\}, \tag{20}
$$

in terms of parameters α and μ. This system of equations is strongly nonlinear, and we propose to solve it using the homotopy perturbation method (HPM) [14,15]. Compared to traditional perturbation methods [16], HPM does not require the definition of a small perturbation parameter and was found in the literature to remain accurate even for strongly nonlinear problems. The method starts by constructing a set of homotopy functions which connect the known configuration of the system when $\mu = 0$ (i.e., the undamped structure) into the unknown configuration when $\mu \neq 0$ (i.e., the damped structure):

$$
\begin{array}{rcl}
H_{1M}(\mathbf{X}; p) & = & (1-p) C_1 \,|_{\mu=0,\, g=g_M} + p\, C_1 \,|_{g=g_M}, \\
H_{1N}(\mathbf{X}; p) & = & (1-p) C_1 \,|_{\mu=0,\, g=g_N} + p\, C_1 \,|_{g=g_N}, \\
H_2(\mathbf{X}; p) & = & (1-p)\, C_2 \,|_{\mu=0,\, g=g_M} + p\, C_2 \,|_{g=g_M}, \\
H_3(\mathbf{X}; p) & = & (1-p)\, C_2 \,|_{\mu=0,\, g=g_N} + p\, C_2 \,|_{g=g_N}, \\
H_4(\mathbf{X}; p) & = & (1-p)\, C_3 \,|_{\mu=0} + p\, C_3,
\end{array} \tag{21}
$$

where $p \in [0, 1]$ is the homotopy parameter. The condition C_1 can be written for both resonance peaks, and two sets of equations are thus created, namely set $M = \{H_{1M},\, H_2,\, H_3,\, H_4\}$ and set $N = \{H_{1N},\, H_2,\, H_3,\, H_4\}$. To minimize both peaks simultaneously, the average value of the two solutions, $\mathbf{X}_M$ and $\mathbf{X}_N$, is considered as the final solution. By enforcing $H_i = 0$ and varying p from 0 to 1, a family of algebraic equations is created for which the solution evolves continuously from that without damping to that with damping.

Expanding the solution $\mathbf{X}$ around $p = 0$, the approximate solution $\widehat{\mathbf{X}}(p)$ is expressed as a so-called homotopy series:

$$\widehat{\mathbf{X}}(p) = \sum_{i=0}^{\infty} p^i \, \widehat{\mathbf{X}}_i, \tag{22}$$

Assuming that this series converges at $p = 1$ [17], i.e., $\widehat{\mathbf{X}}(1) = \mathbf{X}$, the analytical solution is described as:

$$\mathbf{X} = \sum_{i=0}^{\infty} \widehat{\mathbf{X}}_i. \tag{23}$$

Substituting the approximation (22) into (21), and equating terms of the same power of p lead to calculating the analytical expressions of $\widehat{\mathbf{X}}_i$'s at different orders 'i'.

Zero-order solution

Solving the system of Equations (21) for $p = 0$, the zero-order solution is obtained:

$$\widehat{\mathbf{X}}_0 = \left\{ \begin{array}{c} \theta_0 \\ r_0 \\ g_{M0} \\ g_{N0} \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ \frac{\sqrt{3}\,\alpha}{\sqrt{2-\alpha^2}} \\ 1 - \frac{\sqrt{2}}{2}\,\alpha \\ 1 + \frac{\sqrt{2}}{2}\,\alpha \end{array} \right\} \tag{24}$$

This zero-order $\widehat{\mathbf{X}}_0$ is the solution for the undamped case and is found to be similar to the rule proposed by Yamada [8]. However, this rule is approximate and does not enforce equal peaks in the receptance curve, especially for great values of α [10]. A comparison between the performance of these rules has been given in Figure (4) for $\alpha = 0.2$. Hence, to have an accurate zero-order solution, we propose to utilize the exact solution described by Equations (10):

$$\widehat{\mathbf{X}}_0 = \left\{ \begin{array}{c} \theta_0 \\ r_0 \\ g_{M0} \\ g_{N0} \end{array} \right\} = \left\{ \begin{array}{c} \overline{\delta}_{opt}^2 \\ \overline{r}_{opt} \\ (\overline{\gamma}_M \, \overline{\delta}_{opt})^2 \\ (\overline{\gamma}_N \, \overline{\delta}_{opt})^2 \end{array} \right\}. \tag{25}$$

Higher-order solutions

According to the two sets of homotopy functions, the corresponding solutions at order i are summarized as:

$$(p^i) : \widehat{\mathbf{X}}_{iM} = \mathbf{L_M}^{-1} \times \mathbf{F}_{iM}; \;\; \widehat{\mathbf{X}}_{iN} = \mathbf{L_N}^{-1} \times \mathbf{F}_{iN}. \;\; i = 1, 2, ... \tag{26}$$

The final solution of $\widehat{\mathbf{X}}_i$ is obtained as:

$$\widehat{\mathbf{X}}_i = \frac{\widehat{\mathbf{X}}_{iM} + \widehat{\mathbf{X}}_{iN}}{2}, \; i = 1, 2, ... \tag{27}$$

Unlike $\mathbf{L}_M$ and $\mathbf{L}_N$ which are independent of the order i (see Appendix B), $\mathbf{F}_{iM}$ and $\mathbf{F}_{iN}$ have to be calculated at each order. At first order,

$$\mathbf{F}_{1M} = \left\{ \begin{array}{c} \mu \, (-g_{M0} \, {r_0}^2 \theta_0 + {g_{M0}}^2 - 2\, g_{M0} + 1) \\ F_1(2) \\ F_1(3) \\ F_1(4) \end{array} \right\}, \tag{28}$$

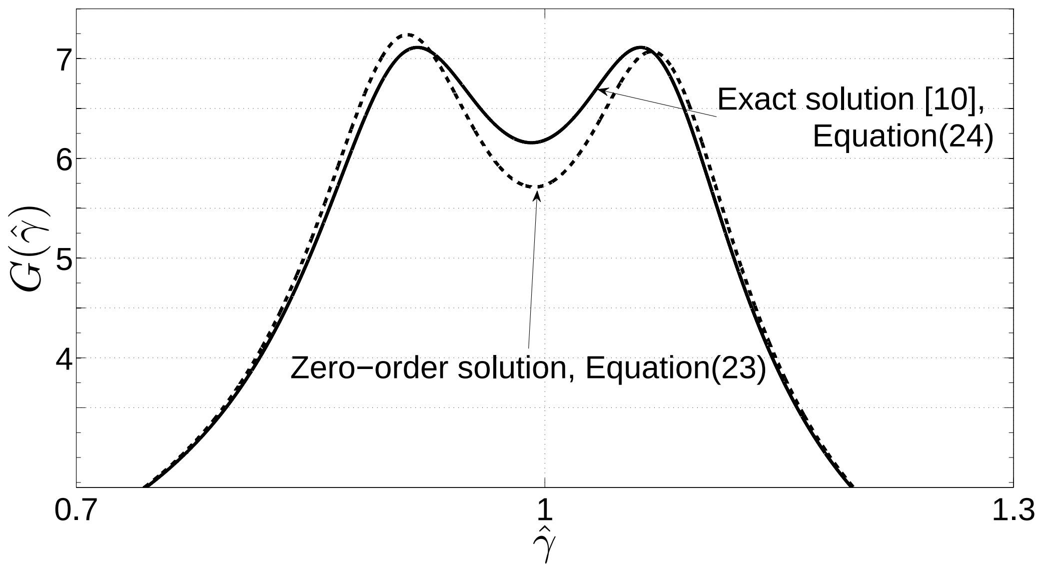


Figure 4: A performance comparison between the zero-order solution proposed by Equation(24) (dashed line) and the exact solution in Equation(24)(solid line) for un-damped structure with $\alpha = 0.2$.

$$\mathbf{F}_{1N} = \left\{ \begin{array}{c} -g_{N0}\, {r_0}^2 \mu\, \theta_0 + {g_{N0}}^2 \mu - 2\, g_{N0}\, \mu + \mu \\ F_1(2) \\ F_1(3) \\ F_1(4) \end{array} \right\}, \tag{29}$$

where

$$\begin{array}{l} F_1(2) = \mu \left[\left(1 + {g_{M0}}^2 + ({r_0}^2\theta_0 - 2)\, g_{M0}\right)^2 \mu - 2\, r_0\, \alpha^2 {g_{M0}}^2 + 2\, \alpha^2 r_0 \right] \theta_0, \\ F_1(3) = \mu \left[\left(1 + {g_{N0}}^2 + ({r_0}^2\theta_0 - 2)\, g_{N0}\right)^2 \mu - 2\, r_0\, \alpha^2 {g_{N0}}^2 + 2\, \alpha^2 r_0 \right] \theta_0, \\ F_1(4) = -\mu \left\{ \begin{array}{c} \left[1 + {g_{M0}}^2 + ({r_0}^2\theta_0 - 2)\, g_{M0}\right] (g_{N0}\, {r_0}^2 \theta_0 + {g_{N0}}^2 - 2\, g_{N0} + 1)\, \mu \\ -2\, r_0\, \alpha^2\, (g_{M0}\, g_{N0} - 1) \end{array} \right\} \theta_0. \end{array} \tag{30}$$

