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Original citation & hyperlink:
Kataev, DE, Kutyakov, EY & Vershinin, Y 2018, The sub-Gramian method as a tool for the automotive suspension tuning. in 2018 Eleventh International Conference "Management of large-scale system development" (MLSD. IEEE, 2018 Eleventh International Conference "Management of large-scale system development" (MLSD) , Moscow, Russian Federation, 1/10/18.
https://dx.doi.org/10.1109/MLSD.2018.8551798

DOI 10.1109/MLSD.2018.8551798

Publisher: IEEE

The sub-Gramian method as a tool for the automotive suspension tuning

Dmitry E. Kataev, Evgeniy Y. Kutyakov
V.A. Trapeznikov Institute of Control Sciences RAS
Moscow, Russia
dekataev@gmail.com, evgeniykutyakov@gmail.com

Yuri Vershinin
Engineering, Environment and Computing
Coventry University
Coventry, UK
vershy@coventry.ac.uk

***Abstract*—The sub-Gramian method allows individual numerical estimation of inner interactions in large-scale systems. However, it may give results that appear difficult to interpret. In this paper we apply the sub-Gramian method to a low-order system with active inner interactions - a vehicle suspension model.**

***Index Terms*—sub-Gramian, Lyapunov differential equation, vehicle suspension, inner interaction**

I. Introduction

The simplest version of the sub-Gramian method is based on the spectral decomposition of a square H_2 norm of the transfer function. This case applies when a system can be described as an LTI dynamic one and therefore has a corresponding algebraic Lyapunov equation. In order to facilitate specific studies, e.g small-signal stability analysis, even large-scale systems like power grids can be considered as LTI systems [1]. The finite sub-Gramian method uses the spectral decomposition for the differential Lyapunov equation solution instead of algebraic one. Potentially it may allow its application to time-variant and certain types of non-linear systems. However, its results appear to be difficult to interpret and use [2]. This created a need to find a relatively simple, yet highly variable and containing multiple feedbacks technical system in order to study practical features of the finite sub-Gramian method.

A vehicle suspension appears to fit this role as it has all the needed physical properties. Its model has low order and can be modified with time-variant, non-linear and controlled elements if needed. Modern literature contains detailed description of its dynamics as well. Vehicle suspension parameters tuning problem is still not fully formalized and its solution mostly exist as a set of empirically derived rules, even despite this problem exists for a long time and has a certain economical significance [3], [4]. Modern handbooks and studies provide a variety of tuning and control strategies for passive and semi-active suspension systems, both regular [5] and heuristic [6], [7], but there is no universal approach yet. This leads to another task of this study - to investigate the suitability of the sub-Gramian method as a base for the formal automotive suspension tuning problem statement.

The authors would like to thank the Russian Foundation for Basic Research (RFBR) for supporting this research by project No.17-08-01107.

II. Problem Statement

Let us consider a vehicle suspension model with four wheels, springs, dampers an tires, absolutely rigid body and no sway bars. The following forces affect front right (FR, fr) wheel and suspension:

$$\begin{aligned} F_{fr} &= K_{fr}(-L_f\theta_d + L_{fr}\theta_l + z - z_f rU) + \\ &\quad + C_{fr}((-L_f\dot{\theta}_d + L_{fr}\dot{\theta}_l + \dot{z}) - \dot{z}_{frU}) \\ \ddot{z}_{frU} &= \frac{1}{M_{frU}}(F_{fr} - K_{frU}\dot{z}_{frU} - C_{frU}z_{frU}) \end{aligned} \tag{1}$$

where F_{fr} is a force of FR suspension, K_{fr} - FR suspension stiffness, L_f - a distance from the center of gravity (COG) to the front axle, θ_d - vehicle body pitch, L_{fr} - lateral distance from COG to FR suspension, θ_l - vehicle body roll, z - COG vertical position, z_{frU} - FR unsprung mass position (m), C_{fr} - FR suspension damping, M_{frU} - FR unsprung mass, K_{frU} - FR tire stiffness (N/m), C_{frU} - FR tire damping. Note that vehicle body and unsprung masses positions are in different coordinate systems. Similar equations define other (FL, RR, RL) suspension and tire forces.

