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Classical phase transitions in a one-dimensional short-range spin model

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Abstract. Ising's solution of a classical spin model famously demonstrated the absence of a positive-temperature phase transition in one-dimensional equilibrium systems with short-range interactions. No-go arguments established that the energy cost to insert domain walls in such systems is outweighed by entropy excess so that symmetry cannot be spontaneously broken. An archetypal way around the no-go theorems is to augment interaction energy by increasing the range of interaction. Here we introduce new ways around the no-go theorems by investigating entropy depletion instead. We implement this for the Potts model with invisible states. Because spins in such a state do not interact with their surroundings, they contribute to the entropy but not the interaction energy of the system. Reducing the number of invisible states to a negative value decreases the entropy by an amount sufficient to induce a positive-temperature classical phase transition. This approach is complementary to the long-range interaction mechanism. Alternatively, subjecting positive numbers of invisible states to imaginary or complex fields can trigger such a phase transition. We also discuss potential physical realisability of such systems.

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1. Introduction

As is widely known, in his famous 1925 paper [1], and following a suggestion by Wilhelm Lenz, Ernst Ising sought a positive-temperature phase transition in a one-dimensional (1D) classical equilibrium system with short-range interactions [2]. To some disappointment [3], there was none. This was the start of a vast amount of literature on the statistical mechanics of critical phenomena, including a number of studies on why it is impossible to have a phase transition in such systems [4, 6–8]. The lower critical dimension is now defined as that below which a phase transition cannot occur at positive temperature and at least two physical dimensions are required for many short-range classical equilibrium models.

Landau and Lifschitz gave heuristic arguments suggesting that entropic excesses prevent phase transitions below the upper critical dimension [4]; later similar reasoning was given rigorously by Simon and Sokal [5]; van Hove's approach was based on proofs of analyticity of the transfer matrix eigenvalues and free energy [6]; Ruelle extended this giving rigorous theorems [7] and, more recently, Cuesta and Sánchez [8] presented more general results about the non-existence of phase transitions in 1D short-ranged systems. For such classical, equilibrium models with short range interactions in 1D second-order phase-transition type phenomena can only occur at zero temperature. The essence of early no-go arguments is that there is an entropy excess in 1D systems relative to interaction energy so that the delicate balance that gives a phase transition is not achieved. The role played by domain walls was further investigated in Ref. [9].

To escape the limitations of no-go theorems, interactions with sufficiently long range can be introduced [10–12]. Another way out is provided by non-equilibrium systems [13] and further exceptions are discussed in Ref. [8]. In particular one can consider models with complex couplings [14–16]. In Ref. [17] we introduced a new way around the no-go theorems involving models with a negative number of invisible states or complex fields acting on them. Here we develop this circumvention through a Lee-Yang-zero analysis [18], an approach described as fundamental to the theory of phase transitions [19]. We also discuss potential physical realisations in real-world systems.

The Potts model with invisible states was introduced nearly a decade ago in Refs. [20, 21] to explain discrepancies between theoretical predictions and experimental observations of phase transitions in some two-dimensional systems where the Z_3-symmetry is spontaneously broken [22]. It differs from the ordinary Potts model [23] in that spins in an invisible state do not interact with their neighbours but they do contribute to entropy. The corresponding Hamiltonian takes the form

$$H = -\sum_{<i,j>} \delta_{s_i,s_j} \sum_{\alpha=1}^{q} \delta_{s_i,\alpha}\delta_{s_j,\alpha}, \tag{1}$$

where q and r are the numbers of visible and invisible states respectively, $s_i = 1, \dots, q, q+1, \dots, q+r$ is the Potts variable and δ is the Kronecker delta symbol. The first sum in Eq. (1) is taken over all distinct pairs of interacting particles, and the second sum requires both of the interacting spins to be in the same visible state. We henceforth use the term "(q, r)-state Potts model" for systems with q visible and r invisible states.

The usual concept of universality means that critical behaviour is determined by dimensionality, the range of the interaction and the symmetries of the system. Although the number of invisible states r does not change any of these properties, it was shown to control the order of the phase transition in Refs. [20, 21, 24]. For example although the two-dimensional $(2, 0)-$state Potts model (which is the ordinary Ising model) is the archetypal example of a continuous phase transition, the model with $(2, 30)$ states undergoes a first-order transition.

We confirm that the Potts model with a positive number of invisible states adheres to the no-go theorems in one dimension in that the only possibility for a phase transition is at zero temperature. However, if external magnetic field is allowed to be complex we obtain positive temperature phase transitions [14–16]. The same phenomenon can be achieved through introducing a negative number of invisible states. Although some of these concepts are unphysical in and of themselves, they can be linked with physicality in a number of interesting ways.

Long thought to be entirely mathematical constructs whose roles, while important, are restricted only to fundamental theories that underlie phase transitions, the notion of complex magnetic fields has recently gained physical traction too. Following on from an earlier theoretical proposal [25], Peng et. al. have shown that complex fields associated with a spin bath are related to the quantum coherence of a probe spin coupled to the bath [26]. The results demonstrate that the times at which quantum coherence reaches zero are equivalent to the complex values of magnetic fields at which the partition function vanishes: i.e., Lee-Yang zeros [18]. The zeros method is considered of fundamental importance in understanding phase transitions [19] and a powerful tool to analyse critical behaviour [27]. The Lee-Yang zeros give direct access to the partition function itself and as such give important information on the nature of phase transitions. Very recently exact results on the classical antiferromagnetic Ising chain in a magnetic field showed an infinite cascade of thermal phase transitions, the origins of which were traced to the lines of the Lee-Yang zeros, opening a way to relate to observable and potentially measurable quantities [28].

Low-dimensional models are of continued theoretical and physical interest [8]. A new combinatorial approach was used to solve Ising's model in Ref. [29] and it was suggested that the method could be applied to the 2D problem. The first experimental verification of Onsager's 1943 solution of the 2D Ising model [30] is also a very recent development, offering a "promising candidate for numerous applications" [31]. This recent experimental advance inspired the question "are we going to see a transition ... in a chain of ... molecules (1D)" [32]. This is the question addressed in this paper.

For these reasons, we analyse 1D models with invisible states using Lee-Yang zeros. There have been other approaches to access phase transitions in 1D models. Following on from suggestions by Anderson [10], Dyson [11] proved that systems with long range order can have positive-temperature phase transitions and Fröhlich and Spencer proved the existence of a spontaneous magnetization at positive low temperature for the one-dimensional Ising model with long-range interactions. Later, Asorey and collaborators showed that in 1D short range models with complex values of the interaction constant, phase transitions at positive temperatures are possible [14–16]. Cuesta and Sánchez gave three further examples of phase transitions, both of purely academic interest and with importance for phenomena such as surface growth and DNA denaturation [8]. For quantum phase transitions the critical dimensionality is also reduced relative to the corresponding classical transition [33]. Some chemical compounds are well described by quasi one dimensional models [34–39]. Here we are interested in pure 1D equilibrium models which are both classical and short-range.

The Potts model with invisible states describes a number of models of physical interest. Notably, the $(1, r)$-state case is equivalent to the Ising model in a temperature dependent field and can be mapped to the Zimm-Bregg model for the helix-coil transition [40]. The long-range extension of the $(1, r)-$state model possesses a reentrant phase transition and is in good agreement with experimental observations for polymer transitions [41]. The $(2, r)-$state Potts model without external fields is equivalent to the Blume-Emery-Grifiths model [20,42,43]. The general q and r case can be interpreted as a diluted Potts model [20, 24].

In Section 2 we present the exact solution for the 1D Potts model with invisible states using the well-known transfer matrix approach [44–51]. Some results from such well-established material were presented in the letter [17] where the focus was on Fisher-zeros [52] (zeros in the complex temperature plane) and we elaborate on these in Appendix A. Here our focus is on Lee-Yang zeros (zeros in the complex magnetic-field plane). We present a duality relation between field and temperature in Section 3. In Section 4 we investigate the Lee-Yang zeros for the ordinary Potts model and for its counterpart with a positive number of invisible states. Singular behaviour of the Lee-Yang zeros is discussed in Appendix B. In Section 5 we relax the constraints involved in conventional studies by allowing the number of invisible states to be negative and/or the magnetic fields to be complex. These enable positive-temperature phase transitions to be achieved — of the type sought by Ising nearly 100 years ago. We draw our conclusions in Section 6.