For the second-order approximation,

$$\mathbf{F}_{2M} = \left\{ \begin{array}{c} F_{2M}(1) \\ F_2(2) \\ F_2(3) \\ F_2(4) \end{array} \right\}, \quad \mathbf{F}_{2N} = \left\{ \begin{array}{c} F_{2N}(1) \\ F_2(2) \\ F_2(3) \\ F_2(4) \end{array} \right\}. \tag{31}$$

with

$$\begin{array}{rl} F_{2M}(1) & = \left[(-{r_0}^2\theta_1 - 2\, r_0\, r_1\, \theta_0 + 2\, g_{M1})\, g_{M0} + (-{r_0}^2\theta_0 - 2)\, g_{M1}\right] \mu + \\ & 2\, r_0\, {g_{M1}}^2 \theta_0 + \left[(4\, g_{M0}\, \theta_0 - 2\, \theta_0 - 2)\, r_1 + 4\, r_0\, \left(g_{M0} - \frac{1}{2}\right) \theta_1\right] g_{M1} + \\ & 2\, r_1\, g_{M0}\, \theta_1\, (g_{M0} - 1)\,. \end{array} \tag{32}$$

and

$$
\begin{aligned}
F_{2N}(1) \; = & \left[\left(-{r_0}^2\theta_1 - 2\,r_0\,r_1\,\theta_0 + 2\,g_{N1}\right) g_{N0} + \left(-{r_0}^2\theta_0 - 2\right) g_{N1}\right]\mu + \\
& 2\,r_0\,{g_{N1}}^2\theta_0 + \left[\left(4\,g_{N0}\,\theta_0 - 2\,\theta_0 - 2\right) r_1 + 4\,r_0\,\left(g_{N0} - \tfrac{1}{2}\right)\theta_1\right] g_{N1} + \\
& 2\,r_1\,g_{N0}\,\theta_1\,\left(g_{N0} - 1\right).
\end{aligned} \tag{33}
$$

For brevity, $F_2(2)$, $F_2(3)$, and $F_2(4)$ are given in Appendix C.

Eventually, by substituting Equations (26) into (27), and using Equation (23), the final approximation is obtained:

$$
\mathbf{X} = \widehat{\mathbf{X}}_0 + \frac{{\mathbf{L_M}}^{-1} \times \sum_{i=1}^{\infty} \mathbf{F}_{iM} + {\mathbf{L_N}}^{-1} \times \sum_{i=1}^{\infty} \mathbf{F}_{iN}}{2}, \tag{34}
$$

where the optimum tuning parameters are:

$$
\delta_{opt} = \sqrt{\mathbf{X}(1)},\; r_{opt} = \mathbf{X}(2). \tag{35}
$$

For illustration, Table (1) lists the resulting tuning parameters for $\alpha = 0.2$ and for different damping values.The receptance function of the PTVAs defined by Equation (9) is very sensitive to the numerical values of tuning parameters, especially δ and as it can be seen in Table (1), enough numbers of decimal digits should be applied in defining δ_{opt} and r_{opt} , to get high accuracy tuning of the peaks. From Figure 5, it is clear that equal peaks can now be maintained in the damped case. The improvement with respect to the undamped formulas is quantified in Figure 6, and the comparison between the first- and second-order approximations is also depicted in this figure. It is seen that the first-order solution significantly outperforms the undamped formulas. For reasonable damping values (i.e., on the order of a few percents), the peak amplitude difference is practically zero for the second-order solution, which validates our developments.

Table 1: Tuning parameters based on (35) for $\alpha = 0.2$.

	1^{st} order approximation		2^{nd} order approximation	
μ	δ_{opt}	r_{opt}	δ_{opt}	r_{opt}
0	1.000176	0.249152	1.000176	0.249152
2.5 %	0.998372	0.254798	0.998513	0.254569
5 %	0.995948	0.260851	0.996576	0.260
7.5 %	0.992899	0.267311	0.994461	0.265046
10 %	0.989220	0.274178	0.992263	0.270

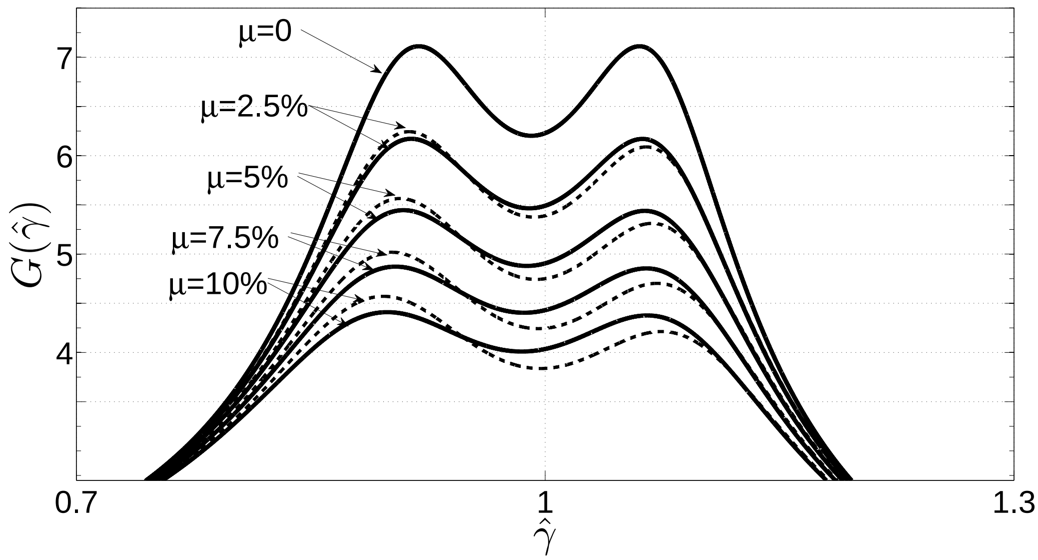


Figure 5: Frequency response of a damped primary structure coupled to a PTVA tuned using Equations (10) (dashed lines) and the proposed analytical formulas (35) (solid lines) for $\alpha = 0.2$.

4 Simplification of the tuning rule

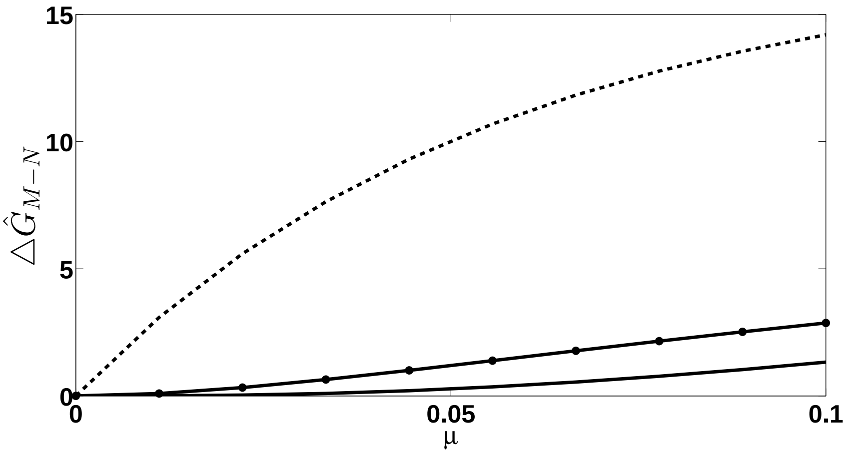
In view of the complexity of the previous developments, a simplification of the analytical formulas (35) is proposed in this section. As shown in Figure 7, the optimum parameters vary according to the electromechanical coupling factor α and the damping μ. These variations were fitted for realistic values of α and μ, i.e., $\alpha \leq 0.4$ and $\mu \leq 0.1$, using polynomial regressions:

$$\begin{aligned} \hat{\delta} &= \hat{\delta}_4\,\mu^4 + \hat{\delta}_3\,\mu^3 + \hat{\delta}_2\,\mu^2 + \hat{\delta}_1\,\mu + \hat{\delta}_0 \\ \hat{r} &= \hat{r}_4\,\mu^4 + \hat{r}_3\,\mu^3 + \hat{r}_2\,\mu^2 + \hat{r}_1\mu + \hat{r}_0 \end{aligned} \tag{36}$$

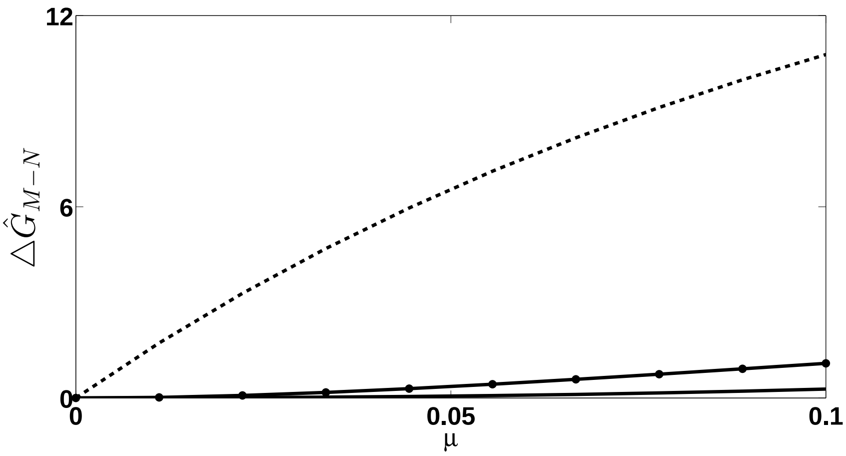
with

$$\begin{aligned} \hat{\delta}_j &= \sum_{i=0}^{n} \alpha^i\,\tilde{\delta}_{ij}, \\ \hat{r}_j &= \sum_{i=0}^{n} \alpha^i\,\tilde{r}_{ij}, \quad j = 0, 1, ..., 4. \end{aligned} \tag{37}$$

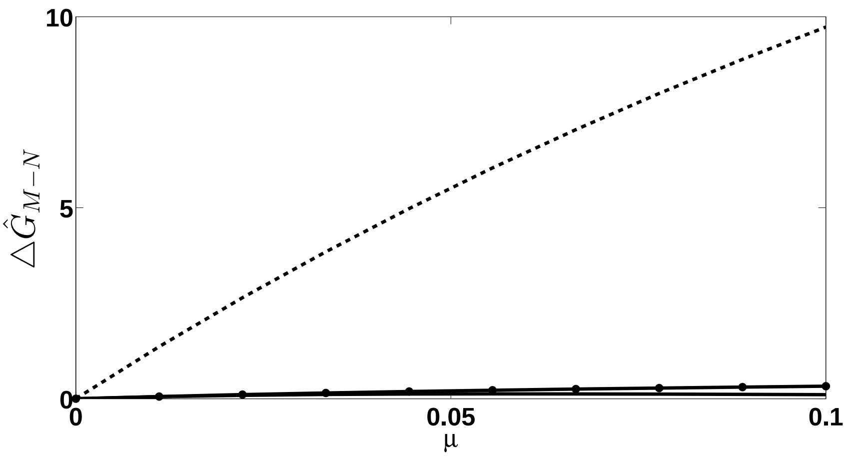
The coefficients of these regressions are given in Tables 2 and 3.