The vehicle body dynamics is the following:

$$\begin{aligned} M_b\ddot{z} &= F_{fr} + F_{fl} + Frr + Frl \\ I_y\ddot{\theta}_d &= L_f(-F_{fr} - F_{fl}) + L_r(F_{rr} + F_{rl}) \\ I_x\ddot{\theta}_l &= L_{fr}F_{fr} + L_{rr}F_{rr} - L_{fl}F_{fl} - L_{rl}F_{fl}, \end{aligned} \tag{2}$$

where M_b is a body mass, F_{fl} - FL suspension force, F_{rr} - RR suspension force, F_{rl} - RL suspension force, I_y - body moment of inertia about y-axis, L_r - distance from COG to rear axle, I_x - body moment of inertia about x-axis (lateral), L_{rr} - lateral distance from COG to RR suspension, L_{fl} - lateral distance from COG to FL suspension, L_{fl} - lateral distance from COG to FL suspension. Figure 1 clarifies arms L and angles θ in this model.

Such physical system being described by (1) for each wheel and (2) can be modeled as a space-state LTI system

$$\dot{x} = Ax + Bu,\ y = Cx, \tag{3}$$

where state vector $x \in R^n$ represents body and suspension relative positions and velocities, as well as body pitch and roll, $u \in R^1$ is an external force caused by road bumps, vehicle acceleration and steering, $y \in R^1$ is an output signal like body position or roll. $A_{[n\times n]}, B_{[n\times 1]}, C_{[1\times n]}$ are real matrices

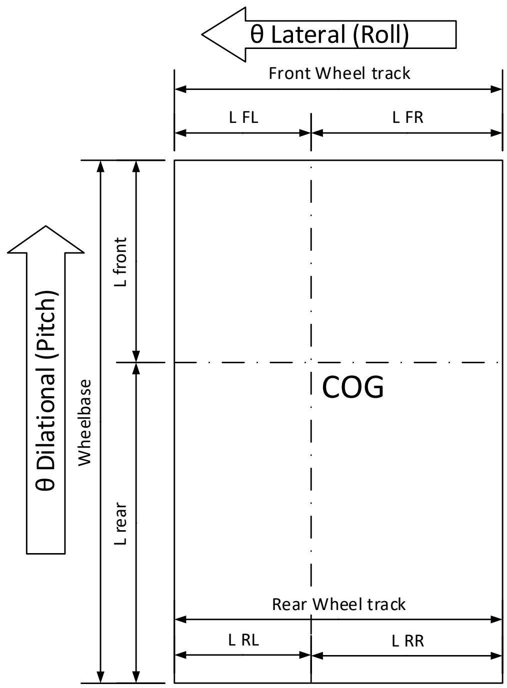

Fig. 1. Vehicle body dimensions and angles.

defining the system dynamics. While further equations would be given for a SISO system, the method description and case study suggest implicit SIMO and MIMO systems analysis by presenting their transfer functions as sets of SISO ones with different matrices C.

The solution of a road vehicle suspension tuning problem is finding an optimal set of adjustable chassis parameters resulting in both minimal transient time and peak body accelerations. Both are important for passenger comfort and vehicle steering but require opposite technical solutions, e.g. soft shock absorbers are good for keeping lower body acceleration values but lead to longer transients.

Considering a suspension system as an LTI system with a body position as its output allows using its transfer function to build comfort and handling functionals. Such functionals can be sensitive both for high peak values and long transient durations. A transfer function square H_2-norm represent total output energy after a δ-function input. The controllability Gramian can express it in the following way [1]:

$$||W||_2^2 = tr(C^T PC) \tag{4}$$

A controllability Gramian is defined as a solution of the following Lyapunov equation:

$$0 = AP + PA^T + BB^T \tag{5}$$

$$P = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau \tag{6}$$

A controllability Gramian on finite time interval is a solution of the following differential Lyapunov equation:

$$\frac{dP}{dt} = AP + PA^T + BB^T, P(0) = 0. \tag{7}$$

While transfer function $||H_2||$-norm is defined only for an infinite time interval, by Parseval's theorem it coincide with its time-domain counterpart - impulse response function $||H_2||$-norm. This allows building finite time version of (4) and potentially adapt the proposed method for linear time-variant systems.