2. Potts model with invisible states

We consider the Potts model with invisible states [the $(q, r)-$Potts model] with nearest-neighbour interactions on a 1D chain of N spins with periodic boundary conditions. The partition function is

$$Z = \sum_s \exp\left(-\beta H_{(q,r)}\right), \tag{2}$$

where $\sum\limits_s$ denotes the sum over all possible spin configurations. With periodic boundary conditions the Hamiltonian can be rewritten as a sum of terms representing one bond each, namely

$$H_{(q,r)} = \sum_i H_i, \quad \text{where} \quad H_i = -\delta_{s_i,s_{i+1}} \sum_{\alpha=1}^{q} \delta_{s_i,\alpha} - h_1\delta_{s_i,1} - h_2\delta_{s_i,q+1}, \tag{3}$$

where the variable i spans the N sites of the chain, $s_i = 1, \ldots, q, q+1, \ldots, q+r$ is a Potts variable and h_1 and h_2 are two ordering fields acting on the first visible and first invisible states respectively, so that

$$Z = \sum_s \prod_i \exp\left(-\beta H_i\right). \tag{4}$$

The final term in Eq. (3) selects only one of the r invisible states as interacting with the external field h_2. As such, it contributes to the energy if $h_2 \neq 0$. The $r-1$ remaining identical invisible states contribute only to the entropy, as do all invisible states if h_2 vanishes. This means that different microscopic configurations could be understood as the same macroscopic configuration. In terms of the partition function the effect is to multiply some of the terms by $(r-1)$. Similarly, as was done in Ref. [20], we can collect all invisible states into a single one with appropriate weight and consider the equivalent Hamiltonian of a diluted Potts model:

$$H^{eq}_{(q,r)} = -\sum_i \delta_{\sigma_i,\sigma_{i+1}} \sum_{\alpha=1}^{q} \delta_{\sigma_i,\alpha} - h_1 \sum_i \delta_{\sigma_i,1} - h_2 \sum_i \delta_{\sigma_i,q+1} - T\ln(r-1)\sum_i \delta_{\sigma_i,q+2}, \tag{5}$$

where $\sigma_i = 1,\ldots,q,q+1,q+2$ is a new Potts variable and all (except the one along the field h_2) the invisible states are gathered into one with the appropriate weight. The Hamiltonian (5) is, of course, different to that in Eq. (3). But the corresponding partition functions are the same.

2.1. Transfer matrix

To develop the formalism to solve the model (3) exactly, we define the transfer matrix $\mathbf{T} = \mathbf{T}(s_i, s_j)$ as [44–47]

$$\mathbf{T}(s_i, s_j) = \exp\left[\beta(\delta_{s_i,s_j} \sum_{\alpha=1}^{q} \delta_{s_i,\alpha} + h_1\delta_{s_i,1} + h_2\delta_{s_i,q+1})\right], \tag{6}$$

so that, in the explicit form of a $(q+r)\times(q+r)$ matrix,

$$\mathbf{T} = \begin{pmatrix} yz_1 & 1 & 1 & \cdots & 1 & z_2 & 1 & \cdots & 1 \\ z_1 & y & 1 & \cdots & 1 & z_2 & 1 & \cdots & 1 \\ z_1 & 1 & y & \cdots & 1 & z_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & 1 & 1 & \cdots & y & z_2 & 1 & \cdots & 1 \\ z_1 & 1 & 1 & \cdots & 1 & z_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & 1 & 1 & \cdots & 1 & z_2 & 1 & \cdots & 1 \end{pmatrix} \tag{7}$$

where the columns are given by the values of s_i, the rows are given by the values of s_{i+1} and where temperature and field dependencies have been absorbed into the variables

$$y = e^{\beta}, \quad z_1 = e^{\beta h_1}, \quad z_2 = e^{\beta h_2}. \tag{8}$$

Along with the temperature variable y we use another variable $t = y^{-1} = e^{-\beta}$ to map the infinite range $0 \le T < \infty$ to the finite region $0 \le t \le 1$.

The partition function can then be recast as

$$Z = \prod_{i=1}^{i=N} \sum_{\{s_i\}} \mathbf{T}(s_i, s_{i+1}) = \mathrm{Tr}\, \mathbf{T}^N = \sum_i \lambda_i^N, \tag{9}$$

where λ_i are the eigenvalues of $\mathbf{T}$.

Some of the eigenvalues can be found using the symmetry of the transfer matrix. It is easy to show that matrix (7) has five different eigenvalues. On the one hand, because the final r columns of the matrix are proportional, one eigenvalue is zero and is $r-1$ times degenerate. On the other hand, because $(q-1)$ elements of the main diagonal are equal to y, choosing $\lambda = y-1$ leads to $q-2$ linearly independent eigenvectors. This leaves only three unknown eigenvalues. They can be found using invariant permutations. This approach leads to the equation for the three remaining eigenvalues:

$$(r-1-\lambda+z_2)(yz_1-\lambda-z_1)(y-\lambda-1)-\lambda z_1(y-\lambda-1)-(q-1)(yz_1-\lambda-z_1)\lambda = 0. \tag{10}$$

This is an equation of third power and can, therefore, be solved exactly. Since the partition function (9) is defined by eigenvalues and all the λ's have been found, the problem is solved exactly [17].

2.2. Partition function zeros

Critical behaviour of equilibrium systems can be extracted from the partition function. In our case, the latter is described by the eigenvalues of the transfer matrix (9) and, since we have shown that they can all be found explicitly, the critical properties of the Potts model with invisible states can, in principle, also be found explicitly. This allows us to access the Lee-Yang zeros in complex magnetic field [18]. For completeness, we also discuss the Fisher zeros [52] in the complex temperature planes in Appendix A.

The standard approach is to label the eigenvalues of the transfer matrix in such a way that they are ordered in magnitudes; $|\lambda_1| \le |\lambda_2| \le |\lambda_3| \le \dots$. The partition function zeros are then found using the condition that (at least) two eigenvalues are largest by modulus [53]

$$|\lambda_1| = |\lambda_2|\,. \tag{11}$$

Since the partition function is analysed in the complex (T or h) plane, the eigenvalues are complex as well. Therefore condition (11) can be written as

$$\lambda_2 = \lambda_1 e^{i\phi}\,. \tag{12}$$

From Eq. (9), the partition function is a sum of eigenvalues to the power N. In our case the eigenvalue $\lambda = 0$ makes no contribution so that the partition function takes the form $Z = \lambda_1^N + \lambda_2^N + \lambda_3^N + \lambda_4^N$. Of these four eigenvalues, three are roots of the polynomial (10) and the fourth equals $y-1$. Taking into account the orders of magnitude of the eigenvalues, the partition function may be rewritten in such a way as to single out the main terms:

$$Z = \lambda_1^N \left[1 + e^{iN\phi} + \left(\frac{\lambda_3}{\lambda_1}\right)^N + \left(\frac{\lambda_4}{\lambda_1}\right)^N \right] . \tag{13}$$

In the limit of large N only the leading two terms in the expression in parentheses on the right-hand side of Eq. (13) contribute so that we obtain the phase ϕ given by

$$1 + e^{iN\phi} = 0 \quad \text{or} \quad \phi = \frac{2k-1}{N}\pi, \quad k = 1 \dots N . \tag{14}$$

In the thermodynamic limit values of the phase ϕ span the whole region $0 \le \phi \le 2\pi$. Therefore the coordinates of the partition function zeros are found solving Eq. (12) with the phase given by Eq. (14). This method is appropriate when all the eigenvalues are given explicitly. However, when they are given as the roots of the polynomial, one can use the method suggested in Ref. [54] for models with three non-zero eigenvalues or that put forward in Ref. [39] adapted for models with four non-zero eigenvalues.

Following this method, the four eigenvalues of the transfer matrix are presented as the roots of the polynomial of fourth order ‡. In the most general case it has the form

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 . \tag{15}$$

For the Potts model with invisible states Eq. (15) is obtained by multiplying Eq. (10) by $[\lambda - (y-1)]$. The corresponding coefficients have the form

$$\begin{aligned}
a_0 &= (y-1)^3 z_1 (r + z_2 - 1) ; \\
a_1 &= -(y-1)^2 (z_1(q + y - 1) + (2z_1 + 1)(r + z_2 - 1)) ; \\
a_2 &= (y-1)((z_1 + 1)(q + y - 1) + (z_1 + 2)(r + z_2 - 1) + yz_1 - 1) ; \\
a_3 &= -q - r - yz_1 - 2y - z_2 + 4 .
\end{aligned} \tag{16}$$

The goal of the method is to obtain a $\lambda-$independent equation linking together temperature, fields and other model parameters. To derive this we use four equations from Vieta's theorem together with the condition (12). Excluding all eigenvalues from these five equations we obtain

$$F(q, r, z_1, z_2, y) = F_1 F_2 (f_1 + f_2 + f_3 + f_4), \tag{17}$$

where

$$F_1 = 8a_2 \cos^2\left(\frac{\phi}{2}\right) \cos(\phi) - a_3^2[2\cos(\phi) + 1];$$

$$F_2 = 4a_1[2\cos(\phi)+1] \left[\cos\left(\frac{\phi}{2}\right) + \cos\left(\frac{3\phi}{2}\right) \right]^2 - 32a_2a_3\cos^4\left(\frac{\phi}{2}\right)\cos\phi + a_3^3[2\cos(\phi)+1]^2;$$

‡ Since one does not know in advance which eigenvalue is the maximum one, all four eigenvalues have to be considered.

and

$$
\begin{aligned}
f_1 = {} & 16a_0^3\left[\cos\left(\frac{\phi}{2}\right)+\cos\left(\frac{3\phi}{2}\right)\right]^4;\\
f_2 = {} & a_1^2\Big[a_2^2\left(a_3^2-2a_2(\cos(\phi)+1)\right)-a_1^2(1+2\cos(\phi))^3+\\
& 2a_1a_3\Big(a_2(5\cos(\phi)+\cos(2\phi)+3)-a_3^2(1+\cos(\phi))\Big)\Big];\\
f_3 = {} & -2a_0\left[a_2^3\left(8a_2\cos^4\left(\frac{\phi}{2}\right)-a_3^2(\cos(\phi)+1)\right)+\right.\\
& a_1^2\left(4a_2\cos^2\left(\frac{\phi}{2}\right)(7\cos(\phi)+5\cos(2\phi)+\cos(3\phi)+5)-a_3^2(2\cos(\phi)+\cos(2\phi))\right)+\\
& \left.a_1a_2a_3\left(-4a_2\cos^2\left(\frac{\phi}{2}\right)(6\cos(\phi)+\cos(2\phi)+3)+a_3^2(5\cos(\phi)+\cos(2\phi)+3)\right)\right];\\
f_4 = {} & a_0^2\left[128a_2^2\cos^4\left(\frac{\phi}{2}\right)\cos^2(\phi)+a_3^4(2\cos(\phi)+1)^3-\right.\\
& 8a_2a_3^2\cos^2\left(\frac{\phi}{2}\right)\Big(7\cos(\phi)+5\cos(2\phi)+\cos(3\phi)+5\Big)+\\
& \left.8a_1a_3\left(\cos\left(\frac{\phi}{2}\right)+\cos\left(\frac{3\phi}{2}\right)\right)^2(2\cos(\phi)+\cos(2\phi)+3)\right].
\end{aligned}
$$

For each value of ϕ given by Eq. (14) all the roots of Eq. (17) provide the values of parameters when two eigenvalues are equal by modulus, but not all of them are actual partition function zeros. Actual zeros are characterised by the condition that two largest eigenvalues are equal by modulus.