(a) $\alpha = 0.2$



(b) $\alpha = 0.4$



(c) $\alpha = 0.6$

Figure 6: Normalized difference between the resonant peaks for three values of α. Undamped formulas: dashed line; first-order homotopy solution: solid line with markers and second-order homotopy solution: solid line.

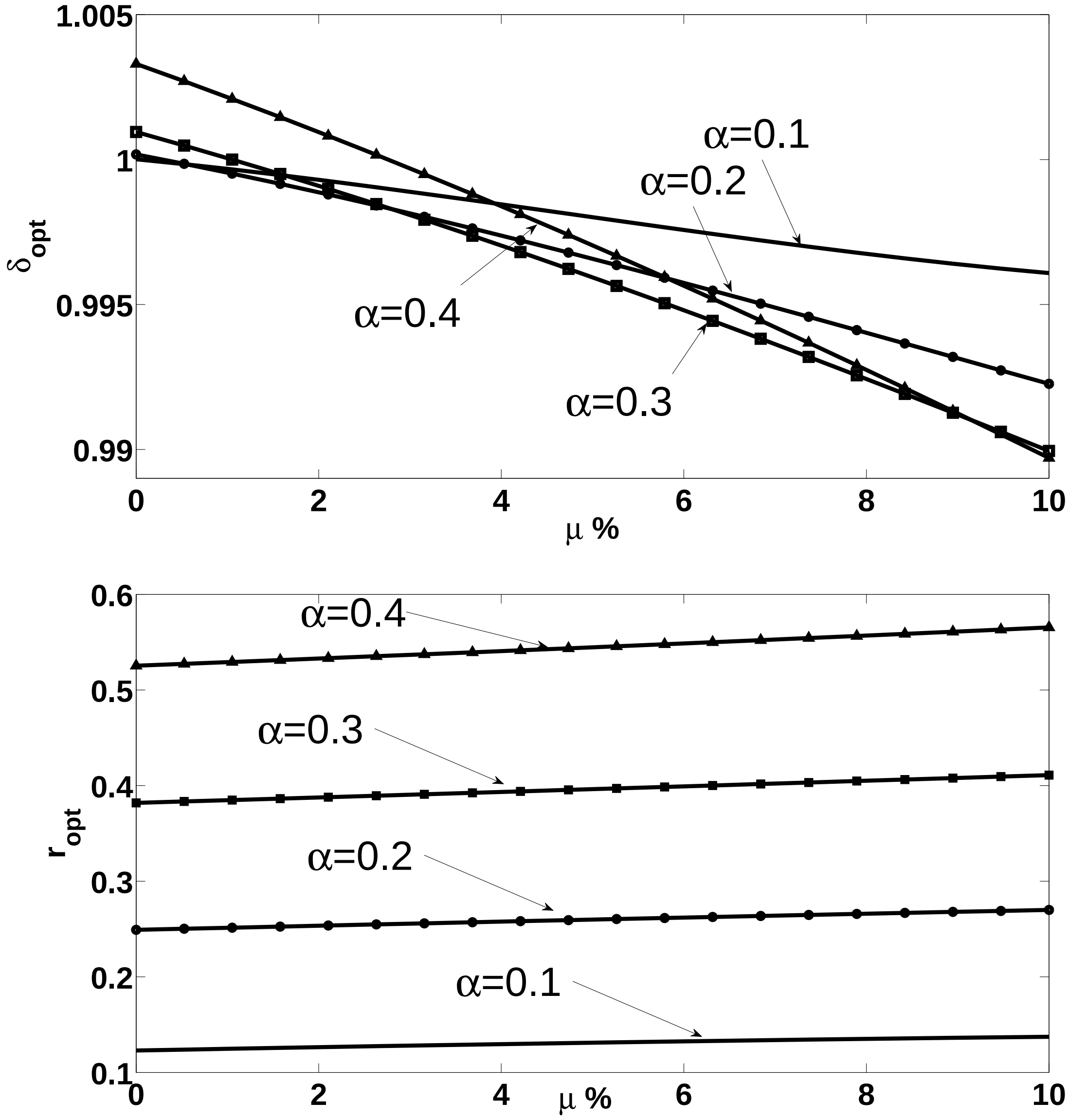


Figure 7: Variations of δ_{opt} and r_{opt} with μ at different values of α.

Table 2: Numerical values of $\tilde{\delta}_{ij}$.

j↓|i→	0	1	2	3	4
0	0.99947	0.013042	-0.099172	0.276174	0
1	0.001839	-0.648595	0.203046	0	0
2	-0.583194	-0.004205	0.036732	-0.161315	1.030
3	14.528682	-180.5251	994.019	-2487.011	2302.699
4	0.235821	-0.045426	0.688615	-2.111	4.072

Table 3: Numerical values of $\tilde{r}_{ij}$.

j↓|i→	0	1	2	3	4
0	-0.000315	1.23240	-0.055608	0.649221	0
1	0.167	-0.007	1.3371	-0.162	0
2	-1.21	15.636	-82.575	208.5644	-189.4
3	-0.874	-0.033277	3.49	-1.66418	6.8932
4	-0.9071	12.89	-61.831	154.971	-141.373

5 Conclusion

Analytical expressions for the optimum frequency and damping ratios of PTVAs coupled to damped systems were developed in this paper. To achieve this, the homotopy perturbation method was carried out up to second order utilizing the exact solution in the undamped case as zero-order solution. The obtained formulas were shown to maintain equal peaks in the receptance function even for relatively great damping values thereby generalizing Den Hartog's equal peak method. Simplified expressions were also provided for the rapid calculation of the frequency and tuning ratios or for the digital implementation of the shunt circuit.The presented formula is recommended to be used as the tuning rule for PTVAs coupling to the structures with damping coefficient higher than 1% especially at the practical range of the electromechanical coupling parameter (i.e. $\alpha \leq 0.2$).

6 Acknowledgment

All authors would like to acknowledge the financial support of the Belgian National Science Foundation FRS-FNRS (PDR T.0028.13). The author G. Kerschen would like to acknowledge the financial support of the European Union (ERC Starting Grant).

References

[1] J. P. Den Hartog, *Mechanical vibrations.* McGraw-Hill, 1934.

[2] J. Brock, "A note on the damped vibration absorber," *Journal of Applied Mechanics*, vol. 13, p. A284, 1946.

[3] O. Nishihara and T. Asami, "Closed-form solutions to the exact optimizations of dynamic vibration absorbers (minimizations of the maximum amplitude magnification factors)," *Journal of Vibration and Acoustics*, vol. 124, pp. 576–582, 2002.

[4] T. Asami, O. Nishihara, and A. Baz, "Analytical solutions to the h_∞ and h_2 optimization of dynamic vibration absorbers attached to damped linear systems," *Journal of Vibration and Acoustics*, vol. 124, pp. 284–295, 2002.

[5] N. W. Hagood and A. von Flotow, "Damping of structural vibrations with piezoelectric materials and passive electrical networks," *Journal of Sound and Vibration*, vol. 146, no. 2, pp. 243–268, 1991.

[6] J. Hogsberg and S. Krenk, "Balanced calibration of resonant shunt circuits for piezoelectric vibration control," *Journal of Intelligent Material, Systems and Structures*, vol. 23, no. 17, pp. 1937–1948, 2012.

[7] J. Hogsberg and S. Krenk, "Balanced calibration of resonant piezoelectric rl shunts with quasi-static background flexibility correction," *Journal of Sound and Vibration*, vol. 341, pp. 16–30, 2015.

[8] K. Yamada, H. Matsuhisa, H. Utsuno, and K. Sawada, "Optimum tuning of series and parallel LR circuits for passive vibration suppression using piezoelectric elements," *Journal of Sound and Vibration*, vol. 329, no. 24, pp. 5036–5057, 2010.

[9] O. Thomas, J. Ducarne, and J. F. Deu, "Performance of piezoelectric shunts for vibration reduction," *Smart Materials and Structures*, vol. 21, p. 015008, 2012.

[10] P. Soltani, G. Kerschen, G. Tondreau, and A. Deraemaeker, "Piezoelectric vibration damping using resonant shunt circuits: an exact solution," *Smart Materials and Structures*, vol. 23, no. 12, p. 125014, 2014.

