

Effect of rossby and alfvén waves on the dynamics of the tachocline

Leprovoſt, N. & Kim, E. J.

Published PDF deposited in Coventry University's Repository

Original citation:

Leprovoſt, N & Kim, EJ 2007, 'Effect of rossby and alfvén waves on the dynamics of the tachocline' *Astrophysical Journal*, vol. 654, no. 2 I, pp. 1166-1170.

<https://dx.doi.org/10.1086/509713>

DOI 10.1086/509713

ISSN 0004-637X

ESSN 1538-4357

Publisher: American Astronomical Society

Copyright © and Moral Rights are retained by the author(s) and/ or other copyright owners. A copy can be downloaded for personal non-commercial research or study, without prior permission or charge. This item cannot be reproduced or quoted extensively from without first obtaining permission in writing from the copyright holder(s). The content must not be changed in any way or sold commercially in any format or medium without the formal permission of the copyright holders.

EFFECT OF ROSSBY AND ALFVÉN WAVES ON THE DYNAMICS OF THE TACHOCLINE

NICOLAS LEPROVOST AND EUN-JIN KIM

Department of Applied Mathematics, University of Sheffield, Sheffield, UK

Received 2006 July 14; accepted 2006 September 22

ABSTRACT

To understand magnetic diffusion, momentum transport, and mixing in the interior of the Sun, we consider an idealized model of the tachocline, namely, magnetohydrodynamic (MHD) turbulence on a β -plane subject to a large-scale shear (provided by the latitudinal differential rotation). This model enables us to self-consistently derive the influence of shear, Rossby, and Alfvén waves on the transport properties of turbulence. In the strong magnetic field regime, we find that the turbulent viscosity and diffusivity are reduced by magnetic fields only, as in the two-dimensional MHD case (without Rossby waves). In the weak magnetic field regime, we find a crossover scale (L_R) from a Alfvén-dominated regime (on small scales) to a Rossby-dominated regime (on large scales). For parameter values typical of the tachocline, L_R is larger than the solar radius so that Rossby waves are unlikely to play an important role in the transport of magnetic field and angular momentum. This is mainly due to the enhancement of magnetic back-reaction by shearing, which efficiently generates small scales, and thus strong currents.

Subject headings: Sun: interior — Sun: rotation — turbulence

1. INTRODUCTION

Data from global helioseismology (Thompson et al. 2003) have shed some light on the internal rotation of the Sun. Throughout the convective envelope, the rotation rate decreases monotonically toward the poles. Near the base of the convection zone, there is a sharp transition between differential rotation in the convective envelope and nearly uniform rotation in the radiative interior. This transition region has become known as the solar tachocline (Spiegel & Zahn 1992), and its thickness has been constrained to be only a few percent of the solar radius by helioseismic data (Charbonneau et al. 1999). The rotation rate of the radiative interior is intermediate between the equatorial and polar regions of the convection zone. Thus, the radial angular velocity gradient across the tachocline is positive at low latitudes and negative at high latitudes. The tachocline is most probably located below the convection zone (Charbonneau et al. 1999), but part of it might be included in the overshooting region where turbulent plumes coming from the convection zone can induce turbulent motions. This overshooting layer contains a prominent latitudinal differential rotation. It is thus of primary importance to understand the physics pertinent to such a turbulent layer in the presence of strong shear. In particular, as the tachocline links two regions of very different transport properties, theoretical predictions of transport coefficients are essential to understand its long-term dynamics (e.g., its thickness). For instance, different mechanisms have been invoked to stabilize this transition layer against the radiative broadening during the solar evolution. First, since the radial mixing is ineffective due to the stable stratification in the tachocline, Spiegel & Zahn (1992) suggested that the efficient turbulent transport of angular momentum (eddy viscosity) in the horizontal direction could be responsible for the confinement of the tachocline. Other models rely on the existence of a magnetic field fully contained in the radiative zone (Rüdiger & Kichatinov 1996; Gough & McIntyre 1998; MacGregor & Charbonneau 1999). Specifically, Rüdiger & Kichatinov (1996) have shown that this configuration could explain the dynamics of the tachocline if the effective viscosity is larger by at least 4 orders of magnitude than its molecular value. Gough & McIntyre (1998) and MacGregor & Charbonneau (1999) have estimated

the thickness of the tachocline by balancing the shearing of the poloidal field with the resistive dissipation of the toroidal magnetic field. However, in this estimate, the turbulent diffusivity and viscosity might be more relevant than the molecular values. The tachocline is also important for the generation of magnetic field inside the Sun (by means of a dynamo process), whose operation crucially depends on the values of effective (eddy) viscosity and magnetic diffusivity (Parker 1993; Tobias 2005).