We analyse the zeros of the partition function in the plane of complex magnetic field (Lee-Yang zeros). According to the Lee-Yang theorem, for the ferromagnetic Ising model on a $d-$dimensional regular lattice, these zeros are purely imaginary [18]. This statement can be generalised to many other models [27]. Transforming to the complex $z = e^{-\beta h}-$plane, the counterpart zeros lie on an arc of the unit circle. Lee-Yang zeros have been called "protocritical points" [53] because they have the potential to become actual critical points. The protocritical point at an end of the arc which lies closest to the positive real axis is referred to as the "Yang-Lee edge" (henceforth also referred to as the "edge") [55]. If the temperature is higher than the critical one, the circular arc is open, i.e. it does not cross the positive real axis. As the temperature is lowered, the arc becomes a circle and the edge pinches the real axis when the critical temperature is reached, precipitating the phase transition. For temperatures lower than the critical value, the arc is closed into a full circle so that zeros cross the real z axis.

In Eq. (17), ϕ takes discrete values for finite systems according to Eq. (14). Having these data, we used finite-size scaling to determine the critical exponents (Appendix A). To establish whether the phase transition occurs at positive or negative temperature, we determine the critical temperature by enabling the zeros pinch the real axis. This only occurs for infinite volume and we achieve this simply by setting $\phi = 0$ in Eq. (17).

Alternatively, one can directly access zeros in the thermodynamic limit and the true (infinite N) Lee-Yang edge by using the Beraha-Kahane-Weiss (BKW) theorem [56–59].

3. Duality relations

To facilitate the analysis of the partition function zeros, we first discuss some properties of the eigenvalues, which will be used later in the text.

In Eq. (10), the quantities r and z_2 appear only in one term together as a sum. We conclude that the magnetic field acting on the invisible states plays the same role as additional invisible states. In this way $r + z_2 - 1$ can be treated as a temperature-dependent number of invisible states. For this reason, in this section z_2 is included in r and is never shown explicitly.

Duality means that under a certain unitary transformation $\mathbf{S}$, the transfer matrix changes according to the rule [60]

$$\mathbf{S}\mathbf{T}(y, z_1)\mathbf{S}^{-1} = \alpha\mathbf{T}^T(y^D, z_1^D), \tag{18}$$

where $y^D = y^D(y, z_1), z_1^D = z_1^D(y, z_1)$ denote variables dual to y, z_1, and $\mathbf{T}^T$ is the transposed transfer matrix. Eq. (18) can be rewritten in terms of the eigenvalues

$$\lambda(y^D, z_1^D) = \frac{1}{\alpha(y, z_1)}\lambda(y, z_1). \tag{19}$$

Eq. (19) is useful when explicit expressions for the eigenvalues are known. But in our case solving the third-order equation (10) results in having cumbersome expressions for each λ and thus Eq. (19) will be hard to handle. Instead, let us derive relations for the coefficients in the third-order polynomial in the left hand side of Eq. (10). In the most general case the equation reads

$$\lambda^3 + A_1(y, z_1)\lambda^2 + A_2(y, z_1)\lambda + A_3(y, z_1) = 0. \tag{20}$$

Eq. (20) holds for both ordinary and dual variables. Substituting (19) into (20) we get the following transformation rules for the coefficients A_1, A_2, A_3:

$$\begin{aligned} A_1(y^D, z_1^D) &= \frac{1}{\alpha}A_1(y, z_1), \\ A_2(y^D, z_1^D) &= \frac{1}{\alpha^2}A_2(y, z_1), \\ A_3(y^D, z_1^D) &= \frac{1}{\alpha^3}A_3(y, z_1). \end{aligned} \tag{21}$$

Using the first two equations of (21) and Eq. (19) with $\lambda = y - 1$ one recovers expressions for the dual variables:

$$\begin{aligned} \alpha &= \frac{(y-1)(q(z_1-1)+(r-1)z_1+1)}{q^2+q(2r-1)+(r-2)r}, \\ y^D &= \frac{(q+r-1)(q+r+z_1-1)}{q(z_1-1)+(r-1)z_1+1}, \\ z_1^D &= \frac{q^2+q(2r+y-2)+r^2-2r-y+1}{(y-1)(q+r-1)}. \end{aligned} \tag{22}$$

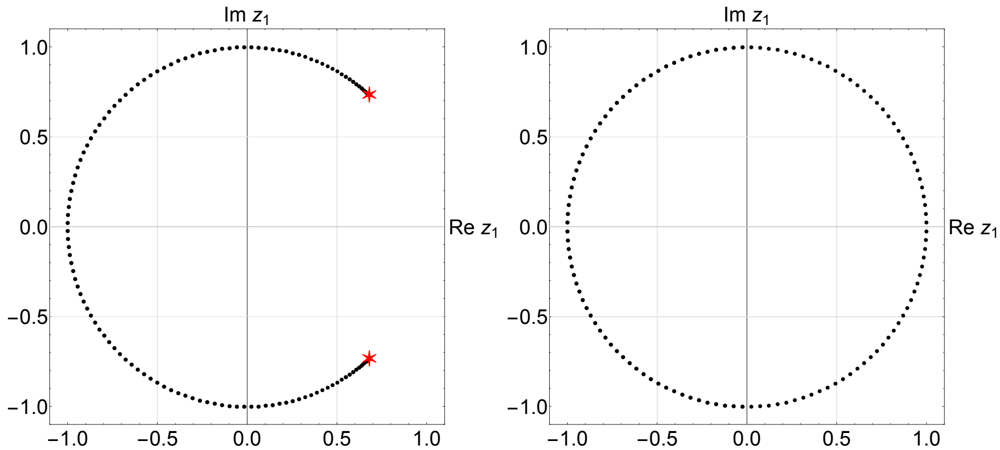


Figure 1. Lee-Yang zeros in the complex $z_1 = e^{-\beta h_1}$−plane of the 1D Ising model ($q = 2, r = 0$) for fixed values of T. The plots are for finite N but the curves become continuous in the thermodynamic limit. The left panel displays Lee-Yang zeros with a positive value of the temperature ($T = 1.091$). The Yang-Lee edges, represented as large red stars, do not reach the positive real axis indicating no phase transition. The right panel displays zeros with with vanishing temperature ($T = 0$). The approach of the zeros to the real axis at $z_1 = 1$ at infinite N indicates a zero-field ($h_1 = 0$) phase transition.

These expressions allow us to substitute temperature by field and vice versa without changing the behaviour of the system. Substituting $r = 0$ into (22) one recovers the duality relations for the ordinary 1D Potts model obtained in [61].

4. Lee-Yang zeros for models with direct physical realisability

4.1. Lee-Yang zeros for the ordinary Potts model

We first consider the ordinary q−state Potts model when the number of invisible states and, correspondingly, the second magnetic field are set to zero ($r = 0$ and $h_2 = 0$). This ordinary Potts model is thoroughly investigated so our results can be compared to those previously obtained [61]. In this case one of the roots of Eq. (10) becomes $\lambda = y - 1$, reducing the number of different eigenvalues to three. The remaining two eigenvalues are found as the roots of Eq. (10) and fully recover results obtained in Ref. [61]. The three eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[(y(z_1+1)+q-2) \pm \sqrt{(y(1-z_1)+q-2)^2+(q-1)4z_1}\right], \; \lambda_3 = y-1. \quad (23)$$

In Ref. [53] it was shown that the edge can be recovered from the condition that the largest eigenvalues of the transfer matrix are degenerate. Two of these eigenvalues in Eq. (23) are degenerate when the expression under the square root sign vanishes.