[11] P. Soltani and G. Kerschen, "The nonlinear piezoelectric tuned vibration absorber," *Smart Materials and Structures*, vol. 24, p. 075015, 2015.

[12] S. Schwarzendahl, M. Neubauer, and J. Wallaschek, "Optimization of a passive piezoelectric damper for a viscously damped main system," in *Proceedings of SPIE*, vol. 8341, 2012.

[13] T. Meeker, "Publication and proposed revision of ANSI/IEEE standard 176-1987, ANSI/IEEE standard on piezoelectricity," *IEEE Transactions on Ultrasonics Ferroelectrics and Frequency Control*, vol. 43, no. 5, pp. 717–772, 1996.

[14] J. H. He, "Homotopy perturbation technique," *Computer methods in applied mechanics and engineering*, vol. 178, no. 3, pp. 257–262, 1999.

[15] J. H. He, "The homotopy perturbation method for nonlinear oscillators with discontinuities," *Applied Mathematics and Computation*, vol. 151, no. 1, pp. 287–292, 2004.

[16] A. H. Nayfeh, *Perturbation methods.* John Wiley & Sons, 2008.

[17] S. Liao, "Comparison between the homotopy analysis method and homotopy perturbation method," *Applied Mathematics and Computation*, vol. 169, no. 2, pp. 1186–1194, 2005.

Appendix A: Analytical formulas for the PTVA coupled to an undamped structure [10]

The analytical expression for the amplitude of the resonance peaks h_0 of an undamped host structure is:

$$h_0 = \frac{8}{\alpha\sqrt{2\sqrt{54\,\alpha^4 - 144\,\alpha^2 + 64} + 9\,\alpha^2 + 16}}, \tag{A-1}$$

and the auxiliary parameters in Equations (10) are:

$$\begin{aligned}
\chi &= \frac{1}{8}\sqrt{64 - 2\,\alpha^2\sqrt{54\,\alpha^4 - 144\,\alpha^2 + 64} + 55\,\alpha^4 - 144\,\alpha^2},\\
\tilde{a} &= \frac{\left(h_0^2 - 1\right)^2}{h_0^6},\\
\tilde{b} &= -2\,\frac{\left(2\,\chi + \alpha^2\right)\left(h_0^2 - 1\right)}{h_0^4},\\
S &= \frac{1}{2}\sqrt{\frac{1}{3\tilde{a}}\left(Q + \frac{\Delta_0}{Q}\right) - \frac{2}{3}p}.
\end{aligned} \tag{A-2}$$

The parameters appearing in the expression of variable S are

$$\begin{aligned}
Q &= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}},\\
\Delta_0 &= \tilde{c}^2 - 3\tilde{b}\tilde{d} + 12\tilde{a}\tilde{e},\\
p &= \frac{8\tilde{a}\tilde{c} - 3\tilde{b}^2}{8\tilde{a}^2},
\end{aligned} \tag{A-3}$$

with

$$\begin{aligned}
\Delta_1 &= 2\tilde{c}^3 - 9\tilde{b}\tilde{c}\tilde{d} + 27\tilde{b}^2\tilde{e} + 27\tilde{a}\tilde{d}^2 - 72\tilde{a}\tilde{c}\tilde{e},\\
\tilde{c} &= 5\,\frac{\alpha^4}{{h_0}^2} + \left(\frac{4\chi}{{h_0}^2} - \frac{8}{{h_0}^2}\right)\alpha^2 + \frac{6}{{h_0}^2} - \frac{6}{{h_0}^4},\\
\tilde{d} &= 2\,\chi^3 + \left(4\,\alpha^2 - 2 - \frac{2}{{h_0}^2}\right)\chi - 2\,\frac{\alpha^2}{{h_0}^2} + 2\,\alpha^6 - 2\,\alpha^4,\\
\tilde{e} &= \frac{1}{{h_0}^2}.
\end{aligned} \tag{A-4}$$

The two resonant frequencies γ_M and γ_N are also defined as:

$$\overline{\gamma}_M = \delta\sqrt{-\frac{b_3}{4\,b_4} - \bar{S}}\,, \overline{\gamma}_N = \delta\sqrt{-\frac{b_3}{4\,b_4} + \bar{S}}. \tag{A-5}$$

where

$$
\begin{aligned}
b_4 &= \delta^4\\
b_3 &= \delta^6 r^2 - 2\,\delta^4 - 2\,\delta^2,\\
b_2 &= -2\,\delta^4 r^2 - 2\,\alpha^2\delta^2 + \delta^4 + 4\,\delta^2 + 1 - \frac{1}{h_0^2},\\
b_1 &= 2\,\alpha^2\delta^2 + \delta^2 r^2 + 2\,\alpha^2 - 2\,\delta^2 - 2 - \frac{\delta^2 r^2 - 2}{{h_0}^2},\\
b_0 &= 1 + \alpha^4 - 2\,\alpha^2 - \frac{1}{h_0^2},\\
\bar{S} &= \frac{1}{2}\sqrt{\frac{-2}{3}\,\bar{p} + \frac{1}{3b_4}(\bar{Q} + \frac{\bar{\Delta}_0}{\bar{Q}})},\\
\bar{Q} &= \sqrt[3]{\frac{\bar{\Delta}_1 + (\bar{\Delta}_1^2 - 4\,\bar{\Delta}_0^3)^{\frac{1}{2}}}{2}},\\
\bar{p} &= \frac{8\,b_2\,b_4 - 3\,{b_3}^2}{8\,{b_4}^2},\\
\bar{q} &= \frac{8\,b_1\,{b_4}^2 - 4\,b_2\,b_3\,b_4 + {b_3}^3}{8\,{b_4}^3},\\
\bar{\Delta}_0 &= 12\,b_0\,b_4 - 3\,b_1\,b_3 + {b_2}^2,\\
\bar{\Delta}_1 &= -72\,b_0\,b_2\,b_4 + 27\,b_0\,{b_3}^2 + 27\,{b_1}^2 b_4 - 9\,b_{1\ 2}\,b_3 + 2\,{b_2}^3.
\end{aligned}
\tag{A-6}
$$

Appendix B: Components of $\mathbf{L_M}$ and $\mathbf{L_N}$ in Equations (26)

$$\mathbf{L}_M = \begin{bmatrix} -2\,r_0\,g_{M0}\,(g_{M0}-1) & -2\,{g_{M0}}^2\theta_0+(2\,\theta_0+2)\,g_{M0}+\alpha^2-2 & (-4\,r_0\,g_{M0}+2\,r_0)\,\theta_0+2\,r_0 & 0 \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \tag{B-1}$$

$$\mathbf{L}_N = \begin{bmatrix} -2\,r_0\,g_{N0}\,(g_{N0}-1) & -2\,{g_{N0}}^2\theta_0+(2\,\theta_0+2)\,g_{N0}+\alpha^2-2 & 0 & (-4\,r_0\,g_{N0}+2\,r_0)\,\theta_0+2\,r_0 \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \tag{B-2}$$

$$\begin{aligned} L_{21} &= -4\,{g_{M0}}^5\theta_0+\left(-12\,{r_0}^2{\theta_0}^2+16\,\theta_0+2\right)\,{g_{M0}}^4+ \\ &\left[-8-8\,{r_0}^4{\theta_0}^3+24\,{r_0}^2{\theta_0}^2+(8\,{r_0}^2-24)\,\theta_0\right]\,{g_{M0}}^3+ \\ &\left\{(6\,{r_0}^4-12\,{r_0}^2)\,{\theta_0}^2+[16+(4\,\alpha^2-16)\,{r_0}^2]\,\theta_0-2\,\alpha^2+12\right\}\,{g_{M0}}^2+ \\ &(8\,{r_0}^2\theta_0+4\,\alpha^2-4\,\theta_0-8)\,g_{M0}+(\alpha^4-2\,\alpha^2)\,{r_0}^2-2\,\alpha^2+2, \end{aligned}$$

$$\begin{aligned} L_{22} &= -8\,{g_{M0}}^3{r_0}^3{\theta_0}^4-\left[(8\,{g_{M0}}^2-4\,{r_0}^2-16\,g_{M0}+8)\,g{M_0}^2r_0-4\,{g_{M0}}^2{r_0}^3\right]\,{\theta_0}^3 \\ &-2\,g_{M0}\,\left[-4\,{g_{M0}}^2r_0+(-2\,\alpha^2+8)\,r_0\,g_{M0}-4\,r_0\right]\,{\theta_0}^2+ \\ &(2\,\alpha^4-4\,\alpha^2)\,r_0\,\theta_0, \end{aligned}$$

$$\begin{aligned} L_{23} &= -6\,{g_{M0}}^2{r_0}^4{\theta_0}^4-[(8\,{g_{M0}}^2-4\,{r_0}^2-16\,g_{M0}+8)\,gM_0+(8\,g_{M0}-8)\,{g_{M0}}^2]\,{r_0}^2{\theta_0}^3- \\ &\left\{\begin{array}{c} 2\,g_{M0}\,\left[4\,g{M_0}^3-12\,{g_{M0}}^2+(-4\,{r_0}^2+12)\,g_{M0}-4+(-\alpha^2+4)\,{r_0}^2\right]+ \\ 2\,{g_{M0}}^4-8\,{g_{M0}}^3+2\,(-2\,{r_0}^2+6)\,{g_{M0}}^2+ \\ 2\,(-4+(-\alpha^2+4)\,{r_0}^2)\,g_{M0}+2-4\,{r_0}^2 \end{array}\right\}{\theta_0}^2+ \\ &(8\,{g_{M0}}^3-24\,{g_{M0}}^2+(-4\,\alpha^2+24)\,g_{M0}+4\,\alpha^2-8)\,\theta_0+2\,\alpha^2\,(g_{M0}-1)+2\,\alpha^2\,(\alpha^2+g_{M0}-1)\,, \end{aligned}$$

