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A Fractional Fokker-Planck Model for Anomalous Diffusion

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In this paper we present a study of anomalous diffusion using a Fokker-Planck description with fractional velocity derivatives. The distribution functions are found using numerical means for varying degree of fractionality observing the transition from a Gaussian distribution to a Lévy distribution. The statistical properties of the distribution functions are assessed by a generalized expectation measure and entropy in terms of Tsallis statistical mechanics. We find that the ratio of the generalized entropy and expectation is increasing with decreasing fractionality towards the well known so-called sub-diffusive domain, indicating a self-organising behavior.

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I. INTRODUCTION

In the early 20th century Einstein studied classical diffusion in terms of Brownian motion. In this process, the mean value of the process vanishes whereas the second moment or variance grows linearly with time $\langle \delta x^2 \rangle = 2Dt$. Anomalous diffusion, however, is in contrast to classical diffusion in terms of the variance that exhibit a non-linear increase with time $\langle \delta x^2 \rangle = 2Dt^\alpha$. There is no mechanism that inherently constrains $\lim_{\delta x, \delta t \rightarrow 0} \frac{\delta x^2}{\delta t}$ to be finite or non-zero. In more general terms, there are two limits of interest where the first is super-diffusion $\alpha > 1$ and the second is sub-diffusion with $\alpha < 1$. Such strange kinetics¹⁻⁶ may be generated by accelerated or sticky motions along the trajectory of the random walk⁷. The cause of anomalous diffusion is the existence of long-range correlations in the dynamics and/or the presence of anomalously large particle displacements or trapping.

Anomalous sub-diffusive properties has been studied in many different contexts, among them are that of holes in amorphous semiconductors where a waiting time distribution with long tails was introduced⁸ and the sub-diffusive processes within a single protein molecule described by generalized Langevin equation with fractional Gaussian noise⁹. Moreover, it has been recognized that the nature of the transport processes common to plasma physics is dominated by turbulence with a significant ballistic or non-local component where a diffusive description is improper. The basic mechanism underlying plasma transport is a very complex process and not very well understood. As a contrast, in plasma physics, super-diffusive properties are often found with $\alpha > 1$ such as the thermal and particle flux in magnetically confined plasmas or transport in Scrape-Off Layer (SOL) dominated by coherent structures¹⁰⁻¹⁴. In this paper, we will mainly concern ourselves with super-diffusion modeled by a Fractional Fokker-Planck equation (FFPE).

A salient component describing the suggestive non-local features of plasma turbulence is the inclusion of a fractional velocity derivative in the Fokker-Planck (FP) equation leading to an inherently non-local description as well as giving rise to non-Gaussian probability density functions (PDFs) of, e.g., densities and heat flux. Note that, the non-Gaussian features of the PDFs heat or particle flux generated by non-linear dynamics in plasmas may be reproduced by a linear, though, fractional model. The non-locality is introduced through the

integral description of the fractional derivative, and the non-Maxwellian distribution function drives the observed PDFs of densities and heat flux far from Gaussian¹⁵ as well as shear flow dynamics¹⁶. Some previous papers on plasma transport have used models including a fractional derivative where the fractional derivative is introduced on phenomenological premises¹⁷. In the present work we introduce the Lévy statistics into the Langevin equation thus yielding a fractional FP description. This approach is similar to that of Ref. 17–19 resulting in a phenomenological description of the non-local effects in plasma turbulence. Using fractional generalizations of the Liouville equation, kinetic descriptions have been developed previously^{20–23}.

In investigations of the anomalous character of transport a useful tool is the non-extensive statistical mechanics which provides distribution functions intermediate to that of Gaussians and Lévy distributions adjustable by a continuous real parameter $q^{25–27}$. The parameter q describes the degree of non-extensivity in the system. Non-extensive statistical mechanics has a solid theoretical basis for analysing complex systems out of equilibrium. For systems comprised of independent or parts interacting through short-range forces the Boltzmann-Gibb statistical mechanics is sufficient however for systems exhibiting fractal structure or long range correlations this approach becomes unwarranted. Tsallis statistics is now widely applied e.g. to solar and space plasmas such as the heliosphere magnetic field and the solar wind^{28–30}.