We plot the resulting loci of Lee-Yang zeros of the 1D Ising model ($q = 2$) in Fig. 1. The $T > 0$ case is illustrated in the left panel. There the edge is strictly complex meaning there is no phase transition in the symmetric phase. As the temperature decreases, the

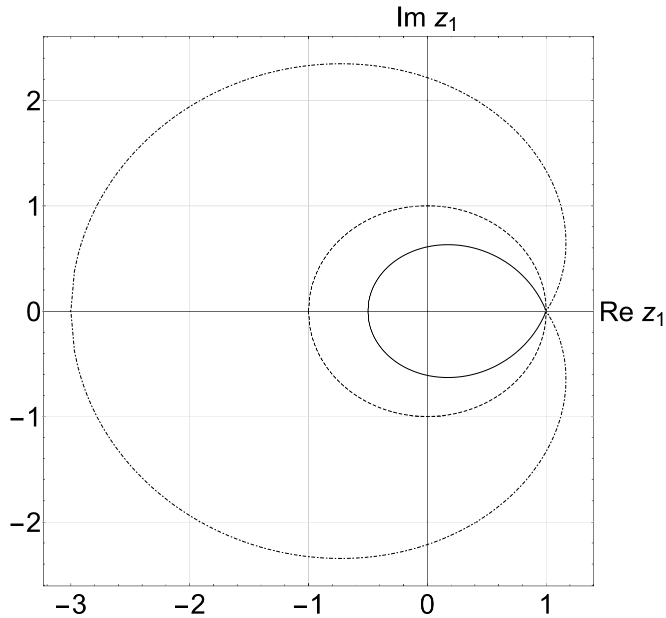


Figure 2. Edge loci in the complex z_1−plane for three Potts models ($q = 1.5, q = 2, q = 4$, moving inside out) without invisible states ($r = 0$). Each locus spans the full range of temperature values ($0 \leq T \leq \infty$) and intersects the $\mathrm{Re}\, z_1$ axis at $T = 0$.

edge approaches the real axis. The limiting case of $T = 0$ is represented in the right panel, albeit for a finite-size system (the circle is complete for an infinite chain of sites). In the thermodynamic limit the approach of the edge to the critical point ($h = 0$ or $z = 1$) triggers the zero-temperature spontaneous (zero-field) phase transition. Fig. 1 illustrates the $q = 2$ case only, for which the Lee-Yang unit-circle theorem is obeyed. Altering the number of Potts states alters the loci of zeros (not shown in the plot); while they remain circular, their radii are q-dependent for positive temperature. If $q < 2$ the radii of these circles are less than 1 and if $q > 2$ the radii exceed 1. However, at $T = 0$ all Lee-Yang arcs close into circles and cross the real axis at $\mathrm{Re}\, z_1 = 1$.

To more compactly illustrate the dependencies of zeros on the both temperature and on the number of states, instead of plotting the loci of the full sets of zeros as in Fig. 1, we plot the coordinates of the edges for different values of T and q in Fig. 2. We call these "edge loci". Such plots allow us to capture a greater span of q and T values while keeping the essential information because where the edge loci cross the real axis is where a phase transition can happen. For different given values of q the edge loci form different closed curves. But in each case the real axis is crossed at $T = 0$ confirming that the only possibility is for a phase transition at zero temperature, as observed nearly a hundred years ago by Ising (in the $q = 2$ case) [1].

To summarise, in this subsection we have recovered known results, supporting the viability of the approach.

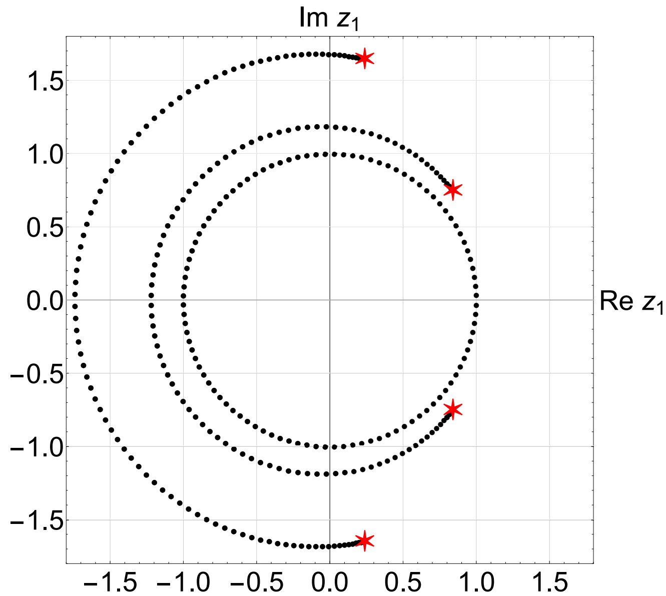


Figure 3. Lee-Yang zeros for the $(2, 2)-$state Potts model in the complex $z_1 = e^{-\beta h_1}-$plane at $y = e^{\beta} = 2$ (outer locus), $y = 4$ and $y \to \infty$ (inner, closed circle, representing $T = 0$) for systems of size $N = 100$. The inner circle is identical to the right panel of Fig. 1 for the Ising model. As in the left panel of Fig. 1, the outer two loci indicate there is no transition at non-zero temperature. The Yang-Lee edges are highlighted in red.

4.2. Lee-Yang zeros for the Potts model with invisible states

Having recovered well-known results for the ordinary Ising model in Fig. 1 and the ordinary Potts model in Fig. 2, and illustrated how the loci depend on q, we turn our attention to the Potts model with invisible states. As follows from Eq. (10) an external field acting on invisible states effectively works as an additional number of such states. We elaborate on this duality in Section 3 where we present a similar relationship between field and temperature. Therefore, without loss of generality, we can set $h_2 = 0$ (or $z_2 = e^{-\beta h_2} = 1$) in Eq. (10) arriving at

$$(r - \lambda)(yz_1 - \lambda - z_1)(y - \lambda - 1) - \lambda z_1(y - \lambda - 1) - (q - 1)(yz_1 - \lambda - z_1)\lambda = 0. \quad (24)$$

Setting $z_2 = 1$ in Eq. (17), and using the method described earlier, we extract the Lee-Yang zeros in the complex z_1-plane for any value of T at fixed values of q and r. As a counterpart of Fig. 1 for the ordinary Potts model we plot Lee-Yang zeros of the $(2, 2)$-state Potts model for different temperatures in Fig. 3.

As seen from the plot, zeros form circular arcs, but their radii are not unity and increase with increasing temperature (lower y-values) as the system is driven further away from the (zero-temperature) phase transition. The same behaviour was observed even in the ordinary Potts model (Fig. 2). The difference is that even in the Ising case $(q = 2)$ the presence of the invisible states changes the radius of the circle. It is only at $T = 0$ that zeros lay on the closed circle of unit radius.

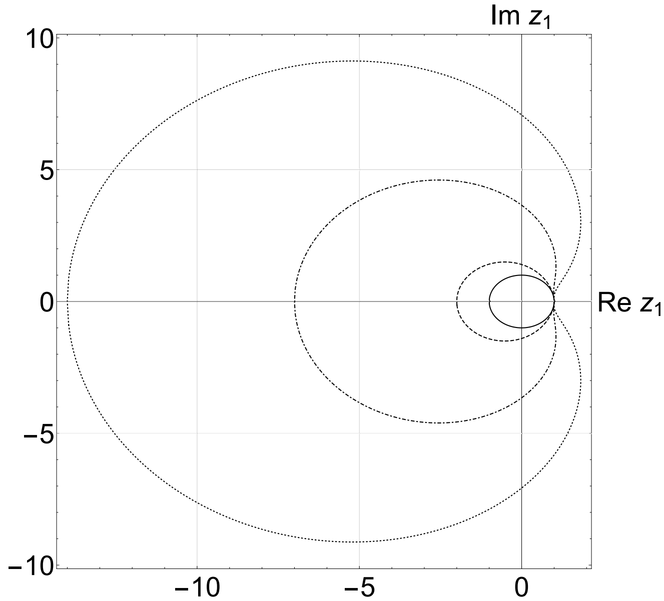


Figure 4. The edge loci in the complex z_1-plane for $q = 2$ and $r = 0$, 1, 6, 13 moving inside out. All plots cross at the real axis at $z_1 = 1$ indicating only zero temperature phase transitions. This is the invisible-states counterpart of Fig. 2.

It also is worth noting here that Lee-Yang zeros become increasingly dense as the edge is approached. This behaviour is quantified by the edge singularity exponent σ [62]. The precise value of this exponent is discussed in Appendix B.

To find the Yang-Lee edge one has to identify when the two largest eigenvalues of the transfer matrix are equal. This means that the polynomial (10) also has degenerate roots. This condition is equivalent to the discriminant $\mathbb{D}(y, z_1, z_2, q, r)$ of Eq.(10) vanishing [63]:

$$\mathbb{D}(y, z_1, z_2, q, r) = 0\,. \tag{25}$$

The discriminant $\mathbb{D}$ is a polynomial function of its arguments y, z_1, z_2, q and r. By setting $z_2 = 1$ and fixing the numbers of visible and invisible states (q and r, respectively) we can scan values of the temperature ($y = e^{\beta}$) to determine the coordinates of the edge points. Finding these coordinates for all possible temperatures $0 \leq T < \infty$ (or $1 \leq y < \infty$) we obtain the edge loci as shown in Fig. 4. This is the counterpart of Fig. 2 (it is the Ising model with invisible states). Comparison between the two figures illustrates that invisible states (entropy augmentation the energy lives in the density) are manifest in Lee-Yang-zero terms by widening the edge loci. But the behaviour of Lee-Yang zeros discussed above signals that the presence of invisible states does not change the fact that there is only a zero-temperature phase transition.

To summarise, in this subsection we have shown that systems with a positive number of invisible states also fail to manifest a positive-temperature phase transition in 1D.

5. Phase transitions at positive temperatures

In this section we introduce new ways to instigate positive-temperature phase transitions in a one-dimensional classical model with short-range interactions.

We start with analysing Lee-Yang zeros in the complex z_2-plane in Subsection 5.1. This will give us an insight into what is needed to shift the phase transition to positive temperatures. Earlier we have already mentioned that r and z_2 contribute only as a sum. It appears that having a negative value of this sum is key to achieving a positive-temperature phase transition in the current context. While mathematically identical, this mechanism can be interpreted physically in two different ways. Negative values of z_2 lead to complex values of the external magnetic field h_2. The effect of complex model parameters on the phase transition in 1D was already discussed in Refs. [14–16]. In our case similar behaviour is achieved by tuning h_2 as is discussed in Subsection 5.2. Alternatively, negative values of invisible states $r < 0$, based on the duality discussed in Section 3, have the same effect and are discussed in Subsection 5.3. Although both of these conditions are exotic, they either have connections to physical systems or have potential to be manifested physically in the future [25, 26, 64].