$$L_{24} = 0,$$

$$\begin{aligned} L_{31} &= -8\,{g_{N0}}^3{r_0}^4{\theta_0}^3-(12\,{g_{N0}}^2-6\,{r_0}^2-24\,g_{N0}+12)\,{g_{N0}}^2{r_0}^2{\theta_0}^2- \\ &[4\,{g_{N0}}^4-16\,{g_{N0}}^3+4\,(-2\,{r_0}^2+6)\,{g_{N0}}^2+4\,(-4+(-\alpha^2+4)\,{r_0}^2)\,g_{N0}+4-8\,{r_0}^2]\,g_{N0}\,\theta_0+ \\ &2\,{g_{N0}}^4-8\,{g_{N0}}^3+(-2\,\alpha^2+12)\,{g_{N0}}^2+(4\,\alpha^2-8)\,gN_0+(\alpha^4-2\,\alpha^2)\,{r_0}^2+2-2\,\alpha^2, \end{aligned}$$

$$\begin{aligned} L_{32} &= -8\,{g_{N0}}^3{r_0}^3{\theta_0}^4-[(8\,{g_{N0}}^2-4\,{r_0}^2-16\,g_{N0}+8)\,{g_{N0}}^2r_0-4\,{r_0}^3{g_{N0}}^2]\,{\theta_0}^3- \\ &2\,g_{N0}\,[-4\,r_0\,{g_{N0}}^2+(-2\,\alpha^2+8)\,r_0\,g_{N0}-4\,r_0]\,{\theta_0}^2+(2\,\alpha^4-4\,\alpha^2)\,r_0\,\theta_0, \end{aligned}$$

$$L_{33} = 0,$$

$$\begin{aligned} L_{34} &= -6\,{g_{N0}}^2{r_0}^4{\theta_0}^4-((8\,{g_{N0}}^2-4\,{r_0}^2-16\,g_{N0}+8)\,g_{N0}+(8\,g_{N0}-8)\,{g_{N0}}^2)\,{r_0}^2{\theta_0}^3- \\ &\left\{\begin{array}{c} 2\,g_{N0}\,\left[4\,{N_{N0}}^3-12\,{g_{N0}}^2+(-4\,{r_0}^2+12)\,g_{N0}-4+(-\alpha^2+4)\,{r_0}^2\right]+ \\ 2\,{g_{N0}}^4-8\,{g_{N0}}^3+2\,(-2\,{r_0}^2+6)\,{g_{N0}}^2+ \\ 2\,[-4+(-\alpha^2+4)\,{r_0}^2]\,g_{N0}+2-4\,{r_0}^2 \end{array}\right\}{\theta_0}^2+ \\ &(8\,{g_{N0}}^3-24\,{g_{N0}}^2+(-4\,\alpha^2+24)\,g_{N0}+4\,\alpha^2-8)\,\theta_0+2\,\alpha^2\,(g_{N0}-1)+ \\ &2\,\alpha^2\,(\alpha^2+g_{N0}-1)\,, \end{aligned}$$

$$
\begin{aligned}
L_{41} \quad &= 4\, {r_0}^4 g_{M0}\, g_{N0}\, \left(g_{M0} + gN_0\right) {\theta_0}^3 + \\
&\left\{ \begin{array}{c} 3\, g_{N0}\, {g_{M0}}^3 + 3\, \left(2\, {g_{N0}}^2 - 4\, g_{N0} + 1\right) {g_{M0}}^2 \\ +3\, g_{N0}\, \left({g_{N0}}^2 - 2\, {r_0}^2 - 4\, g_{N0} + 2\right) g_{M0} + 3\, {g_{N0}}^2 \end{array} \right\} {r_0}^2 {\theta_0}^2 + \\
&\left\{ \begin{array}{c} 2\, \left(g_{N0} - 1\right)^2 {g_{M0}}^3 + 2\, \left[{g_{N0}}^3 - 4\, {g_{N0}}^2 - 2 + \left(-2\, {r_0}^2 + 5\right) g_{N0}\right] {g_{M0}}^2 + \\ 2\, \left(-2\, {g_{N0}}^3 + \left(-2\, {r_0}^2 + 5\right) {g_{N0}}^2 + \left(-4 + \left(-2\, \alpha^2 + 8\right) {r_0}^2\right) g_{N0} + 1 - 2\, {r_0}^2\right) g_{M0} + \\ 2\, g_{N0}\, \left({g_{N0}}^2 - 2\, {r_0}^2 - 2\, g_{N0} + 1\right) \end{array} \right\} \theta_0 \\
&-2\, \left(g_{N0} - 1\right)^2 {g_{M0}}^2 + \left[4\, {g_{N0}}^2 + \left(2\, \alpha^2 - 8\right) g_{N0} + 4 - 2\, \alpha^2\right] g_{M0} - \\
&2\, {g_{N0}}^2 + \left(-2\, \alpha^2 + 4\right) g_{N0} + \left(-\alpha^4 + 2\, \alpha^2\right) {r_0}^2 - 2 + 2\, \alpha^2,
\end{aligned}
$$

$$
\begin{aligned}
L_{42} \quad &= 4\, g_{M0}\, {r_0}^3 g_{N0}\, \left(g_{M0} + g_{N0}\right) {\theta_0}^4 + \\
&\left\{ \left[\begin{array}{c} 2\, g_{N0}\, {g_{M0}}^3 + 2\, \left(2\, {g_{N0}}^2 - 4\, g_{N0} + 1\right) {g_{M0}}^2 + \\ 2\, g_{N0}\, \left({g_{N0}}^2 - 2\, {r_0}^2 - 4\, g_{N0} + 2\right) g_{M0} + 2\, {g_{N0}}^2 \end{array} \right] r_0 - 4\, g_{M0}\, {r_0}^3 g_{N0} \right\} {\theta_0}^3 + \\
&\left(-4\, g_{N0}\, r_0\, {g_{M0}}^2 + \left[-4\, r_0\, {g_{N0}}^2 + \left(-4\, \alpha^2 + 16\right) r_0\, g_{N0} - 4\, r_0\right] g_{M0} - 4\, g_{N0}\, r_0\right) {\theta_0}^2 + \\
&\left(-2\, \alpha^4 + 4\, \alpha^2\right) r_0\, \theta_0,
\end{aligned}
$$

$$
\begin{aligned}
L_{43} \quad &= \left[{r_0}^4 g_{M0}\, g_{N0} + {r_0}^4 g_{N0}\, \left(g_{M0} + g_{N0}\right)\right] {\theta_0}^4 + \\
&\left[3\, g_{N0}\, {g_{M0}}^2 + \left(4\, {g_{N0}}^2 - 8\, g_{N0} + 2\right) g_{M0} + g_{N0}\, \left({g_{N0}}^2 - 2\, {r_0}^2 - 4\, g_{N0} + 2\right)\right] {r_0}^2 {\theta_0}^3 + \\
&\left[\begin{array}{c} 3\, \left(g_{N0} - 1\right)^2 {g_{M0}}^2 + \left[2\, {g_{N0}}^3 - 8\, {g_{N0}}^2 - 4 + 2\, \left(-2\, {r_0}^2 + 5\right) g_{N0}\right] g_{M0} - \\ 2\, {g_{N0}}^3 + \left(-2\, {r_0}^2 + 5\right) {g_{N0}}^2 + \left(-4 + \left(-2\, \alpha^2 + 8\right) {r_0}^2\right) g_{N0} + 1 - 2\, {r_0}^2 \end{array} \right] {\theta_0}^2 + \\
&\left[-4\, \left(g_{N0} - 1\right)^2 g_{M0} + 4\, {g_{N0}}^2 + \left(2\, \alpha^2 - 8\right) g_{N0} + 4 - 2\, \alpha^2\right] \theta_0 - 2\, \alpha^2 \left(\tfrac{1}{2}\, \alpha^2 - 1 + g_{N0}\right),
\end{aligned}
$$

$$
\begin{aligned}
L_{44} \quad &= \left[{r_0}^4 g_{M0}\, g_{N0} + {r_0}^4 g_{M0}\, \left(g_{M0} + g_{N0}\right)\right] {\theta_0}^4 + \\
&\left\{ \begin{array}{c} {g_{M0}}^3 + \left(4\, g_{N0} - 4\right) {g_{M0}}^2 + \\ \left[g_{N0}\, \left(2\, g_{N0} - 4\right) + {g_{N0}}^2 - 2\, {r_0}^2 - 4\, g_{N0} + 2\right] g_{M0} + 2\, g_{N0} \end{array} \right\} {r_0}^2 {\theta_0}^3 + \\
&\left\{ \begin{array}{c} \left(2\, g_{N0} - 2\right) {g_{M0}}^3 + \left(3\, {g_{N0}}^2 - 2\, {r_0}^2 - 8\, g_{N0} + 5\right) {g_{M0}}^2 + \\ \left[-6\, {g_{N0}}^2 + \left(-4\, {r_0}^2 + 10\right) g_{N0} - 4 + \left(-2\, \alpha^2 + 8\right) {r_0}^2\right] g_{M0} + g_{N0}\, \left(2\, g_{N0} - 2\right) + \\ {g_{N0}}^2 - 2\, {r_0}^2 - 2\, g_{N0} + 1 \end{array} \right\} {\theta_0}^2 + \\
&\left[\left(-4\, g_{N0} + 4\right) {g_{M0}}^2 + \left(2\, \alpha^2 + 8\, g_{N0} - 8\right) g_{M0} - 4\, g_{N0} - 2\, \alpha^2 + 4\right] \theta_0 + \\
&-2\, \alpha^2 \left(g_{M0} + \tfrac{1}{2}\, \alpha^2 - 1\right).
\end{aligned}
\tag{B-3}
$$