The purpose of this paper is to provide a consistent theory of momentum transport and magnetic field diffusion in the tachocline. In view of the tendency of two-dimensional (2D) turbulence in a stably stratified layer, we focus on the dynamics in the (local) horizontal plane orthogonal to the density gradient, by incorporating the effects of rotation and toroidal magnetic fields. Specifically, the latitudinal differential rotation is represented by a large-scale shear flow, while the latitudinal variation in the Coriolis force is captured by using a local β -plane. This model permits us to study the influence on turbulent transport of shear flow, and Rossby and Alfvén waves. The shear flow efficiently generates small scales by shearing, thereby enhancing the back-reaction of fluctuating magnetic fields. We show that in the limit of a strong magnetic field, the turbulent viscosity and magnetic diffusivity are reduced by magnetic fields only, with the same results as in the 2D MHD case (Kim & Dubrulle 2001). In the weak magnetic field regime, we find a crossover from a Alfvén-dominated regime (on small scales) to a Rossby-dominated regime (on large scales). Using parameter values typical of the tachocline, we identify these two regimes and show that Rossby waves are unlikely to play an important role in the tachoclinic turbulent transport. This work complements our previous study of three-dimensional (3D) hydrodynamic (HD) turbulence in the tachocline (Kim 2005; Leprovost & Kim 2006), within which the effects of shear flows (due to radial and latitudinal differential rotations) and the average rotation were investigated.

The remainder of the paper is organized as follows: we provide our model of the tachocline and the governing equations in § 2, and the results for the HD and MHD cases in §§ 3 and 4, respectively. In § 5, we discuss the implication of our findings for the tachocline.

2. MODEL

To model the tachocline, we consider the motion in a plane (y, z) independent of the third coordinate x . In the tachocline case, the x -, y -, and z -directions stand respectively for the radial, azimuthal, and latitudinal directions. For simplicity, we assume that the Coriolis force varies linearly with latitude: $\boldsymbol{\Omega} = (\Omega_0 + \beta z)\hat{\mathbf{x}}$. This model has originally been introduced by Rossby (1939) to study a homogeneous sheet of fluid on a sphere, the effect of the sphericity being only in the variation of the rotation rate with latitude. The term containing β is responsible for the appearance of the so-called Rossby waves. We also assume that, on large scales, there are a linear shear flow (due to latitudinal differential rotation), $\mathbf{U}_0 = -(Az)\hat{\mathbf{y}}$, and a uniform toroidal magnetic field, $\mathbf{B}_0 = B_0\hat{\mathbf{y}}$, where $A > 0$ and $B_0 > 0$ without loss of generality. This magnetic field supports Alfvén waves in the system, whereas a shear flow will favor the creation of small scales in the direction of the shear and thus enhance the dissipation (Kim & Dubrulle 2001; Kim & Diamond 2003; Kim 2006). The generation of small scales by a shear flow enhances the effect of Lorentz force via the formation of strong current. Consequently Alfvén waves tend to become dynamically more important than Rossby waves, as is shown below.

Because of the geometry of the problem, it is convenient to work with the vorticity of the fluctuations $\boldsymbol{\omega} = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{x}}$ and their magnetic potential defined by $\mathbf{b} = \nabla \times (a\hat{\mathbf{x}})$. We study these fluctuations in the framework of the quasi-linear approximation (Moffatt 1978), where the interaction between the small-scale fields can be neglected:

$$\begin{aligned} (\partial_t + U_0 \partial_y) \boldsymbol{\omega} &= -\beta v_z - (B_0 \partial_y) \nabla^2 a + \nu \nabla^2 \boldsymbol{\omega} + f_w, \\ (\partial_t + U_0 \partial_y) a &= -v_z B_0 + \eta \nabla^2 a. \end{aligned} \quad (1)$$

Here, we assume that the turbulence is driven mainly hydrodynamically (no force in the equation for the magnetic field). For simplicity, we assume a unit magnetic Prandtl number ($\nu = \eta$) and introduce a time-dependent Fourier transform to capture the shearing effect:

$$F(\mathbf{x}, t) = \frac{1}{(2\pi)^2} \int d^2 \mathbf{k} e^{i[k_y y + k_z(t)z]} \tilde{F}(\mathbf{k}, t), \quad (2)$$

where F stands for $\boldsymbol{\omega}$, a , or f_w and the azimuthal wavenumber evolves as $\partial_t k_z = k_y A$. By using the transformation, $\tilde{F} = \tilde{F} \exp[\nu(k_y^2 t + k_z^2/3k_y A)]$, to absorb the diffusion term and by transforming the time variable from t to $\tau = k_z(t)/k_y = k_z(t_0)/k_y + A(t - t_0)$, equation (1) can be rewritten as

$$\begin{aligned} \partial_\tau \hat{\boldsymbol{\omega}} &= i \frac{\alpha \hat{\boldsymbol{\omega}}}{(1 + \tau^2)} + i k_y^2 \gamma (1 + \tau^2) \hat{a} + \frac{\hat{f}_w}{A}, \\ \partial_\tau \hat{a} &= i \frac{\gamma \hat{\boldsymbol{\omega}}}{k_y^2 (1 + \tau^2)}. \end{aligned} \quad (3)$$