The sub-Gramian method is based on the decomposition of Lyapunov equation solution as a sum of parts corresponding to system eigenvalues and their pairs, therefore to certain suspension components and component sets. Elements of such decomposition are called sub-Gramians. This allows to decompose energy-based functionals proposed above and possibly make them more descriptive.

III. Sub-Gramians

One way to define the finite controllability sub-Gramian corresponding to a particular eigenvalue involves the solution of the Sylvester differential equation [8]

$$\begin{aligned} &\frac{d}{dt} P_k^c(t) = s_k I P_k^c(t) + P_k^c(t) A^T + A_{(k)} BB^T, \\ &P_k^c(0) = 0_n. \end{aligned} \tag{8}$$

Its general solution is

$$P_k^c(t) = \int_0^t A_{(k)} BB^T e^{s_k\tau} e^{A^T\tau} d\tau, \tag{9}$$

$$A_{(k)} = Res\,(Is - A)^{-1}|_{s=s_k}. \tag{10}$$

The infinite sub-Gramian P_k trace for a diagonalized SISO LTI system:

$$||W||_2^2 = \sum_k^n trP_k = \sum_k^n tr(C^T P_k^c C), \tag{11}$$

$$trP_k = tr(C^T P_k^c C) = \sum_l^n p_{k,l}, \tag{12}$$

$$p_{k,l} = -\sum_l^n \frac{1}{s_k + s_l} b_k b_l c_k c_l, \tag{13}$$

where s_k is k-th eigenvalue of A, b_k is k-th element of vector B and c_k is k-th element of vector C. The finite sub-Gramian trace for a diagonalized SISO LTI system:

$$\begin{aligned} &tr(CP^c(t_0,t)C^T) = \\ &\sum_{k=1}^n \sum_{\lambda=1}^n b_k b_\lambda c_k c_\lambda \frac{1}{s_k + s_\lambda} (e^{(s_k+s_\lambda)t} - e^{(s_k+s_\lambda)t_0}). \end{aligned} \tag{14}$$

IV. Case Study

A. Model Description and Methodology

Table I shows the default values of all model parameters. COG position is typical for compact front wheel drive cars with only a driver onboard.

TABLE I
DEFAULT MODEL PARAMETERS

Parameter	Value	Unit
L_f	0.84	m
L_{fr}, L_{rr}	0.88	m
L_r	1.26	m
L_{fl}, L_{rl}	0.72	m
K_{fr}, K_{fl}	28000	N/m
C_{fr}, C_{fl}	2500	$N/(m/s)$
K_{rr}, K_{rl}	21000	N/m
C_{rr}, C_{rl}	2000	$N/(m/s)$
M_{frU}, M_{flU}	35	kg
K_{frU}, K_{flU}	$10K_{fr}$	N/m
C_{frU}, C_{flU}	$10C_{fr}$	$N/(m/s)$
M_{rrU}, M_{rlU}	25	kg
K_{rrU}, K_{rlU}	$10K_{rr}$	N/m
C_{rrU}, C_{rlU}	$10C_{rr}$	$N/(m/s)$
M_b	1200	kg
I_y	2100	kgm^2
I_x	900	kgm^2

TABLE II
SYSTEM MODES

Mode	Dominant States	Mode	Dominant States
M1, M2	θ_d, $\dot{\theta}_d$	M6	z_{flU}, z_{frU}
M3	z_{rlU}, z_{rrU}	M7, M8	θ_l, $\dot{\theta}_l$, z, $\dot{z}$
M4	z_{rlU}, z_{rrU}	M9, M10	θ_l, $\dot{\theta}_l$, z, $\dot{z}$
M5	z_{flU}, z_{frU}	M11-M14	$\dot{z_{rlU}}$, $\dot{z_{rrU}}$, $\dot{z_{flU}}$, $\dot{z_{frU}}$

The input is front right unsprung mass instantly moving 1cm down. Such input signal represents an idealized case of front right wheel falling into a 1cm deep road dent with instant tire deformation. Outputs are body COG position z, pitch θ_d and roll θ_l angles. Studying these outputs allows analyzing basic suspension-dependent comfort and handling properties of a vehicle.