Note that due to the obtained Lévy type distribution functions, higher moments will diverge thus it is of interest to define convergent statistical measures of the underlying process. We will employ the generalized q -moments or q -expectations as $\langle v^p \rangle_q = \int dv F(v)^q v^p$. The q -expectation can be a convergent moment of the distribution function although the regular moments diverges. This also gives us the opportunity to have a convergent pseudo-energy that is always convergent.

The aim of this study is to elucidate on the non-extensive properties of the velocity space statistics and characterization of the fractal process in terms of Tsallis statistics. We show that the PDFs derived from the Tsallis statistic is in good agreement with those found using the FFPE. Moreover, we find that self-organising behavior is present in the system where the ratio of the entropy and energy expectation is decreasing with decreasing fractionality.

The remainder of the paper is organized as in Sec. II the Fractional Fokker-Planck Equation is introduced whereas in Sec. III a numerical study of the probability distribution functions obtained is presented. In Sec. IV, the paper is concluded by results and discussions.

II. FRACTIONAL FOKKER-PLANCK EQUATION

The motion of a colloidal particle, i.e. Brownian motion, is described by a stochastic differential equation also known as the Langevin equation³¹. It is assumed that the influence of the background medium can be split into a dynamical friction and a fluctuating part, $A(t)$, represented by Gaussian white noise. The Gaussian white noise assumption is usually imposed in order to obtain a Maxwellian velocity distribution describing the equilibrium of the Brownian particle. This connection is due to the relation between the Gaussian central limit theorem and classical Boltzmann-Gibbs statistics³². However, the Gaussian central limit theorem is not unique and a generalization was done by Lévy³³, Khintchine³² and Seshadri³⁴ by using long tailed distributions.

The underlying physical reasoning is to allow for the non-negligible probability of direction preference and long jumps, i.e., Lévy flights, which therefore allows for asymmetries and long tails in the equilibrium PDFs, respectively. In the present work we introduce the Lévy statistics into the Langevin equation thus yielding a fractional FP description. Following the approach used by Barkai³⁵ and^{18,19} a Fractional Fokker-Planck Equation (FFPE) with a fractional velocity derivative in the presence of a constant external force is obtained as

$$\frac{\partial F}{\partial t} + \mathbf{v} \frac{\partial F}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \frac{\partial F}{\partial \mathbf{v}} = \nu \frac{\partial}{\partial \mathbf{v}} (\mathbf{v} F) + D \frac{\partial^\alpha F}{\partial |\mathbf{v}|^\alpha}, \quad (1)$$

where $0 \leq \alpha \leq 2$. Here, the term $\frac{\partial^\alpha F}{\partial |\mathbf{v}|^\alpha}$ is the fractional Riesz derivative. The diffusion coefficient, D , is related to the damping term ν , according to a generalized Einstein relation³⁵

$$D = \frac{2^{\alpha-1} T_\alpha \nu}{\Gamma(1 + \alpha) m^{\alpha-1}}. \quad (2)$$

Here, T_α is a generalized temperature, and taking force \mathbf{F} to represent the Lorentz force acting on the particles with mass m and $\Gamma(1 + \alpha)$ is the Euler gamma function.

In order to analytically investigate the effects of the fractional derivative on the diffusion

we consider the force-less homogeneous one dimensional Fokker-Planck equation of the form,

$$\frac{\partial F}{\partial t} = \nu \frac{\partial}{\partial v}(vF) + D \frac{\partial^\alpha}{\partial |v|^\alpha} F. \quad (3)$$

The solution is found by Fourier transforming and treating the fractional derivative in the same manner as in Ref.³⁶ we find,

$$\frac{\partial \hat{F}}{\partial t} = -\nu k \frac{\partial}{\partial k} \hat{F} - D |k|^\alpha \hat{F}. \quad (4)$$

The stationary PDF is now readily obtained by integration and an inverse Fourier transform,

$$\hat{F}(k) = F_0 \exp\left(-\frac{D}{\nu\alpha} |k|^\alpha\right), \quad (5)$$

$$F(v) = \frac{F_0}{2\pi} \int_{-\infty}^{\infty} dk \exp\left(-\frac{D}{\nu\alpha} |k|^\alpha + ikv\right). \quad (6)$$