Appendix C: Components of $\mathbf{F}_{2M}$ and $\mathbf{F}_{2N}$ in Equations (26)

$$
\begin{aligned}
F_2(2) \;&= 2\,{g_{M0}}^5{\theta_1}^2+\left[4\,{r_1}^2{\theta_0}^3+24\,r_0\,r_1\,\theta_1\,{\theta_0}^2+\left(12\,{r_0}^2{\theta_1}^2+20\,g_{M1}\,\theta_1\right)\theta_0+\theta_1\,\left(\mu^2-8\,\theta_1\right)\right]{g_{M0}}^4+\\
&\left\{\begin{array}{c}
12\,{r_0}^2{r_1}^2{\theta_0}^4+32\,\left({r_0}^3\theta_1+r_0\,g_{M1}-\frac{r_1}{4}\right)r_1\,{\theta_0}^3+\\
\left[12\,{r_0}^4{\theta_1}^2+48\,{r_0}^2g_{M1}\,\theta_1+4\,r_1\,\left(\mu^2-12\,\theta_1\right)r_0-4\,{r_1}^2+20\,{g_{M1}}^2\right]{\theta_0}^2+\\
\left[\left(4\,\mu^2\theta_1-24\,{\theta_1}^2\right){r_0}^2-16\,r_0\,r_1\,\theta_1+4\,g_{M1}\,\left(\mu^2-16\,\theta_1\right)\right]\theta_0+\\
-4\,\theta_1\,\left({r_0}^2\theta_1+\mu^2+2\,g_{M1}-3\,\theta_1\right)
\end{array}\right\}{g_{M0}}^3+\\
&\left\{\begin{array}{c}
24\,{r_0}^3r_1\,g_{M1}\,{\theta_0}^4+\\
\left[\begin{array}{c}
24\,g_{M1}\,\theta_1\,{r_0}^4+\\
4\,\mu^2{r_0}^3r_1+\left(24\,{g_{M1}}^2-12\,{r_1}^2\right){r_0}^2+\\
-48\,r_1\,g_{M1}\,r_0+4\,{r_1}^2
\end{array}\right]{\theta_0}^3+\\
\left[\begin{array}{c}
3\,\mu^2{r_0}^4\theta_1-24\,{r_0}^3r_1\,\theta_1+\\
6\,g_{M1}\,\left(\mu^2-12\,\theta_1\right){r_0}^2+\\
-8\,r_1\,\left(\mu^2+3\,g_{M1}-3\,\theta_1\right)r_0-48\,{g_{M1}}^2+\\
\left(-2\,\alpha^2+8\right){r_1}^2
\end{array}\right]{\theta_0}^2+\\
\left[\begin{array}{c}
-6\,{r_0}^4{\theta_1}^2-8\,\left[\mu^2+3\,g_{M1}-\frac{3}{2}\,\theta_1\right]\theta_1\,{r_0}^2+\\
\left(-8\,\alpha^2+32\right)r_1\,\theta_1\,r_0+\\
-2\,\alpha^2\mu\,r_1-12\,\mu^2g_{M1}+\\
-12\,{g_{M1}}^2+72\,g_{M1}\,\theta_1
\end{array}\right]\theta_0+\\
6\,\theta_1\,\left[\theta_1\,\left(-\frac{1}{3}\,\alpha^2+\frac{4}{3}\right){r_0}^2-\frac{1}{3}\,\alpha^2\mu\,r_0+\mu^2+4\,g_{M1}-\frac{4}{3}\,\theta_1\right]
\end{array}\right\}{g_{M0}}^2+\\
&\left\{\begin{array}{c}
6\,{r_0}^4{g_{M1}}^2{\theta_0}^4+\\
2\,r_0\,g_{M1}\,\left(\mu^2{r_0}^3-8\,{r_0}^2r_1-12\,r_0\,g_{M1}+8\,r_1\right){\theta_0}^3+\\
\left[\begin{array}{c}
-12\,g_{M1}\,\theta_1\,{r_0}^4+\\
-8\,\left(\mu^2+\frac{3}{2}\,g_{M1}-3\,\theta_1\right)g_{M1}\,{r_0}^2+\\
4\,\left[\left(-2\,\alpha^2+8\right)g_{M1}+\mu^2\right]r_1\,r_0+\\
-4\,{r_1}^2+36\,{g_{M1}}^2
\end{array}\right]{\theta_0}^2+\\
\left[\begin{array}{c}
4\,\theta_1\,\left(-2\,\alpha^2g_{M1}+\mu^2+8\,g_{M1}\right){r_0}^2+\\
\left(-4\,\alpha^2g_{M1}\,\mu-16\,r_1\,\theta_1\right)r_0+\\
12\,\left(\mu^2+2\,g_{M1}-\frac{8}{3}\,\theta_1\right)g_{M1}
\end{array}\right]\theta_0+\\
-4\,\theta_1\,\left({r_0}^2\theta_1-\alpha^2g_{M1}+\mu^2+6\,g_{M1}-\theta_1/2\right)
\end{array}\right\}g_{M0}+\\
&-2\,{r_0}^2{g_{M1}}^2\left({r_0}^2-2\right){\theta_0}^3+\\
&2\,\left[\left(-\alpha^2g_{M1}+\mu^2+4\,g_{M1}\right){r_0}^2-4\,r_0\,r_1-4\,g_{M1}\right]g_{M1}\,{\theta_0}^2+\\
&\left\{\begin{array}{c}
-8\,{r_0}^2g_{M1}\,\theta_1+\\
4\,g_{M1}\,\theta_1+\left(2\,\alpha^2-12\right){g_{M1}}^2+\\
-4\,\mu^2g_{M1}+2\,\alpha^2\left[\left(-\frac{1}{2}\,\alpha^2+1\right)r_1+\mu\right]r_1
\end{array}\right\}\theta_0+2\,\alpha^2\theta_1\,\left(-\alpha^2r_1+\mu+2\,r_1\right)r_0+\\
&\left(-4\,\alpha^2g_{M1}+\mu^2+8\,g_{M1}\right)\theta_1-2\,\alpha^2{g_{M1}}^2.
\end{aligned}
\tag{C-1}
$$