The following experiments involve linearizing the model with input and outputs defined above while continuously changing rear suspension damping in the interval from 0.5 to 2 of the default value (1000-4000 $N/(m/s)$). We repeat this procedure for three different front damping values: 1, 1.2 and 1.4 of the default value (2500, 4200 and 5880 $N/(m/s)$). Table II shows dominant connections between system modes and state variables acquired using matrix A eigenvector analysis [9].

B. Numerical Results

At first we study square H_2-norms of the acquired transfer functions in order to clarify the energy-based approach. Figure 2 shows corresponding dependencies and allows to conclude that too low and too high damping values result in undesirably high energy estimations due to long-duration or highly accelerated transients respectively. It is notable, that optimal by the means of energy estimation rear damping values get proportionally higher as similar values for front dampers rise. The situation differ for transfer functions with body roll as output. As the input perturbation is localized in front, firmer rear dampers provide better roll resistance almost without local minimums, and vice versa for the front dampers.

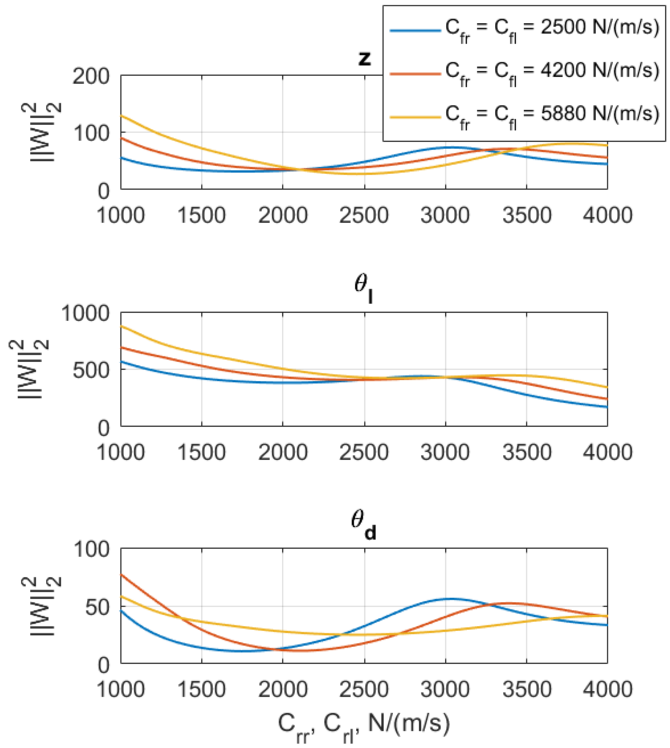

Fig. 2. Transfer functions square H_2-norms for different outputs.

Every mode M1-M10 has a significant impact on the resulting energy estimation as seen on Figure 3 for a transfer function with body bounce as output. At the same time the ratios of their impact are very variable. It is expectable since COG doesn't coincide with the geometrical center of the chassis, so any process in the suspension leads to COG movement.

Figure 4 shows dominant sub-Gramian traces for a transfer function with body roll as output. They correspond to modes M3 and M5 connected to rear and front unsprung masses relative positions respectively. In order to minimize body roll after hitting a road obstacle with one of forward wheels the methods advices for softer front dampers and firmer rear ones. Probably it will work the opposite way if only one rear wheel would hit a dent or bump on a road.

Figure 5 shows that the energy functional has a visible minimum in case of body pitch as output. Additionally, modes M4 and M9 start to take bigger part in forming the energy estimation as front damping values go higher. We may interpret this as more energy being not absorbed by too firm front suspension then going through the body to softer rear suspension.