Due to the the fractal form of the inverse Fourier transform analytical solutions of the PDF for the general case is difficult to obtain, except in particular cases of $\alpha = 1.0$ and $\alpha = 2$ yielding a Lorentz distribution and a Gaussian distribution, respectively. Note, that Eq. (6) is equivalent to what was found in Ref. 23. From a different perspective, for a PDF of a single variable the Tsallis statistics may be generated by an appropriately constructed Langevin equation of the form,

$$\frac{dv}{dt} = K(v) + \frac{dD(v)}{dv} + \sqrt{2D(v)}w(t). \quad (7)$$

This result was obtained in Ref. 37, where $K(v)$, $D(v)$ and $dD(v)/dv$ are non-stochastic functions of v and $w(t)$ is white-in-time Gaussian noise. The PDF generated by Eq. (7) is,

$$F(p) = \frac{N}{(1 + \beta(q-1)v^2)^{1/(q-1)}}. \quad (8)$$

Note that $q > 1$ and β are found from the analytical forms of $D(v)$ and $K(v)$ as well as the v^2 dependence. Here N is normalization factor. Furthermore, it is found that β is not representative of an inverse temperature of the system due to its non-equilibrium nature. There is a seemingly simple relation proposed between the degree of fractionality α and the non-extensivity parameter of the form²⁵,

$$\alpha = \frac{1}{q-1}. \quad (9)$$

Interestingly, here is a direct connection between non-linear dynamics and the fractional FP model. It has been recognized that multi-fractal models stemming from the Tsallis statistical mechanics may well describe isotropic fluid turbulence at high but finite Reynolds number²⁴. In the next Section we will study the solutions to the Eq. (6) in more detail.

III. NUMERICAL SOLUTIONS TO THE FRACTIONAL FOKKER-PLANCK EQUATION

The main topic of this paper is to evaluate the statistical properties in terms of Tsallis statistics dependent of the fractional index (α) in Eq. (6). We will start by numerically computing the PDFs with α as our free parameter, and next we will fit the computed PDFs to the proposed generalized analytical Cauchy-Lorentz PDFs found from Tsallis statistical mechanics. Subsequently, in order to statistically evaluate the numerically found PDFs in the fractal model we will determine the q -expectation and the Tsallis non-extensive entropy. Note that the regular statistical moments of the PDFs will not converge unless the PDFs are considered to have a finite compact support. Thus, we will now focus on solving Eq. (6) numerically, by computing the inverse Fourier transform and compare the found PDFs to previously derived analytical solutions.

In Figure 1, the numerically found PDFs are shown (log-linearly) for $\alpha = 0.25$ (black dashed line), $\alpha = 0.50$ (cyan line), $\alpha = 0.75$ (yellow line), $\alpha = 1.00$ (magenta line), $\alpha = 1.25$ (green line), $\alpha = 1.50$ (red line), $\alpha = 1.75$ (blue line) and $\alpha = 2.00$ (black line). Here, in this study the diffusion coefficient over the dissipation is kept constant $D/\nu = 1.0$. We note as the parameter α decreases, the normalized fourth moment (Kurtosis = m_4/m_2^2 = the ratio of the fourth moment divided by the square of the standard deviation) of the symmetric PDF increases rapidly where PDFs become more and more peaked with elevated tails. It is also found that there is a smooth transition from a Gaussian distribution ($\alpha = 2.0$) to a Lévy type distribution passing through the Lorentz distribution ($\alpha = 1.0$).

In line with the stochastic non-linear analysis presented in Eqs (7) - (9) it has been shown, in Ref. 25, that using generalized statistical mechanics, it was again predicted that