$$
\begin{aligned}
F_2(3) = \; & 2\, {g_{N0}}^5 {\theta_1}^2 + \\
& \left[4\, {r_1}^2 {\theta_0}^3 + 24\, r_0\, r_1\, \theta_1\, {\theta_0}^2 + \left(12\, {r_0}^2 {\theta_1}^2 + 20\, g_{N1}\, \theta_1\right) \theta_0 + \theta_1\, \left(\mu^2 - 8\, \theta_1\right)\right] {g_{N0}}^4 + \\
& \left[\begin{array}{c}
12\, {r_0}^2 {r_1}^2 {\theta_0}^4 + 32\, \left({r_0}^3 \theta_1 + r_0\, g_{N1} - \frac{r_1}{4}\right) r_1\, {\theta_0}^3 + \\
\left[12\, {r_0}^4 {\theta_1}^2 + 48\, g_{N1}\, \theta_1\, {r_0}^2 + 4\, r_1\, \left(\mu^2 - 12\, \theta_1\right) r_0 - 4\, {r_1}^2 + 20\, {g_{N1}}^2\right] {\theta_0}^2 + \\
\left[\left(4\, \mu^2 \theta_1 - 24\, {\theta_1}^2\right) {r_0}^2 - 16\, r_0\, r_1\, \theta_1 + 4\, g_{N1}\, \left(\mu^2 - 16\, \theta_1\right)\right] \theta_0 + \\
-4\, \theta_1\, \left({r_0}^2 \theta_1 + \mu^2 + 2\, g_{N1} - 3\, \theta_1\right)
\end{array}\right] {g_{N0}}^3 + \\
& \left\{\begin{array}{c}
24\, {r_0}^3 r_1\, g_{N1}\, {\theta_0}^4 + \left[\begin{array}{c}
24\, g_{N1}\, \theta_1\, {r_0}^4 + 4\, \mu^2 {r_0}^3 r_1 \\
\left(24\, {g_{N1}}^2 - 12\, {r_1}^2\right) {r_0}^2 + \\
-48\, r_1\, g_{N1}\, r_0 + 4\, {r_1}^2
\end{array}\right] {\theta_0}^3 + \\
\left[\begin{array}{c}
3\, \mu^2 {r_0}^4 \theta_1 - 24\, {r_0}^3 r_1\, \theta_1 + \\
6\, g_{N1}\, \left(\mu^2 - 12\, \theta_1\right) {r_0}^2 + \\
-8\, r_1\, \left(\mu^2 + 3\, g_{N1} - 3\, \theta_1\right) r_0 + \\
-48\, {g_{N1}}^2 + \left(-2\, \alpha^2 + 8\right) {r_1}^2
\end{array}\right] {\theta_0}^2 + \\
\left[\begin{array}{c}
-6\, {r_0}^4 {\theta_1}^2 - 8\, \left(\mu^2 + 3\, g_{N1} - \frac{3}{2}\, \theta_1\right) \theta_1\, {r_0}^2 \\
+ \left(-8\, \alpha^2 + 32\right) r_1\, \theta_1\, r_0 \\
-2\, \alpha^2 \mu\, r_1 - 12\, \mu^2 g_{N1} - 12\, {g_{N1}}^2 + 72\, g_{N1}\, \theta_1
\end{array}\right] \theta_0 + \\
6\, \left[\begin{array}{c}
\theta_1\, \left(-\frac{1}{3}\, \alpha^2 + \frac{4}{3}\right) {r_0}^2 + \\
-\frac{1}{3}\, \alpha^2 \mu\, r_0 + \mu^2 + 4\, g_{N1} - \frac{4}{3}\, \theta_1
\end{array}\right] \theta_1
\end{array}\right\} {g_{N0}}^2 + \\
& \left[\begin{array}{c}
6\, {r_0}^4 {g_{N1}}^2 {\theta_0}^4 + \\
2\, r_0\, g_{N1}\, \left(\mu^2 {r_0}^3 - 8\, {r_0}^2 r_1 - 12\, r_0\, g_{N1} + 8\, r_1\right) {\theta_0}^3 + \\
\left\{\begin{array}{c}
-12\, g_{N1}\, \theta_1\, {r_0}^4 - 8\, \left[\mu^2 + \frac{3}{2}\, g_{N1} - 3\, \theta_1\right] g_{N1}\, {r_0}^2 + \\
4\, \left[\left(-2\, \alpha^2 + 8\right) g_{N1} + \mu^2\right] r_1\, r_0 - 4\, {r_1}^2 + 36\, {g_{N1}}^2
\end{array}\right\} {\theta_0}^2 + \\
\left[\begin{array}{c}
4\, \theta_1\, \left(-2\, \alpha^2 g_{N1} + \mu^2 + 8\, g_{N1}\right) {r_0}^2 + \\
\left(-4\, \alpha^2 g_{N1}\, \mu - 16\, r_1\, \theta_1\right) r_0 + \\
12\, \left(\mu^2 + 2\, g_{N1} - \frac{8}{3}\, \theta_1\right) g_{N1}
\end{array}\right] \theta_0 + \\
-4\, \left({r_0}^2 \theta_1 + -\alpha^2 g_{N1} + \mu^2 + 6\, g_{N1} - \theta_1/2\right) \theta_1
\end{array}\right] g_{N0} \\
& -2\, {r_0}^2 {g_{N1}}^2 \left({r_0}^2 - 2\right) {\theta_0}^3 + 2\, \left[\left(-\alpha^2 g_{N1} + \mu^2 + 4\, g_{N1}\right) {r_0}^2 - 4\, r_0\, r_1 - 4\, g_{N1}\right] g_{N1}\, {\theta_0}^2 + \\
& \left[\begin{array}{c}
-8\, g_{N1}\, \theta_1\, {r_0}^2 + 4\, g_{N1}\, \theta_1 + \\
\left(2\, \alpha^2 - 12\right) {g_{N1}}^2 - 4\, \mu^2 g_{N1} + 2\, \alpha^2 \left[\left(-\frac{1}{2}\, \alpha^2 + 1\right) r_1 + \mu\right] r_1
\end{array}\right] \theta_0 \\
& +2\, \alpha^2 \theta_1\, \left(-\alpha^2 r_1 + \mu + 2\, r_1\right) r_0 + \left(-4\, \alpha^2 g_{N1} + \mu^2 + 8\, g_{N1}\right) \theta_1 - 2\, \alpha^2 {g_{N1}}^2 .
\end{aligned}
\tag{C-2}
$$

$$
F_2(4) = \quad -2\, r_0{}^2 \left\{ \begin{array}{c}
\left(3\, g_{N0}\, r_1{}^2 + 2\, r_0\, r_1\, g_{N1}\right) g_{M0}{}^2 + \\
\left[\begin{array}{c}
3\, r_1{}^2 g_{N0}{}^2 + \\
4\, r_0\, r_1\, \left(g_{M1} + g_{N1}\right) g_{N0} + \\
r_0{}^2 g_{N1}\, \left(g_{M1} + g_{N1}/2\right)
\end{array} \right] g_{M0} + \\
\frac{1}{2}\, r_0\, \left[4\, r_1\, g_{N0} + r_0\, \left(g_{M1} + 2\, g_{N1}\right)\right] g_{N0}\, g_{M1}
\end{array} \right\} \theta_0{}^4 +
$$

$$
\left\{ \begin{array}{c}
\left(-g_{N0}\, r_1{}^2 - 2\, r_0\, r_1\, g_{N1}\right) g_{M0}{}^3 + \\
\left[\begin{array}{c}
-2\, r_1{}^2 g_{N0}{}^2 - \\
16\, \left[\theta_1\, r_0{}^3 + \left(3/8\, g_{M1} + g N_1/2\right) r_0 - \frac{r_1}{4}\right] r_1\, g_{N0} + \\
-4\, \theta_1\, r_0{}^4 g_{N1} + \left[-3\, g_{N1}\, g_{M1} - 2\, g_{N1}{}^2\right] r_0{}^2 + \\
8\, r_0\, r_1\, g_{N1} - r_1{}^2
\end{array} \right] g_{M0}{}^2 + \\
\left[\begin{array}{c}
-r_1{}^2 g_{N0}{}^3 - 16\, \left[\theta_1\, r_0{}^3 + \left(g_{M1}/2 + 3/8\, g_{N1}\right) r_0 - \frac{r_1}{4}\right] r_1\, g_{N0}{}^2 + \\
\left[\begin{array}{c}
-8\, \theta_1\, \left(g_{M1} + g_{N1}\right) r_0{}^4 + \\
-4\, \mu^2 r_0{}^3 r_1 + \left[-3\, g_{M1}{}^2 - 8\, g_{N1}\, g_{M1} - 3\, g_{N1}{}^2 + 12\, r_1{}^2\right] r_0{}^2 + \\
16\, r_0\, r_1\, \left[g_{M1} + g_{N1}\right] - 2\, r_1{}^2
\end{array} \right] g_{N0} + \\
-r_0 \left[\begin{array}{c}
\mu^2 r_0{}^3 g_{N1} + \\
-8\, r_1\, g_{N1}\, r_0{}^2 + \left(-8\, g_{N1}\, g_{M1} - 4\, g_{N1}{}^2\right) r_0 + \\
4\, r_1\, \left(g_{M1} + g_{N1}\right)
\end{array} \right]
\end{array} \right] g_{M0} + \\
-2\, r_0\, r_1\, g_{M1}\, g_{N0}{}^3 + \\
\left[-4\, \theta_1\, r_0{}^4 g_{M1} + \left(-2\, g_{M1}{}^2 - 3\, g_{N1}\, g_{M1}\right) r_0{}^2 + 8\, r_0\, r_1\, g_{M1} - r_1{}^2\right] g_{N0}{}^2 + \\
-r_0 \left[\begin{array}{c}
\mu^2 r_0{}^3 g_{M1} + \\
-8\, r_1\, g_{M1}\, r_0{}^2 - 4\, g_{M1}\, \left(g_{M1} + 2\, g_{N1}\right) r_0 + 4\, r_1\, \left(g_{M1} + g_{N1}\right)
\end{array} \right] g_{N0} + \\
2\, r_0{}^2 \left(r_0{}^2 g_{M1}\, g_{N1} - \frac{1}{2}\, \left(g_{M1} + g_{N1}\right)^2\right)
\end{array} \right\} \theta_0{}^3 +
$$