5.1. Zeros in complex z_2 for $h_1 = 0$

To begin our investigations we take inspiration from the analysis of so-called Potts zeros. In the ordinary Potts model, these are studied by an extended Fortuin-Kasteleyn representation [65] by promoting the Potts variable q to a complex number. Zeros in the complex $q-$plane are used to find the critical number of states for a given temperature [66–68]. While values of q below 2 are unphysical in terms of spin models, they can have physical manifestations — for example $q = 1$ describes percolation, $q = 0$ spanning trees and Abelian sandpile model for self-organised criticality [69, 70].

We have seen earlier that r acts similarly to an external magnetic field h_2. One can therefore interpret zeros in the complex $r-$plane as Lee-Yang-type zeros in the complex plane of z_2. We obtain partition function zeros in the complex z_2-plane by substituting values of q, r, h_1 into Eq. (17) in a similar manner as in the previous section. On the other hand, we can analyse the behaviour of the edge coordinates directly by solving Eq. (25). The corresponding plot is given in Fig. 5 for the particular case $q = 2$.

For positive values of r, the only crossing point with the real z_2 axis is located on the negative part. This is illustrated in Fig. 5. The middle edge locus in Fig. 5 was obtained for $r = 1$. Increasing r serves to shift the locus to the left; but its shape remains the same. This confirms our interpretation of Fig. 4 that increasing the value of r pushes the loci further away from a physical phase transition. One may infer that the converse also holds; decreasing r to a negative number of invisible states may shift the real edge locus from the negative real semi-axis to its positive counterpart. This is supported in the rightmost locus of Fig. 5 where $r = -4$.

To summarise this section, we have observed that zeros in complex h_2 cross the real axis at negative values of z_2 when r is positive, or vice versa, when $r < 0, z_2 \geq 0$.

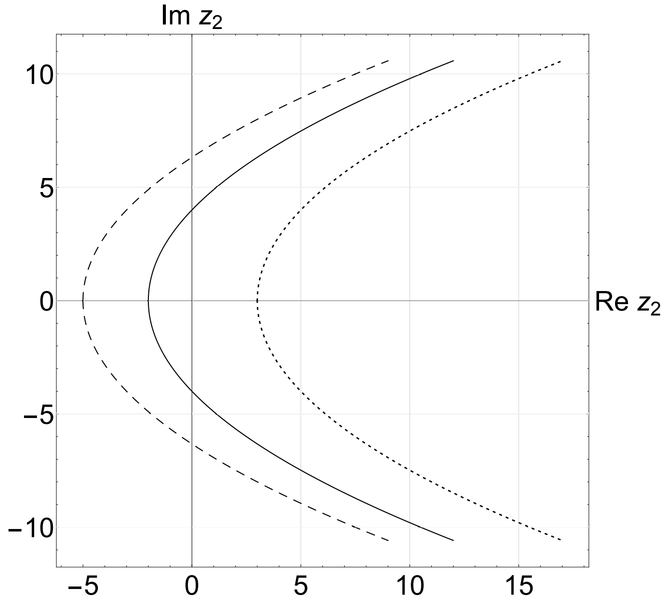


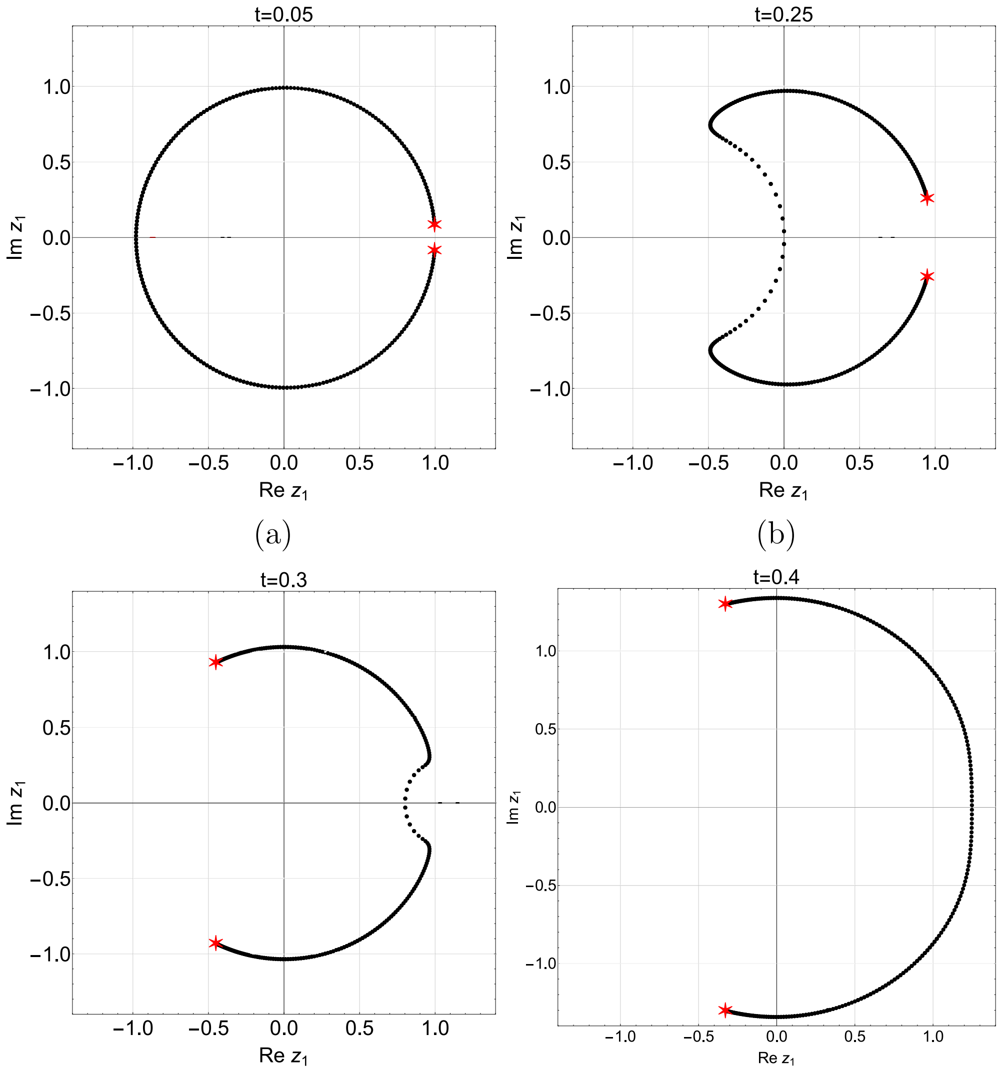
Figure 5. Yang-Lee edge locus corresponding to complex z_2 for $q = 2$, $h_1 = 0$ and $r = 4$ (dashed line), $r = 1$ (solid line) and $r = -4$ (dotted line). Each point of the lines represent a certain temperature in the region $0 \leq t \leq 1$.

Of course, the figure refers to the complex z_2-plane, corresponding to a field acting on entropic (invisible) states only. To connect with previous studies of Lee-Yang zeros we have to examine the complex h_1 or z_1-plane. We start in Subsection 5.2 by considering complex values of h_2 and $r > 0$ and defer $r < 0$ to Subsection 5.3.

5.2. Lee-Yang zeros in the complex z_1-plane with a complex field acting on invisible states: The case $z_2 < 0$ ($h_2 \in \mathbb{C}$).

Following on from the above considerations, we extend our search for positive-temperature phase transitions in one dimension to an analysis of the effects of negative values of z_2 (meaning a complex external magnetic field h_2) through Lee-Yang zeros in the complex h_1-plane. Using the same method as previously deployed, we substitute into Eq. (17) negative value of z_2 and obtain Lee-Yang zeros for various values of q, r and temperatures t. Results are represented in Fig. 6, which is the positive-r and negative-z_2 counterpart of Fig. 1 (which has vanishing values of r and h_2). At small temperatures these zeros lay close to the unit circle of Fig. 1, but increasing the temperature does not leave the Lee-Yang zeros on the unit circle. Instead they assume rather moon-like shapes. Moreover, although the locus of Lee-Yang zeros opens with increasing temperatures as in Fig. 1, the orientation of the arcs is reversed. It is worth noting here, that in order to obtain plots shown in Fig. 6 we fixed z_2 and not h_2, meaning that with the change of temperature t the external magnetic field is changed.

To further analyse the complex values of h_2 that shift the phase transition to positive temperatures, we set $h_1 = 0$ and use the condition (25) for the discriminant to find when



(a) (b) (c) (d)

Figure 6. Lee-Yang zeros in the complex z_1-plane for the $(2,2)$-state Potts model with $z_2 = -5$ for different temperatures (a) $t = 0.05$, (b) $t = 0.25$, (c) $t = 0.3$ and (d) $t = 0.4$. Large red stars correspond to the edges. Overall behaviour of zeros is similar to that described in the previous subsection.

two eigenvalues are equal and largest by modulus. This leads to a relation between the critical temperature and field h_2, namely

$$z_2 = y - q - r \pm 2i\sqrt{q(y-1)}\,. \tag{26}$$

In this equation both y and $z_2 = y^{h_2}$ are temperature dependent. Fixing q, r and sweeping through the region $1 \leq y < \infty$ (meaning a temperature range $0 \leq t \leq 1$), we solve numerically Eq. (26) and obtain complex values for h_2. In Fig. 7 we plot these values for the $(2,3)-$state Potts model in the form $e^{-\beta h_2}$. The curve forms two lines, each point of which corresponds to a certain positive critical temperature. The upper and lower branches correspond to complex conjugate values of the field.