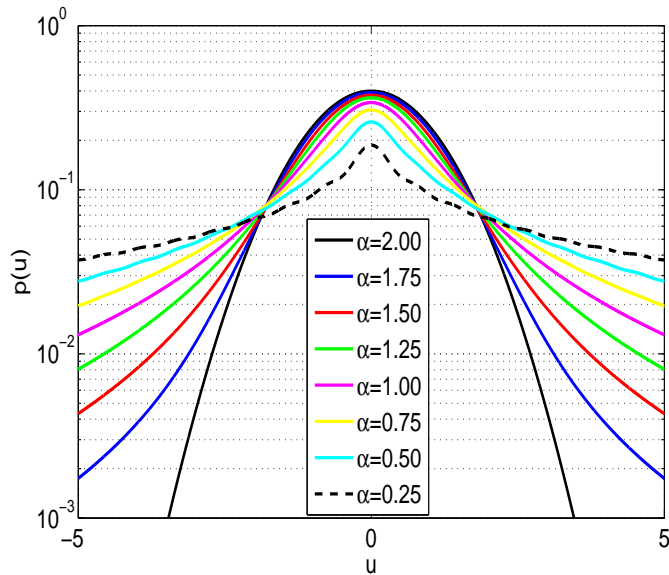


FIG. 1. The PDF(u) as a function of the velocity u for $\alpha = 2.00$ (black line), $\alpha = 1.75$ (blue line), $\alpha = 1.50$ (red line), $\alpha = 1.25$ (green line), $\alpha = 1.00$ (magenta line), $\alpha = 0.75$ (yellow line), $\alpha = 0.50$ (cyan line) and $\alpha = 0.25$ (black dashed line).

the PDFs are of Cauchy-Lorentz form,

$$F(v) = \frac{a}{(1 + b(q-1)v^2)^{1/(q-1)}}. \quad (10)$$

We note that this type of PDF exhibit power law tails that are significantly elevated compared to Gaussian or exponential tails, c.f. Eq. (7). Here, it is interesting to note that the precise analytical relation between the fractality index α and the non-extensivity parameter q is still unclear, one other formal relation have been proposed as²⁵

$$\alpha = \frac{3 - q}{q - 1}. \quad (11)$$

In Figure 2, the PDFs are fitted using Eqs. (9) and (10) with $q = 5/3$ for $\alpha = 3/2$, $q = 2$ for $\alpha = 1$ and $q = 3$ for $\alpha = 1/2$ except for $\alpha = 2.0$ where a Gaussian distribution is utilized. We find an excellent agreement over several orders of magnitude between the proposed analytically derived and the numerically computed PDFs for all values of $0.25 < \alpha < 1.5$. Note that, the process with $\alpha < 0.5$ does not have a convergent PDF, i.e. the total probability diverges. We solve Eq. (6) numerically for $\alpha = 3/2$ and show two fits in

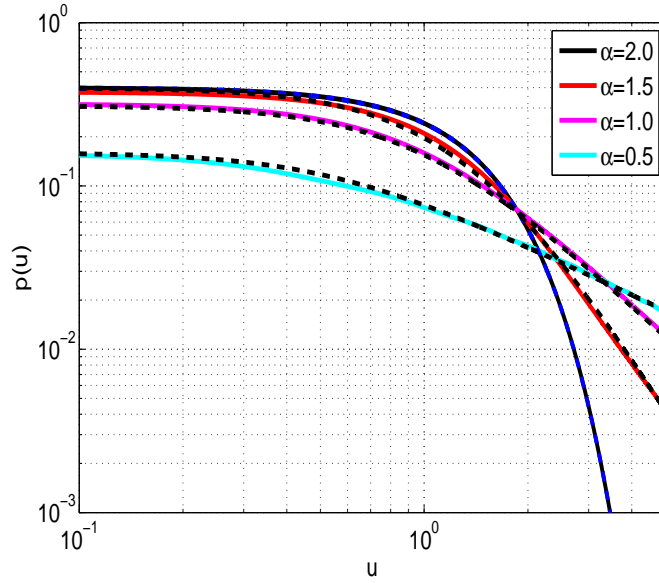


FIG. 2. Fitted PDFs using a Cauchy-Lorentz distribution as a function of the velocity u for $\alpha = 2.00$ (black line) with blue Gaussian fit. The other fits are shown with black dashed lines with $q = 5/3$ for $\alpha = 3/2$ (red line); $q = 2$ for $\alpha = 1$ (purple line); and $q = 3$ for $\alpha = 1/2$ (cyan line).