Here, $\alpha = \beta/Ak_y$ is the ratio between the rate of change in the rotation vector and the shearing rate, and $\gamma = B_0 k_y/A$ is the ratio between the magnetic field and the shear. Note that the velocity and magnetic fields can be obtained from the vorticity and the magnetic potential by the identities: $\hat{v}_y = ik_z \hat{\boldsymbol{\omega}}/k^2$, $\hat{v}_z = -ik_y \hat{\boldsymbol{\omega}}/k^2$, $\hat{b}_y = ik_z \hat{a}$, and $\hat{b}_z = -ik_y \hat{a}$. Equation (3) can be solved with the initial conditions $\hat{\boldsymbol{\omega}} = \hat{a} = 0$ at $\tau = \tau_0 = k_z(t_0)/k_y$. Note that the precise value of the initial conditions is irrelevant as we are mainly interested in the long-time limit, where the memory of the initial conditions has been lost.

We are also interested in the transport of chemical species whose concentration is again split into a large-scale and a fluctuating part: $N = N_0 + n$. In the quasi-linear approximation, by assuming the molecular diffusivity of particles to be the same as the molecular viscosity (for the case of a nonunit Prandtl number, see Kim 2006), we obtain the following equation for the fluctuating density of particles:

$$\partial_\tau \hat{n} = i \frac{\hat{\boldsymbol{\omega}}}{k_y (1 + \tau^2)} \partial_z N_0, \quad (4)$$

which is the same as the second equation in (3) up to a multiplicative constant. Thus, the turbulent diffusivity of the magnetic field is the same as the turbulent particle diffusivity for $D = \eta$.

3. HD RESULTS

We first consider the case without a magnetic field ($\gamma = 0$) to elucidate the influence of Rossby waves on a sheared turbulence. Equation (3) with $\gamma = 0$ can be readily integrated with the following result for the fluctuating vorticity:

$$\hat{\boldsymbol{\omega}} = \int_{\tau_0}^{\tau} \frac{dt}{A} e^{i\alpha[\arctan \tau - \arctan t]} \hat{f}_w(t). \quad (5)$$

To calculate the turbulence intensity and the turbulent transport of angular momentum (i.e., the eddy viscosity ν_T defined as $\langle v_y v_z \rangle = -\nu_T \partial_z U_0 = \nu_T A$), we prescribe the statistical properties of the forcing f_w to be spatially homogeneous and temporally short correlated with the correlation time τ_f :

$$\langle \tilde{f}_w(\mathbf{k}_1, t_1) \tilde{f}_w(\mathbf{k}_2, t_2) \rangle = \tau_f (2\pi)^2 \delta(\mathbf{k}_1 + \mathbf{k}_2) \delta(t_1 - t_2) \phi(\mathbf{k}_2). \quad (6)$$

Here, $\phi(\mathbf{k}_2)$ is the power spectrum of the forcing. We focus on the strong-shear limit: $\xi = \nu k_y^2/A \ll 1$, which is valid on characteristic scales of fluctuation $L \gg L_s = 2\pi(\nu/A)^{1/2}$. The calculation of the turbulence intensity and turbulent viscosity is similar to that of Kim (2005), to which we refer the readers for technical details. The following solutions are obtained:

$$\begin{aligned} \langle \omega^2 \rangle &\sim \frac{\tau_f}{(2\pi)^2 A} \int d^2 \mathbf{k} \phi(\mathbf{k}) \frac{1}{3} \left(\frac{3}{2\xi} \right)^{1/3} \Gamma\left(\frac{1}{3}\right), \\ \nu_T &\sim -\frac{\tau_f}{2(2\pi)^2 A^2} \int d^2 \mathbf{k} \frac{\phi(\mathbf{k})}{k^2}. \end{aligned} \quad (7)$$

The results in equation (7) are the same as those in the 2D turbulence with $\beta = 0$ (Kim & Dubrulle 2001). That is, turbulence amplitude and transport are reduced only by shear stabilization, and Rossby waves have no influence on the intensity of turbulence or the transport of angular momentum. In the kinematic regime ($\gamma = 0$), we can also compute the magnetic diffusivity η_T , which enters the induction equation for the mean magnetic field and is defined as $\langle av_z \rangle = -\eta_T \partial_z A_0 = -\eta_T B_0$:

$$\begin{aligned} \eta_T &\sim \frac{2\tau_f}{(2\pi)^2 A^2} \int d^2 \mathbf{k} \frac{\phi(\mathbf{k})}{\alpha^2 k_y^2} \sin^2 \left(\alpha \arctan \frac{1}{a} \right) \\ &\sim \frac{2\tau_f}{(2\pi)^2 \beta^2} \int d^2 \mathbf{k} \phi(\mathbf{k}) \sin^2 \left(\alpha \arctan \frac{1}{a} \right). \end{aligned} \quad (8)$$