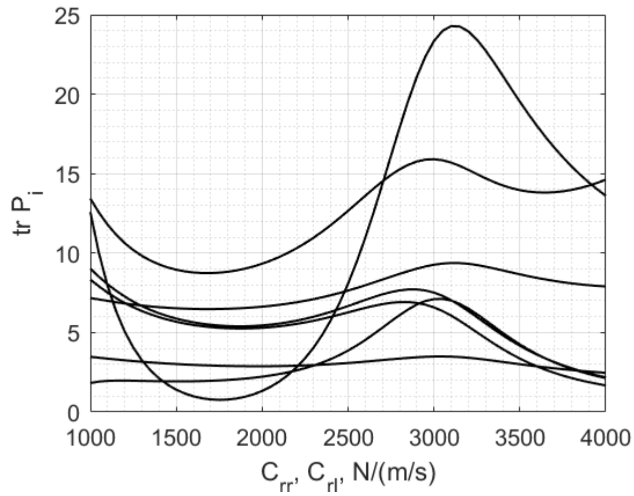

Fig. 3. General view of the spectral decomposition of transfer functions square H_2 norms. Body bounce output. $c_f = 2500$.

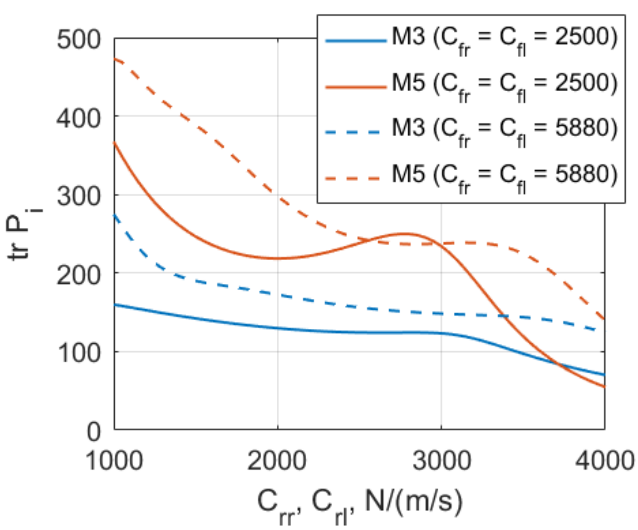

Fig. 4. Dominant Sub-Gramian traces. Body roll output. $c_f = 2500$.

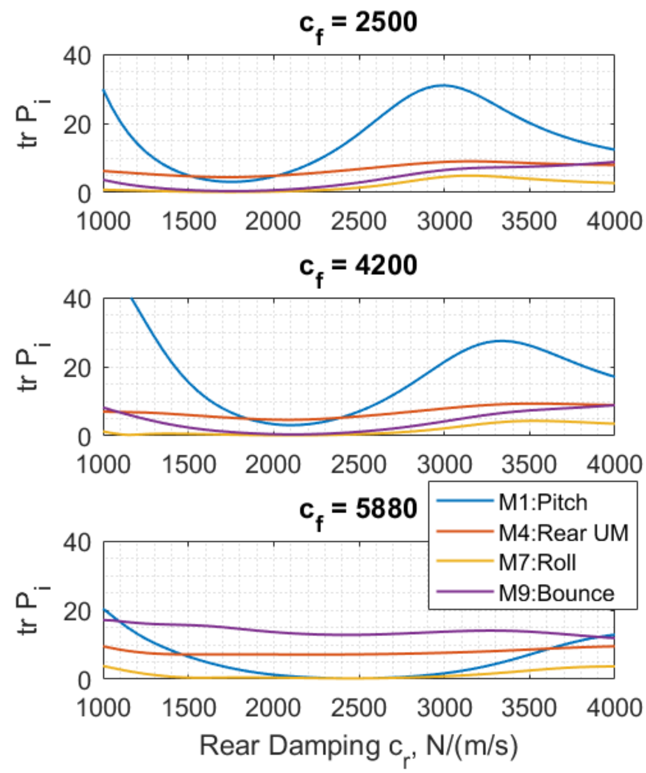

Fig. 5. The most changing Sub-Gramian traces. Body pitch output.