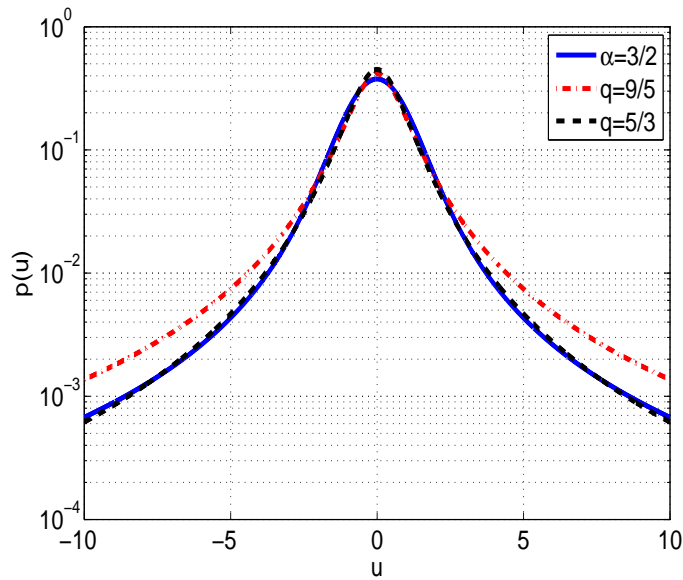


FIG. 3. PDF for $\alpha = 3/2$ (blue line) fitted to $q = 5/3$ (Eq. (9) in black) and $q = 9/5$ (Eq. (11) in red).

Figure 3 with $q = 5/3$ and $q = 9/5$ using Eqs (9) and (11), respectively. We find that using $q = 5/3$, gives a significantly better fit compared to the higher q value. In comparison using the relation in Eq. (11) yields $q = 9/5$ for $\alpha = 3/2$, $q = 2$ for $\alpha = 1$ and $q = 7/3$ for $\alpha = 1/2$ which seems to provide excessive tails for smaller α , thus we will use the relation in Eq. (9) throughout the rest of the paper.

We adopt the definition that any diffusive process that diverges from the form $\langle x^2 \rangle(t) \propto t$ is called anomalous however in most cases we will deal with super-diffusion where $\langle x^2 \rangle(t)$ may be divergent. In order to find a useful statistical measure of the super-diffusive or fractal process we introduce,

$$\langle v^2 \rangle_q = \int_{-\infty}^{\infty} dv (F(v))^q v^2, \quad (12)$$

which we will call the q -expectation. Note, that e.g. the exactly solvable case with $\alpha = 1.0$ we find that the ordinary expectation diverges, however as q increases a finite measure is found. Moreover, it can be shown that the q -expectation converges if $\alpha q > 3/2$, this gives also the opportunity to define a pseudo-energy in the system as the smallest possible value αq where the q -expectation converges. Naturally this reduces to the classical energy for $\alpha = 2$.

For the numerical estimation, we compute the q -expectation of $F(v)$ $|v| < \infty$, however for numerical tractability we have used a bounded PDF with finite support $F(v)_{num} = F(v) \quad |v| < 10$ and zero everywhere else.

Figure 4, shows the q -expectation as a function of q for $\alpha = 0.5$ (blue stars), $\alpha = 1.0$ (red triangles), $\alpha = 1.5$ (black diamonds) and $\alpha = 2.0$ (magenta rings). We find that just as expected the q -expectation falls off with q , however distributions coming from smaller α (more intermittent) falls off faster than the Gaussian q -expectation. In general, there exist a smallest value for q where the q -expectation is finite. In the case of $\alpha = 2.0$ it converges for all q and we find that the PDF is a Gaussian with a variance $\sigma^2 = 1.0$. Furthermore the minimum q value for $\alpha = 3/2$ is $q_{min} \approx 4/3$, $\alpha = 1$ is $q_{min} \approx 2$ and $\alpha \approx 1/2$ is $q_{min} = 4$. Note that the results q -expectation exhibit a slight bend around q_{min} , where convergent values of the q -expectation is found for the whole real line.