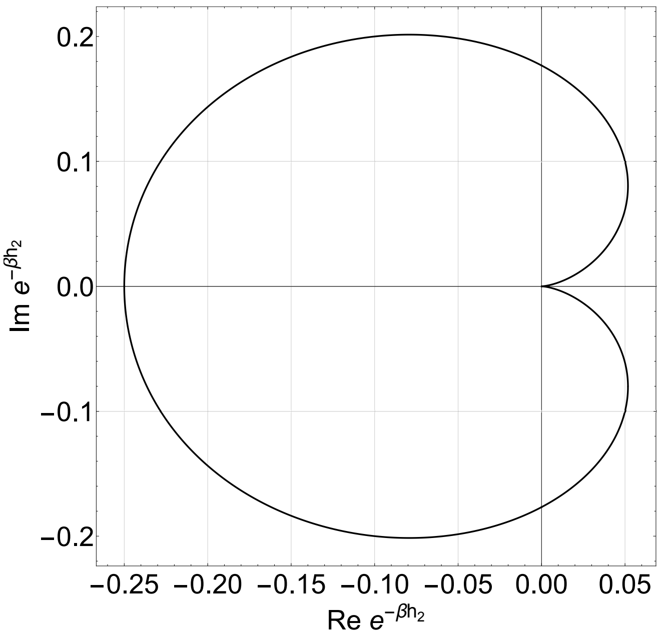


Figure 7. Values of $e^{-\beta h_2}$, for which the phase transition in the (2,3)-state Potts model occurs at positive temperature. Each point of the plot corresponds to a certain physically accessible critical temperature.

To summarise this subsection, we have observed that a complex field acting purely on invisible states can induce a positive-temperature phase transition in 1D. Although this appears exotic, 60 years after their introduction [18], it has recently been established that complex external magnetic fields h_1 can be mapped into physically accessible quantum coherence times [25]. Similarly, tuning complex values of the external field h_2 may one day be accessible, perhaps by changing the "invisible" part of a system's behaviour from classical to quantum.

5.3. Zeros in the complex z_1-plane for $h_2 = 0$ and $r < 0$

We build upon the observation in Fig. 5 that the edges in the complex z_2-plane are horizontally shifted to the right by reducing r to negative values. Moreover, taking into account results of the previous subsection, where the transition is observed for negative values of z_2 and the duality relation between r and z_2, we expect to obtain a phase transition for $r < 0$ as well. Using an extended Fortuin-Kasteleyn representation we relax the condition of positivity on the number of invisible states r. In Fig. 8, we plot the Lee-Yang zeros in complex z_1 for the $(2, -5)-$state Potts model for different values of temperature. Fig. 8 is a negative-r counterpart of Figs. 1 and 3. Our principal result in this subsection is that loci of zeros cross the real axis at positive temperatures for negative values of r. In particular, the left plot in the second line shows that, in the absence of field h_1 (i.e., when $z_1 = 1$), the zeros cross the real axis at a positive value of t (namely $t = 0.25$). This is the sought-after spontaneous, zero-field, positive-temperature phase transition in 1D. The correlation length is infinite at the critical temperature but the entropy has discontinuity. Therefore the phase transition can be interpret as of first order. However, small part of the entropy dependency on the temperature has an unphysical region. This can be as a consequence of unphysical values of the model parameters. Similar behaviour was earlier established in the models with complex interactions [14–16]. This connection is obvious, since we showed earlier that negative values of r have the same effect on the system as complex external magnetic field.

Each of the loci in Fig. 8 crosses the real axis, corresponding to a critical point. Negative values of z_1 represent complex values of physical field h_1. Values $z_1 > 1$ represent $h_1 > 0$ — a positive field acting on the first state $s = 1$. Positive values in the range $0 < z_1 < 1$ correspond to negative values of h_1. Such negative values of the external magnetic field are effectively the same as positive external fields acting on the other ($s \neq 1$) states. For the ordinary Ising case ($q = 2, r = 0$) negative values of the external magnetic field acting on the first state ($s = 1$), say, effectively represent the same physics as a positive external field acting on the other ($s = 2$) state. For the ordinary Potts model with $q > 2, r = 0$, negative external fields disfavour one of the states reducing the symmetry from Z_q to Z_{q-1}. In three dimensional three state Potts model this affects the phase diagram; weak magnetic fields do not change the order of the phase transition, while strong negative magnetic field changes it to the three-dimensional (3D) Ising universality class [71].

The figure also illustrates that, as for the ordinary Potts model, the Lee-Yang circle theorem is violated for the Ising model with a negative number of invisible states; the loci of zeros are not circular.

The set of crossing points for various temperatures in the range $0 \leq t \leq 1$ can be interpreted as a phase diagram and is shown for the $(2, -5)$-state model as solid black line in Fig. 9. The spontaneous transition is identified at $t = 0.25$, $z_1 = 1$. The counterpart for the ordinary Ising model is at $t = 0$, $z_1 = 1$ — i.e., at vanishing instead of positive temperature. To further illustrate this representation, in Fig. 9 we divide the $(t, z_1)-$plane into regions. The different colours represent different eigenvalues which are maximal by absolute values. Where they coincide is where criticality occurs.

A curious phenomenon in Fig. 8 is the flipping with increasing T of the edges of the loci (illustrated as large stars, red online) from the positive to negative half planes. This happens not at the zero-field critical point but at a lower value of T. The reason for this is that the edges in Fig. 8 are each away from the real axis and are pseudocritical points with $\operatorname{Im} z_1 \neq 0$ — not zero-temperature critical points. The phase diagram of Fig. 9 has vanishing imaginary field $\operatorname{Im} z_1 = 0$. To access the LY edges, and their flipping, requires non-zero values of $\operatorname{Im} z_1$ and three examples of this are depicted in Fig. 10. Different colours in the plot represent different eigenvalues. These are basically three slices through of a 3D plot with axes t (temperature), $\operatorname{Re} z_1$ and $\operatorname{Im} z_1$. These plots are given for the fixed temperature, when flipping occurs and coordinates of the edge. Three eigenvalues are equal by modulus exactly at the point where flipping occurs. Such a behaviour signals existence of a point with unusual Lee-Yang edge singularity