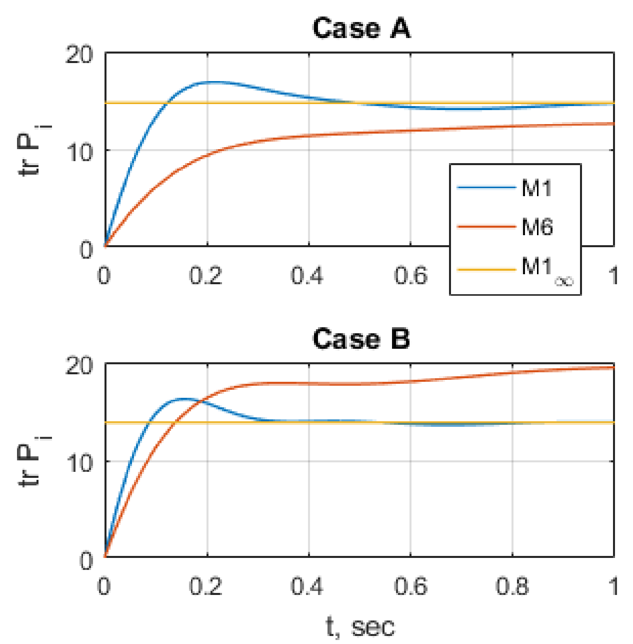

Fig. 6. Finite M1 and M6 sub-Gramian traces, infinite M1 sub-Gramian trace.

Let us study two cases from the transfer function set mentioned above using the finite sub-Gramian method. Let the case A correspond to the values C_{fr}, C_{fl} = 4200 N/(m/s) and C_{rr}, C_{rl} = 1900 N/(m/s), it is suboptimal in terms of proposed energy-based criteria. Let the case B correspond to the values C_{fr}, C_{fl} = 5880 N/(m/s) and C_{rr}, C_{rl} = 1000 N/(m/s), it is clearly unbalanced and far from optimal. Figure 6 presents finite sub-Gramian traces dynamics for modes M1 and M6 as well as infinite sub-Gramian trace for M1 as a reference. It shows that a finite sub-Gramian trace value can temporarily exceed an infinite one. That does not strictly comply with earlier interpretation of finite sub-Gramians as total accumulated energy and likely means that the energy can pass from mode to mode, so it will be accounted in other modes on the infinite time interval.

The nature of physical processes indirectly proves such interpretation. After initial hit the suspension converses several types of kinetic and potential energy of the body, the springs and the unsprung masses into each other, which doesn't

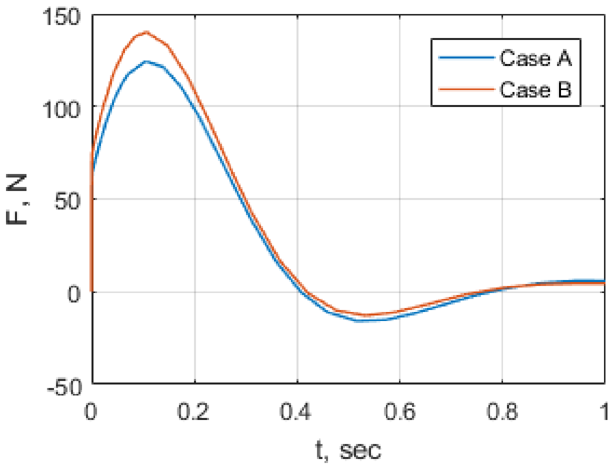

Fig. 7. Force, generated by the FR damper.