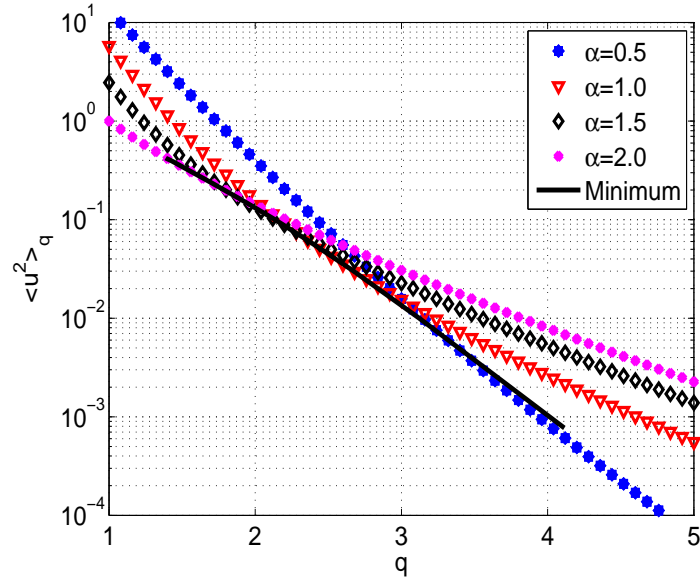


FIG. 4. The q -expectation as a function of q for $\alpha = 0.5$ (blue stars), $\alpha = 1.0$ (red triangles), $\alpha = 1.5$ (black diamonds) and $\alpha = 2.0$ (magenta rings).

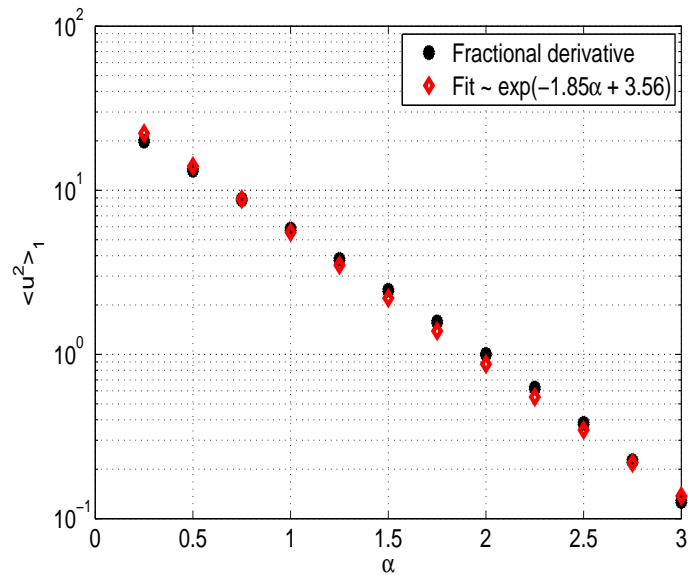


FIG. 5. The 1-expectation as a function of α for the numerically bounded PDFs with the numerical result in black dots and a fit in red diamonds is shown.

The q -expectation can be computed analytically in a few cases,

$$\begin{aligned} \langle u^2 \rangle_q(\alpha = 2) &= \int_{-\infty}^{\infty} u^2 \left(\frac{e^{-u^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \right)^q du = \lim_{R \rightarrow \infty} (2\pi)^{-q/2} \sigma^2 q^{-3/2} \left(\frac{e^{-R^2/(2\sigma^2)}}{\sqrt{\sigma^2}} \right) \\ &\times \left(-2\sqrt{q}R + e^{R^2/(2\sigma^2)} \sqrt{2\pi\sigma} \operatorname{erf} \left(\frac{\sqrt{q}R}{\sqrt{2\sigma^2}} \right) \right), \quad \text{and} \end{aligned} \quad (13)$$

$$\begin{aligned} \langle u^2 \rangle_q(\alpha = 1) &= \lim_{R \rightarrow \infty} \left(\frac{\gamma}{\pi} \right)^q \int_{-R}^R \frac{u^2 du}{(\gamma^2 + u^2)^q} \\ &= \frac{\gamma^q}{\pi^q} \lim_{R \rightarrow \infty} \frac{1}{(3 + 4(-2 + q)q)R} \times \\ &\quad (\gamma^2(R^2 + \gamma^2))^{-q} (-2\gamma^{2q}(R^2 + \gamma^2) ((-1 + 2q)R^2 + \gamma^2) \\ &\quad + 2\gamma^4(R^2\gamma^2)^q {}_2F_1(-1/2, q; 1/2, -R^2/\gamma^2)). \end{aligned} \quad (14)$$

Here ${}_2F_1(a, b; c; d)$ is the Hypergeometric function and γ is normalization constant. The integrals 13 and 14 are computed using Mathematica and we observe that Eq. (14) can be divergent depending on the value of q . Note that, for $q = 1$ Eq. (13) reduces to the usual value and that the singularity in the denominator in Eq. (14) is in the sub-diffusive domain of $\alpha > 2$. It is interesting to note that the value at the crossings of the results for different α -s are close to the predicted q -values of the models in Eq. (9), however it is difficult to obtain the exact value due to the functional form of Eq. (14). The solid black line shows the numerically found crossing determining the minimum q -expectation, it should be noted that for $q < 2.25$ the minimum falls off as a power-law proportional to $q^{3.08}$.