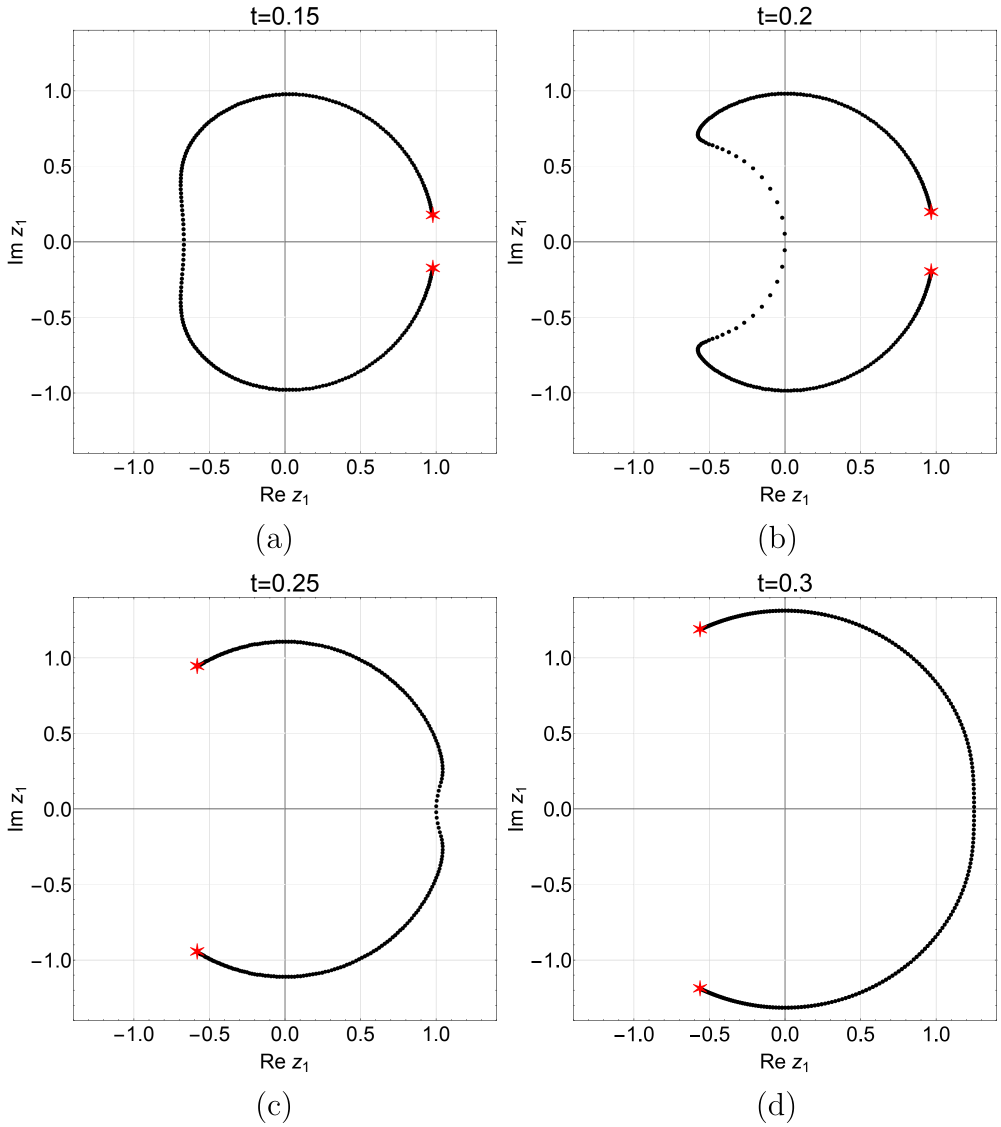


Figure 8. Lee-Yang zeros of the $(2,-5)$-state Potts model at different values of temperature (a) $t = 0.15$, (b) $t = 0.2$, (c) $t = 0.25$, (d) $t = 0.3$ in the complex z_1-plane for the system size $N = 256$. Panels (a) and (b) illustrate zeros below the zero-field critical temperature $t_c = 0.25$, panel (c) and (d) illustrate zeros at and above t_c. Large red stars show edges and black dots ordinary Lee-Yang zeros. For small temperatures the edge is located in the positive semi plane $\mathrm{Re}\, z_1 > 0$, while at higher temperature it jumps to the region $\mathrm{Re}\, z_1 < 0$. This jump occurs below the critical temperature t_c.

exponent [72].

To summarise this section, while positive values of r do not change the order and temperature of the phase transition, negative numbers of invisible states shift it to positive temperatures. In addition, there is a curious phenomenon involving the flipping of the locations of the edges relative to the other Lee-Yang zeros. This occurs at a value of temperature below the critical one and is explained by complex fields.

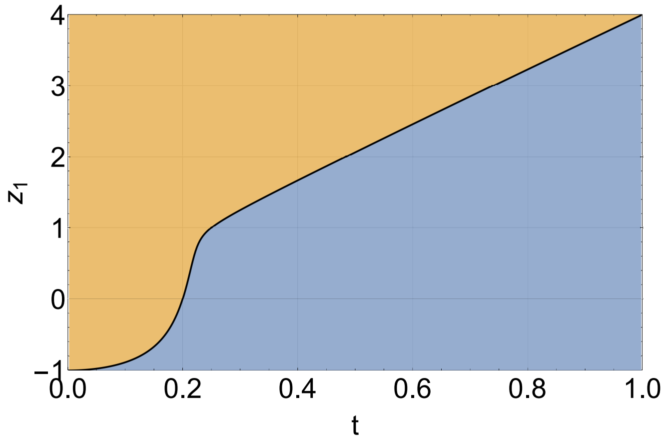


Figure 9. Phase diagram of the $(2,-5)$-state Potts model. $(t, z_1)-$plane is divided into regions according to the maximal eigenvalue. Values $z_1 < 0$ correspond to the complex values of magnetic field h_1, while $0 < z_1 < 1$ corresponds to negative values of physical field h_1.

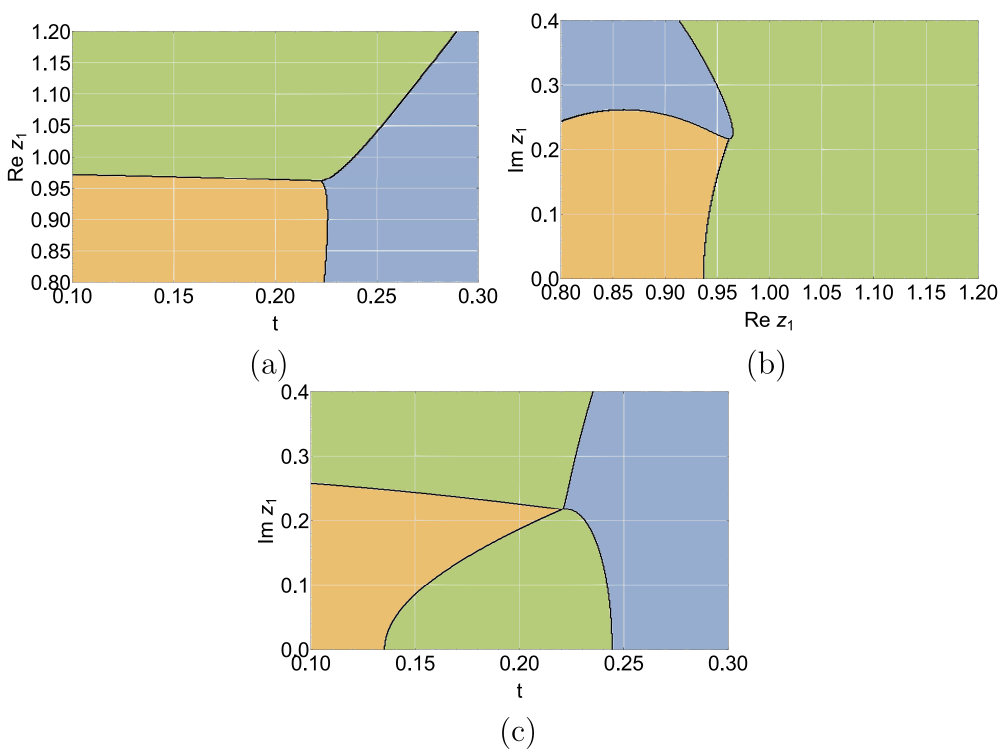


Figure 10. Three cross-sections of the 3D phase diagram. Each section is given at the fixed values of parameters when flipping of the edge occurs: (a) - fixed $\operatorname{Im} z_1$, (b) - fixed t, (c) - fixed $\operatorname{Re} z_1$. Colour of the region represent the eigenvalue, which is the largest by modulus inside this region. When three colours meet is the point where flipping of the edge occurs.

6. Conclusions

The possibility of classical, equilibrium phase transitions below the lower critical dimension is important at a fundamental level as well as for potential manifestations in real-world systems. Onsager's solution of the 2D Ising model [30] was only recently confirmed experimentally [31] and theoretical investigations have shown that adding invisible states can alter the type of phase transition present in such models [20, 21, 24]. Other recent theoretical and experimental developments include the establishment of a link between complex fields and quantum coherence times [25, 26], opening up new ways to access complex fields physically [32]. Still more recent experiments in entropy depletion involve realising Maxwell's demon by sorting ultracold atoms in an optical lattice [77]. Here we combine some of these developments with an exact solution of the Potts model with invisible states on a 1D chain with two distinct ordering fields. We use a Lee-Yang zeros analysis to investigate the effects of negative numbers of invisible states and the complex fields acting on them.

We found that the loci of Lee-Yang zeros and of the Yang-Lee edges strongly depend on the number of invisible states r. The area covered by the locus of the Yang-Lee edge increases with r but, if the convention of a positive number of invisible states and real fields are adhered to, the real part of the z_1-axis is only crossed at zero temperature. This extends the original result of Ising to generalisations of the eponymous model; there is no positive-temperature phase transition. Nonetheless, the fact that invisible states alter the locus of the partition function zeros opens up new possibilities to achieve this end. Interesting results are obtained when either number of invisible states is negative or the external field h_2 is complex. Both these cases are shown to deliver the possibility for the system to undergo phase transition at positive temperature. Ref. [9] contains a review of 1D lattice models with entropic stabilities, showing that the temperature at which the energy cost of producing a domain wall is balanced by gain of entropy matches the temperature at which the transitions occur, thus asking the question if all phase transitions in 1D are driven by the formation of domain walls. Here, we have established that 1D phase transitions are achievable by negative numbers of invisible states and by complex magnetic fields. These would appear to be outside the domain-wall criterion.

The reason why our results are not governed by the rigorous theorems [4, 7] is the following. A negative number of invisible states is equivalent to complex values of the external magnetic field. Ruelle in his proof requires all physical parameters to be real. Thus the conditions of this well-known theorem are violated, making it non-applicable in our case. Fields acting on conventional states have been shown to be mappable to quantum coherence time. This suggests that allowing h_2 to be complex may similarly endow our classical system with an element of quantum properties. Moreover, based on the energy-entropy arguments, the addition of a negative number of invisible states can be interpreted as a sort of indirect or artificial ordering mechanism, whereby the entropy is decreased by an amount sufficient to bypass the no-go theorems. This suggests that the introduction of a more physically direct ordering mechanism might overcome the no-go theorems in a new manner. Finally, according to the equivalent Hamiltonian (5), a negative number of invisible states plays the same role as a complex chemical potential, suggestive of a possible link between both bypass mechanisms.

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Appendix A. Fisher zeros

In Ref. [17], we reported on an analysis of the Fisher zeros in the complex temperature plane [52]. Here we present a brief version of that study because (i) it shows how we extract zero field critical points needed for Section 5 and (ii) it provides information on phase transition strength.

Fisher zeros are usually considered at the critical value of the external field. For the spontaneous phase transition the critical value of the field is $h_1 = 0$ corresponding to $z_1 = 1$. In this case one of the roots of the polynomial (10) becomes $\lambda = y - 1$ so that the polynomial (10) has only three different eigenvalues. Fisher zeros can be obtained from the condition that (at least) two eigenvalues of the transfer matrix are largest by modulus [53]. This approach allows to obtain zeros of the finite-size system and thus use the finite-size scaling technique (FSS) for the Fisher zeros coordinates. In the thermodynamic limit the line of zeros crosses the real axis at the transition point.