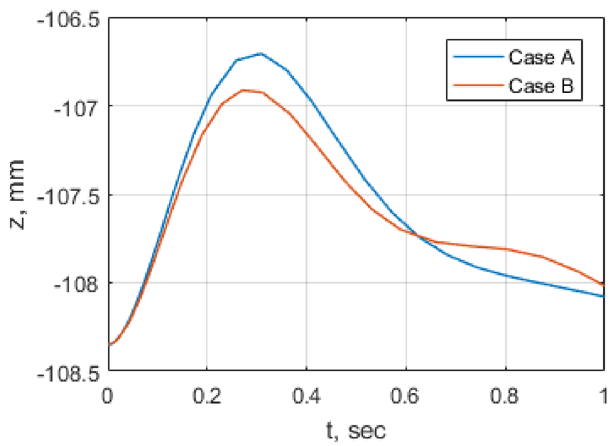

Fig. 8. Vertical body movement.

happen instantly. Figure 7 shows that in case B the damper generates greater force in shorter time during the first 400ms. Its main goal is to slow down the spring, therefore the vertical movement of the body was reduced both by amplitude and speed during the first 600ms, as seen on Figure 8. However, after these 600ms the delayed reaction from underdamped rear suspension takes place, which Figure 6 also does reflect.

As the spring and damper exert their forces on both the body and unsprung mass, it is inevitable for certain modes finite-time energy estimations to temporarily exceed their final value. Therefore, the time to reach this value for the first time may be criterion for estimating the process speed. In the future work the system can be improved by the adding adaptive controller. And such additional process speed criterion may be of use for off-line adaptation algorithm tuning or short-term performance assessment. This may lead to further improvements in the adaptive control abilities to compensate the parameters change of the vehicle suspension system due to the change of the environmental conditions or the fault of the system's components [10].

V. Conclusion

Both versions of the sub-Gramian method provide results that agree with engineering practice under given conditions. In particular, the method recommends correct parameters changes. The finite sub-Gramian analysis allows to locate undesirably fast or slow processes and to monitor energy conversion inside the system.

This will allow to connect finite sub-Gramian traces with physical energies in the system in further studies. More importantly, the results of this study provide a base for the formal suspension tuning problem statement using presented energy functionals.

References

[1] I. B. Yadykin, D. E. Kataev, A. B. Iskakov and V. A. Shipilov, "Characterization of power systems near their stability boundary using the sub-Gramian method," Control Engineering Practice. 2015. Vol. 53. pp. 173-183.

[2] I. B. Yadykin, A. A. Grobovoy, A. B. Iskakov, D. E. Kataev, M. S. Khmelik, "Stability analysis of electric power systems using finite Gramians," IFAC-PapersOnLine. 2015. Vol. 48, Issue 30. pp. 548-553.

[3] R. Stone and J. K. Ball, Automotive Engineering Fundamentals. PA: SAE, 2004.

[4] M. Guiggiani, The Science of Vehicle Dynamics. Springer Netherlands, 2014.

[5] S. Kashem, R. Nagarajah, M. Ektesabi, "Vehicle suspension systems and electromagnetic dampers," Springer Nature Singapore Pte Ltd. 2018

[6] A. Seifi, R. Hassannejad, M. A Hamed, "Optimum design for passive suspension system of a vehicle to prevent rollover and improve ride comfort under random road excitations," Proceedings of the Institution of Mechanical Engineers Part K Journal of Multi-body Dynamics, Vol. 230(4), pp. 426-441, 2016.

[7] R. R. Das, V. K. Elumalai, R. G. Subramanian, K. V. A. Kumar, "Adaptive predatorprey optimization for tuning of infinite horizon LQR applied to vehicle suspension system," Applied Soft Computing, 2018, in press.

[8] D. E. Kataev and I. B. Yadykin, "Solution of the Lyapunov Matrix Differential Equations by the Frequency Method," Journal of Computer and Systems Sciences International. Moscow: MAIK "Nauka/Interperiodika'" and Pleiades Publishing, 2016. Vol.55, No.6. pp. 843-855.

[9] S. N. Vassiliev, I. B. Yadykin, A. B. Iskakov, D. E. Kataev, A. A. Grobovoy and N. G. Kiryanova, "Participation factors and sub-Gramians in the selective modal analysis of electric power systems," IFAC-PapersOnLine. 2017. Volume 50, Issue 1. pp. 14806-14811.

[10] Yuri Vershinin, "Adaptive Control System for Solution of Fault Tolerance Problem for MIMO Systems," Lecture Notes in Engineering and Computer Science, Volume 2202, Issue 1, 2013, pp.117120.