Furthermore in Figure 5, the 1-expectation or energy is shown as a function (numerical result in black dots and a fit in red diamonds) of α utilizing the finite support of the PDFs shown in Figure 1. It is found that the 1-expectation decreases exponentially with increasing α , here we also note that the energy is convergent for all α due to the finite support of the PDFs. The main motivation for defining the q -expectation or tempered pseudo-energy is the sharp increase in the second moment for small α , where expectation value is diverging due to the long tails of the PDFs.

In thermodynamics a measure on the number of ways a system can be arranged is termed entropy. In terms of generalized statistical mechanics q -entropy or Tsallis entropy can be introduced as,

$$S_q = \frac{1 - \int dv (F(v))^q}{q - 1}. \quad (15)$$

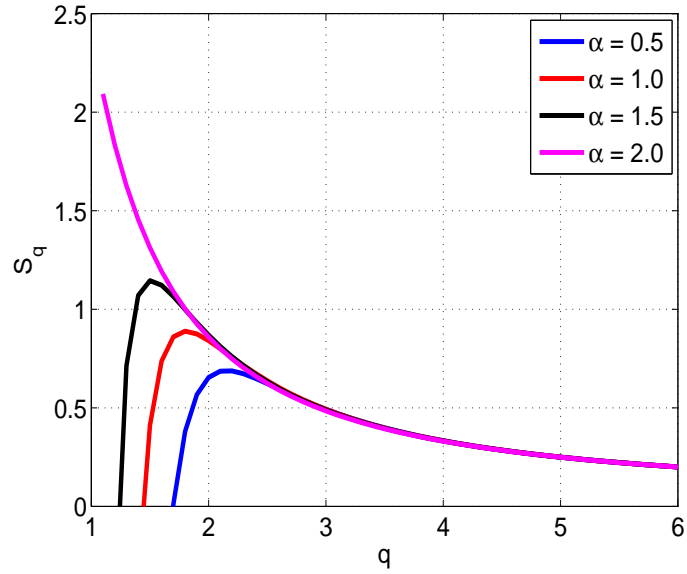


FIG. 6. The q -entropy as a function of q for PDFs with finite support with $\alpha = 0.5$ (blue line), $\alpha = 1.0$ (red line), $\alpha = 1.5$ (black line) and $\alpha = 2.0$ (magenta line) are displayed.

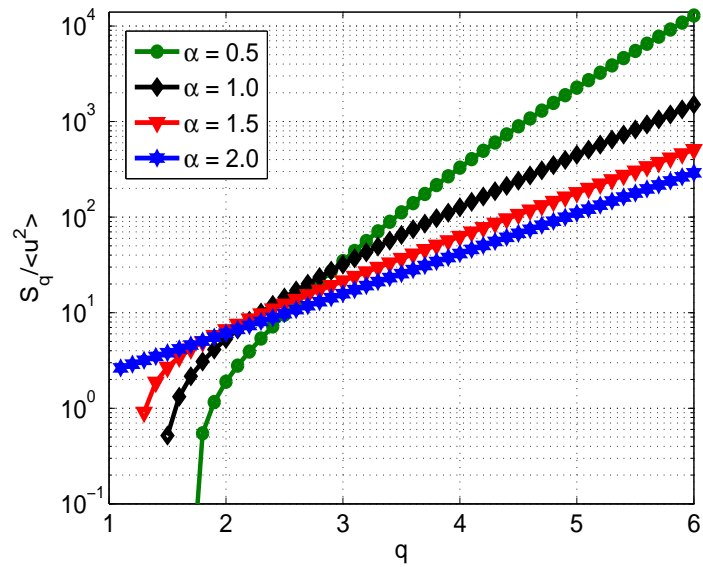


FIG. 7. The q -entropy normalized to the q -expectation $\langle u^2 \rangle_q$ as a function of q for PDFs with finite support.