$$
\left\{
\begin{array}{c}
\left[-6\,g_{N0}\,r_0\,r_1\,\theta_1-3\,g_{N1}\,{r_0}^2\theta_1-{g_{N1}}^2\right]{g_{M0}}^3+ \\[2ex]
\left[
\begin{array}{c}
-12\,\theta_1\,r_0\,r_1\,{g_{N0}}^2+ \\
\left[
\begin{array}{c}
-6\,{\theta_1}^2{r_0}^4-9\,\left(g_{M1}+\frac{4}{3}\,g_{N1}\right)\theta_1\,{r_0}^2 \\
-2\,r_1\,\left(\mu^2+-12\,\theta_1\right)r_0+ \\
2\,{r_1}^2-6\,g_{N1}\,g_{M1}-3\,{g_{N1}}^2
\end{array}
\right]g_{N0}+ \\
-g_{N1}\,\left(\mu^2-12\,\theta_1\right){r_0}^2-6\,\left(\theta_1-2/3\,g_{N1}\right)r_1\,r_0+ \\
6\,g_{N1}\,g_{M1}+4\,{g_{N1}}^2
\end{array}
\right]{g_{M0}}^2+ \\[2ex]
\left[
\begin{array}{c}
-6\,\theta_1\,r_0\,r_1\,{g_{N0}}^3+ \\[2ex]
\left[
\begin{array}{c}
-6\,{\theta_1}^2{r_0}^4+ \\
-12\,\left(g_{M1}+\frac{3}{4}\,g_{N1}\right)\theta_1\,{r_0}^2+ \\
-2\,r_1\,\left(\mu^2-12\,\theta_1\right)r_0+ \\
2\,{r_1}^2-3\,{g_{M1}}^2-6\,g_{N1}\,g_{M1}
\end{array}
\right]{g_{N0}}^2+ \\[2ex]
\left[
\begin{array}{c}
-3\,\mu^2\theta_1\,{r_0}^4+24\,\theta_1\,{r_0}^3r_1+ \\
-2\,\left(g_{M1}+g_{N1}\right)\left(\mu^2-12\,\theta_1\right){r_0}^2+ \\
8\,\left(\mu^2-\frac{3}{2}\,\theta_1+g_{M1}+g_{N1}\right)r_1\,r_0+ \\
6\,{g_{M1}}^2+16\,g_{N1}\,g_{M1}+6\,{g_{N1}}^2+ \\
\left(2\,\alpha^2-8\right){r_1}^2
\end{array}
\right]g_{N0}+ \\[2ex]
6\,\theta_1\,{r_0}^4g_{N1}+\left[\left(-6\,g_{M1}-6\,gN_1\right)\theta_1+4\,\left(\mu^2+g_{M1}+g_{N1}/2\right)g_{N1}\right]{r_0}^2+ \\[2ex]
-2\,\left[\left(-2\,\alpha^2+8\right)g_{N1}+\mu^2\right]r_1\,r_0+2\,{r_1}^2-3\,{g_{M1}}^2-10\,g_{N1}\,g_{M1}-5\,{g_{N1}}^2
\end{array}
\right]g_{M0}+ \\[2ex]
\left(-3\,g_{M1}\,{r_0}^2\theta_1-{g_{M1}}^2\right){g_{N0}}^3+ \\
\left[-g_{M1}\,\left(\mu^2-12\,\theta_1\right){r_0}^2-6\,\left(\theta_1-2/3\,g_{M1}\right)r_1\,r_0+4\,{g_{M1}}^2+6\,g_{N1}\,g_{M1}\right]{g_{N0}}^2+ \\[2ex]
\left[
\begin{array}{c}
6\,\theta_1\,{r_0}^4g_{M1}+\left[\left(-6\,g_{M1}-6\,g_{N1}\right)\theta_1+4\,\left(\mu^2+g_{M1}/2+g_{N1}\right)g_{M1}\right]{r_0}^2+ \\[2ex]
-2\,r_1\,\left(\left(-2\,\alpha^2+8\right)g_{M1}+\mu^2\right)r_0+2\,{r_1}^2-5\,{g_{M1}}^2-10\,g_{N1}\,g_{M1}-3\,{g_{N1}}^2
\end{array}
\right]g_{N0}+ \\[2ex]
\left[\left(2\,\alpha^2g_{N1}-\mu^2-8\,g_{N1}\right)g_{M1}-\mu^2g_{N1}\right]{r_0}^2+4\,r_0\,r_1\,\left(g_{M1}+g_{N1}\right)+2\,\left(g_{M1}+g_{N1}\right)^2
\end{array}
\right\}{\theta_0}^2+
$$

$$
\left\{
\begin{array}{c}
-3\,\theta_1\,\left[\left(\theta_1\,{r_0}^2+\frac{4}{3}\,g_{N1}\right)g_{N0}-\frac{4}{3}\,g_{N1}\right]{g_{M0}}^3+\\[2ex]
\left[
\begin{array}{c}
-6\,\theta_1\,\left(\theta_1\,{r_0}^2+g_{M1}+g_{N1}\right){g_{N0}}^2+\\[2ex]
\left[
\begin{array}{c}
\left(-2\,\mu^2\theta_1+12\,{\theta_1}^2\right){r_0}^2+\\
8\,\theta_1\,r_0\,r_1+\left(12\,gM_1+16\,g_{N1}\right)\theta_1-2\,\mu^2 g_{N1}
\end{array}
\right]g_{N0}+\\[2ex]
\left(4\,g_{N1}\,\theta_1-3\,{\theta_1}^2\right){r_0}^2+\\[2ex]
\left(-6\,g_{M1}-10\,g_{N1}\right)\theta_1+2\,\mu^2 g_{N1}+2\,{g_{N1}}^2
\end{array}
\right]{g_{M0}}^2+\\[2ex]
\left\{
\begin{array}{c}
\left(-3\,{r_0}^2{\theta_1}^2-4\,g_{M1}\,\theta_1\right){g_{N0}}^3+\\[2ex]
\left[\left(-2\,\mu^2\theta_1+12\,{\theta_1}^2\right){r_0}^2+8\,\theta_1\,r_0\,r_1+\left(16\,g_{M1}+12\,g_{N1}\right)\theta_1-2\,\mu^2 g_{M1}\right]{g_{N0}}^2+\\[2ex]
\left[
\begin{array}{c}
6\,{\theta_1}^2{r_0}^4+\\
8\,\left(\mu^2-\frac{3}{4}\,\theta_1+g_{M1}+g_{N1}\right)\theta_1\,{r_0}^2+\\[2ex]
8\,\theta_1\,r_1\,\left(\alpha-2\right)\left(\alpha+2\right)r_0+\\[2ex]
\left(-20\,g_{M1}-20\,g_{N1}\right)\theta_1+\\[2ex]
\left(4\,\mu^2+8\,g_{N1}\right)g_{M1}+\\[2ex]
2\,\alpha^2\mu\,r_1+4\,\mu^2 g_{N1}
\end{array}
\right]g_{N0}+\\[2ex]
-2\,\theta_1\,\left(-2\,\alpha^2 g_{N1}+\mu^2+8\,g_{N1}\right){r_0}^2+\\[2ex]
\left(2\,\alpha^2 g_{N1}\,\mu+8\,r_1\,\theta_1\right)r_0+\\[2ex]
\left(8\,g_{M1}+8\,g_{N1}\right)\theta_1+\left(-2\,\mu^2-8\,g_{N1}\right)g_{M1}+\\[2ex]
-4\,\mu^2 g_{N1}-4\,{g_{N1}}^2
\end{array}
\right\}g_{M0}+\\[2ex]
4\,\theta_1\,g_{M1}\,{g_{N0}}^3+\left[
\begin{array}{c}
\left(4\,g_{M1}\,\theta_1-3\,{\theta_1}^2\right){r_0}^2+\\[2ex]
\left(-10\,g_{M1}-6\,g_{N1}\right)\theta_1+2\,\mu^2 g_{M1}+2\,{g_{M1}}^2
\end{array}
\right]{g_{N0}}^2+\\[2ex]
\left[
\begin{array}{c}
-2\,\theta_1\,\left(-2\,\alpha^2 g_{M1}+\mu^2+8\,g_{M1}\right){r_0}^2+\\
\left(2\,\alpha^2 g_{M1}\,\mu+8\,r_1\,\theta_1\right)r_0+\\
\left(8\,g_{M1}+8\,g_{N1}\right)\theta_1+\\
-4\,{g_{M1}}^2+\left(-4\,\mu^2-8\,g_{N1}\right)g_{M1}-2\,\mu^2 g_{N1}
\end{array}
\right]g_{N0}+4\,\theta_1\,\left(g_{M1}+g_{N1}\right){r_0}^2+\\[2ex]
\left(-2\,g_{M1}-2\,g_{N1}\right)\theta_1+2\,{g_1}^{M2}+\left(-2\,\alpha^2 g_{N1}+2\,\mu^2+8\,g_{N1}\right)g_{M1}\\[2ex]
+2\,{g_{N1}}^2+2\,\mu^2 g_{N1}-2\,\left[\left(-\frac{1}{2}\,\alpha^2+1\right)r_1+\mu\right]\alpha^2 r_1
\end{array}
\right\}\theta_0+
$$

$$
\begin{aligned}
&-{\theta_1}^2\,(g_{N0}-1)^2\,{g_{M0}}^3-\\
&\left[\begin{array}{c}\theta_1\,{g_{N0}}^3+[\mu^2-4\,\theta_1]\,{g_{N0}}^2+\\ [-2\,\theta_1\,{r_0}^2-2\,\mu^2-4\,g_{N1}+5\,\theta_1]\,g_{N0}+\\ \mu^2-2\,\theta_1+4\,g_{N1}\end{array}\right]\theta_1\,{g_{M0}}^2+\\
&2\left[\begin{array}{c}\theta_1\,{g_{N0}}^3+\\ \left(\theta_1\,{r_0}^2+\mu^2-\frac{5}{2}\,\theta_1+2\,g_{M1}\right)\,{g_{N0}}^2+\\ [\theta_1\,\left(\alpha^2-4\right){r_0}^2+\alpha^2\mu\,r_0-2\,\mu^2+2\,\theta_1-4\,g_{M1}-4\,g_{N1}]\,g_{N0}+\\ \theta_1\,{r_0}^2-\alpha^2 g_{N1}+\\ \mu^2-\theta_1/2+2\,g_{M1}+4\,g_{N1}\end{array}\right]\theta_1\,g_{M0}+\\
&-{\theta_1}^2{g_{N0}}^3-\theta_1\,\left(\mu^2+4\,g_{M1}-2\,\theta_1\right)\,{g_{N0}}^2+\\
&2\,\left[\theta_1\,{r_0}^2-\alpha^2 g_{M1}+\mu^2-\theta_1/2+4\,g_{M1}+2\,g_{N1}\right]\theta_1\,g_{N0}+-2\,\theta_1\,\alpha^2\,(-\alpha^2 r_1+\mu+2\,r_1)\,r_0+\\
&(2\,\alpha^2 g_{M1}+2\,\alpha^2 g_{N1}-\mu^2-4\,g_{M1}-4\,g_{N1})\,\theta_1+2\,g_{N1}\,g_{M1}\,\alpha^2.
\end{aligned}
\tag{C-3}
$$