Appendix A.1. Critical temperature

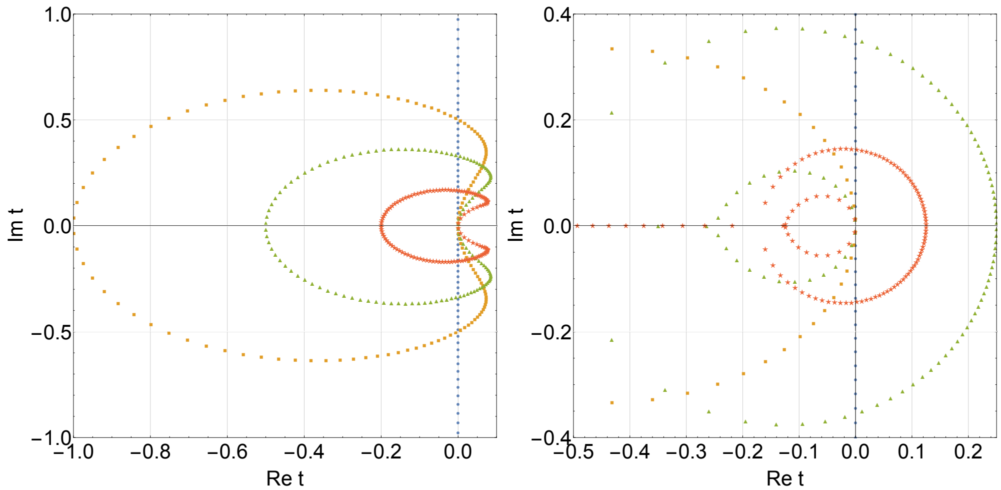
Fixing $z_1 = z_2 = 1$ in Eq. (17) we arrive at the equation for the coordinates of the partition function zeros in the complex $y-$plane at given pair of (q, r). It is most convenient to display Fisher zeros in the complex $t = y^{-1}-$plane. In this case they form closed curves around the origin $t = 0$ (rather than the "run-away" behaviour at $y \to \infty$). The infinite region $0 \leq T < \infty$ corresponds to the section $0 \leq t \leq 1$. In Fig. A1(a) we plot coordinates of the zeros for $q = 2$ and $r = 0, 1, 2, 5$ at fixed $N = 128$. The case $r = 0$ recovers results for the 1D Ising model. For $q = 2, r = 0$ zeros lie on the imaginary axis. With increasing N, the zero closest to the real axis closes in and with $N \to \infty$ it crosses the real axis at $t = 0$ ($T = 0$), implying again that there is only a zero temperature phase transition for the 1D systems [4, 7].

As one can see from Fig. A1(a), the presence of invisible states changes the locus of Fisher zeros. Now the zeros have both real and imaginary parts and in addition one more crossing point of the real $t-$axis appears. However this crossing point is located in the unphysical region $t < 0$ (complex values of T). $t = 0$ remains the point where the Fisher zeros approach the real axis and this confirms that the phase transition in the 1D Potts model is not changed by the presence of the invisible states. The critical exponents of this transition are further discussed in Appendix A.2.

We extend these considerations and show in Fig. A1(b) the locus of zeros for the $(2, r)-$state Potts model with negative values of r. §

As we can see from the figure, the locus of Fisher zeros in case $r = -5, -7$ intersect the real axis at the value $t_c = -\frac{1}{q+r-1}$. This means that besides the ordinary zero temperature phase transition we observe finite temperature phase transition in 1D model. The equivalent representation of the Potts model with invisible states through Eq. (5) indicates that the chemical potential is $\mu = -T \log r$. Therefore the

§ We do not show in the plot some points in the region $t > 1$ that correspond to negative temperatures.



(a) (b)

Figure A1. Fisher zeros of the $(2, r)$−state Potts model in $t = y^{-1} = e^{-\beta}$−plane for system size $N = 128$ with a) $r = 0$(blue), 1(yellow), 2(green), 5(red) and b) $r = 0$(blue), -2 (yellow), -5 (green), -7 (red).

negative number of invisible states is equivalent to a model with complex chemical potential. Again, via the aforementioned relation between the complex external field and decoherence time [25], this gives a connection to the behaviour of quantum systems.

Appendix A.2. Critical exponents

With the coordinates of the Fisher zeros to hand it is possible to obtain values of critical exponents. The first method to use is the scaling of the zero closest to the critical point. The corresponding scaling law has the form

$$\mathrm{Re}\, t = t_c + A \cdot N^{-\Lambda} \tag{A.1}$$

$$\mathrm{Im}\, t = B \cdot N^{-1/\nu}, \tag{A.2}$$

where t_c is the critical temperature, ν is the correlation length critical exponent, Λ is so-called shift exponent and N is the system size [73].

Another approach to analyse the partition function zeros is to use partition function zeros density [74]. This method allows to use not only coordinate of the closest zero but consider zeros density function which scales as

$$G(R) \sim R^{2-\alpha}, \tag{A.3}$$

where R is the distance to zero from the critical point α is the specific heat critical exponent.

We have used both techniques to extract the critical exponents from the coordinates of the zeros. Using system sizes from $N = 500$ to $N = 1000$ with increment $\Delta N = 20$ $q = 2$ and $r = 6$ we obtained $\nu = 0.9998(2)$, $\Lambda = 1.9997(2)$ and $\alpha = 1.002(2)$, which are in a good agreement with the hyperscaling relation $\alpha = 2 - d\nu$. Moreover these values remain close to the values $\nu = 1, \alpha = 1, \Lambda = 2$ with q and r changing.

It is worth mentioning that in the absence of magnetic field exact solution can be obtained and critical exponents are the same as in 1D Ising model ($\nu = 1$, $\alpha = 1$, $\eta = 1$, $\gamma = 1$, $\mu = 0$, $\beta = 0$, $\delta = \infty$).

Appendix B. Yang-Lee edge singularity exponent

In this appendix we show that the value $\sigma = \frac{1}{2}$ remain unchanged by introdusing invisible states. To do so we closely follow the method developed in Refs. [39, 54].

With increase of system size, Lee-Yang zeros $z_1 = |z_1|e^{i\theta}$ terminate in the complex plane at the Yang-Lee edge $z_1^e = |z_1^e|e^{i\theta_e}$. Their density $g(z_1)$ in the vicinity of z_1^e is governed by the edge singularity exponent σ [62]:

$$g(\theta) \propto |\theta - \theta_e|^{\sigma} . \tag{B.1}$$

In circumstances where, for a given value of T, the zeros are located on curves (the so-called singular line [75] as opposed to two dimensional regions [74]), the function g can be written for a fixed $|z_1|$ keeping dependency of the phase θ only. The exponent σ, like the other critical exponents, is characteristic of a given universality class. For the 1D Ising and $q-$state Potts models its exact value is $\sigma = -\frac{1}{2}$ [18, 54, 62]. Another known exact value for the $\sigma-$exponent has been obtained for the spherical model, where $\sigma = \frac{1}{2}$ independently of the type of interaction (short- or long-range) and space dimensionality [76].

The density of the partition function zeros in the region $(\phi, \phi + \Delta\phi)$ is proportional to the number of zeros in this region divided by the length of the part of the cord these zeros occupy. Since for each ϕ there is a certain zero, than the number of zeros in the region $(\phi, \phi + \Delta\phi)$ is proportional to $\Delta\phi$. Therefore, the density can be written as

$$\tilde{g}(\phi, \phi + \Delta\phi) \propto \frac{\Delta\phi}{\int_{\phi}^{\phi+\Delta\phi} \sqrt{(\frac{\partial \mathrm{Re} z_1}{\partial \phi})^2 + (\frac{\partial \mathrm{Im} z_1}{\partial \phi})^2}\, d\phi} . \tag{B.2}$$

In the thermodynamic limit zeros form continuous curve with density at the point ϕ given by

$$\tilde{g}(\phi, \phi + d\phi) = g(\phi) \propto \frac{1}{\sqrt{(\frac{\partial \mathrm{Re} z_1}{\partial \phi})^2 + (\frac{\partial \mathrm{Im} z_1}{\partial \phi})^2}} . \tag{B.3}$$

In the vicinity of the edge θ_e (which corresponds to $\phi = 0$) coordinates of zeros can be expanded into the Taylor series

$$z_1(\phi) \approx z_1^e + \frac{\partial^2 z_1(0)}{\partial \phi^2}\phi^2 + \dots \quad , \tag{B.4}$$

The linear term is absent since Eq. (17) is an even function of ϕ. Substituting the expansion (B.4) into Eq. (B.3) we obtain a simple relation between the density function

$g(\phi)$ and the phase ϕ:

$$g(\phi) \propto |\phi|^{-1} . \tag{B.5}$$

In the thermodynamic limit, close to the edge point, the phase and coordinates of zeros are connected through

$$\theta - \theta_e \propto |z_1 - z_1^e| . \tag{B.6}$$

Using expansion (B.4) in the right-hand side of the Eq. (B.6) we arrive at

$$\phi^2 \propto (\theta - \theta_e) . \tag{B.7}$$

Relation (B.5) together with Eq. (B.7) lead to the power-law behaviour of the density of zeros as a function of their phase close to the edge point

$$g(\theta) \propto |\theta - \theta_c|^{-1/2} . \tag{B.8}$$

Thus the Yang-Lee edge singularity exponent is $\sigma = -\frac{1}{2}$. This value follows immediately from the symmetry of zeros under the substitution $\phi \to -\phi$, which is observed for the models considered in Refs. [39, 43, 54, 61].