Note that for Gaussian statistics the $q = 1.0$ q -entropy is reduced (by L'Hospital's rule) to the regular Boltzmann-Gibbs entropy $S_q = - \int dv \log(F(v))F(v)$. Note that this generalized entropy is non-extensive, i.e. for two systems A and B , the total entropy is not the sum of the entropies of the individual systems, $S_q(A + B) \neq S_q(A) + S_q(B)$.

We will now use the Tsallis entropy to investigate the importance of fractal structure in velocity space. In Figure 6, the dependence of the q -entropy as a function of q for $\alpha = 0.5$ (blue line), $\alpha = 1.0$ (red line), $\alpha = 1.5$ (black line) and $\alpha = 2.0$ (magenta line) are displayed. In the Figure, the scaling with q is represented by $q = 1$ ($\alpha = 2.0$), $q = 5/3$ ($\alpha = 3/2$), $q = 2.0$ ($\alpha = 1.0$) and $q = 3.0$ ($\alpha = 1/2$). We note that the q -Entropy is decreasing with q as a power-law and that local maxima are present close to the corresponding q values for each distribution. This indicated that as fractality index increases the entropy decreases indicating a self-organising behavior in velocity space with long-range correlations. The number of possible microscopical realizations decreases as a power-law.

The q -entropy normalized with the q -expectation as a function of q is displayed in Figure 7 for $\alpha = 1/2$ (green rings), $\alpha = 1.0$ (black diamonds), $\alpha = 3/2$ (red triangles) and $\alpha = 2.0$ (blue stars). The normalized q -entropy is rapidly increasing with increasing q mainly due to the rapid decrease of the q expectation ($\langle u^2 \rangle_q$) with increasing q . This indicates that in a statistical mechanics sense the normalized generalized entropy is increasing with increasing q indicating an increasing coarse-graining process in velocity space into the sub-diffusive domain whereas in the range of small α the high-velocity, small likelihood events are more dominant.

IV. RESULTS AND DISCUSSION

Non-linear processes with non-Gaussian character have attracted significant attention during recent years calling for an efficient model describing such dynamics. In this paper we have investigated one prominent candidate capturing the main features in the dynamics, namely the Fractional extended Fokker-Planck Equation (FFPE). The FFPE is obtained by modifying the velocity derivative to a fractional differential operator allowing for non-local effects in velocity space. The underlying physical reasoning for using the FFPE is to allow

for the non-negligible probability of direction preference and long jumps, i.e., Lévy flights, which therefore allows for asymmetries and long tails in the equilibrium PDFs, respectively.

The aim of this study was to shed light on the non-extensive properties of the velocity space statistics and characterization of the fractal processes of the FFPE in terms of Tsallis statistics. The non-extensive statistical mechanics of Tsallis provides velocity space distribution functions intermediate to that of Gaussians and Lévy distributions adjustable by a continuous real parameter q which seems to be suitable for comparing with the distribution found in FFPE. The parameter q describes the degree of non-extensivity in the system. Non-extensive statistical mechanics has a solid theoretical basis for analysing complex systems out of equilibrium. For systems comprised of independent or parts interacting through short-range forces the Boltzmann-Gibb statistical mechanics is sufficient however for systems exhibiting fractal structure or long range correlations this approach becomes unwarranted.

In this work we have utilized generalized q -moments or q -expectations as $\langle v^p \rangle_q = \int dv F(v)^q v^p$ due to the obtained Lévy type character of the distributions, higher moments may diverge. The q -expectation can have convergent moment although the regular moments diverges. This also gives us the opportunity to have a convergent pseudo-energy that is always convergent. We show that the PDFs derived from the Tsallis statistic is in good agreement with those found using the FFPE. Moreover, we find that self-organising behavior is present in the system where the ratio of the entropy and energy expectation is decreasing with decreasing fractionality or increased α .

To this end, it seems that a FFPE is a viable candidate for explaining certain non-linear features ubiquitous to anomalous plasma transport as well as for other physical processes. Note that in Ref. 24, a relation between Tsallis statistical mechanics and Navier-Stokes turbulence was established. A direct numerical comparison between the Langevin approach and the FFPE using Tsallis statistical mechanics is a possible topic for future work.

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