Here $\alpha = \beta/Ak_y$, and $a = k_z/k_y$. Thus, in the kinematic regime, the magnetic field diffuses more slowly compared to the 2D HD case (the sine cardinal function being always smaller than 1). The last expression in equation (8) even shows that for large enough β , the magnetic diffusivity is reduced by a factor of β^{-2} . Note that the same reduction applies to the transport of particle as $\eta_T = D_T$.

4. MHD TURBULENCE ON A β -PLANE

Combining the two equations of system (3), we can obtain the following equation for the magnetic potential:

$$\partial_\tau [(1 + \tau^2)\partial_\tau \hat{a}] - i\alpha \partial_\tau \hat{a} + \gamma^2(1 + \tau^2)\hat{a} = \frac{i\gamma}{Ak_y^2} \hat{f}_w. \quad (9)$$

The exact solution to equation (9) can be written in terms of Heun’s confluent function; however, we here derive the physically more transparent results in the case of a weak and a strong magnetic field separately. The boundary between the weak ($|\gamma| \gg 1$, where $\gamma = B_0 k_y/A$) and the strong magnetic regime ($|\gamma| \ll 1$) is given by $L_B = 2\pi B_0/A$.

4.1. Weak Magnetic Field

For $|\gamma| \ll 1$, we calculate the correction to the magnetic potential and vorticity up to second order in γ as

$$\begin{aligned} \hat{\omega}(\tau) &= \hat{\omega}_0(\tau) + \gamma \hat{\omega}_1(\tau) + \gamma^2 \hat{\omega}_2(\tau) + \dots \\ \text{and } \hat{a}(\tau) &= 0 + \gamma \hat{a}_1(\tau) + \gamma^2 \hat{a}_2(\tau) + \dots, \end{aligned} \quad (10)$$

where $\hat{\omega}_0(\tau)$ is given by equation (5). Solving order by order, we find that $\hat{\omega}$ is an even function of γ , whereas \hat{a} is an odd function. The first correction to the HD result is

$$\begin{aligned} \hat{a}_1(\tau) &= \int_{\tau_0}^\tau dt \frac{\hat{f}_w(t)}{k_y^2 A \alpha} \left[e^{i\alpha(\arctan \tau - \arctan t)} - 1 \right], \\ \hat{\omega}_2(\tau) &= \frac{i}{A\alpha} \int_{\tau_0}^\tau dt \hat{f}_w(t) e^{i\alpha \arctan \tau} \\ &\quad \times \int_t^\tau dt' \left(e^{-i\alpha \arctan t} - e^{-i\alpha \arctan t'} \right). \end{aligned} \quad (11)$$

Using equation (11), we can easily calculate the second-order (in γ) correction to the turbulence amplitude and turbulent stresses. Assuming that the forcing has the following symmetry property $\phi(k_y) = \phi(-k_y)$, we find that the first correction to the turbulence amplitude vanishes as it is odd in k_y . The turbulence amplitude is thus given by equation (7), showing that neither Alfvén nor Rossby waves change the turbulence amplitude. On the contrary, the Reynolds and Maxwell stresses are modified as follows:

$$\begin{aligned} \langle v_y v_z \rangle &\sim -\frac{\tau_f}{(2\pi)^2 A} \\ &\quad \times \int d^2 \mathbf{k} \phi(\mathbf{k}) \left[\frac{1}{2k^2} + \frac{2\gamma^2}{k_y^2 \alpha^2} \sin^2 \left(\frac{\alpha}{2} \arctan \frac{1}{a} \right) \right], \\ \langle b_y b_z \rangle &\sim -\frac{4\tau_f}{3(2\pi)^2 A} \\ &\quad \times \int d^2 \mathbf{k} \frac{\gamma^2 \phi(\mathbf{k})}{k_y^2 \alpha^2} \sin^2 \left(\frac{\alpha}{2} \arctan \frac{1}{a} \right) \left(\frac{3}{2\xi} \right)^{2/3} \Gamma \left(\frac{2}{3} \right). \end{aligned} \quad (12)$$

Here, $\alpha = \beta/Ak_y$, $a = k_z/k_y$, and $\xi = \nu k_y^2/A$. One can see that the correction to the magnetic stress is much larger than that to the Reynolds stress (recall that $\xi \ll 1$). Comparing the zeroth-order solution for the Reynolds stress and second order for the Maxwell stress, we find that the former is dominant if $\gamma^2 \alpha^{-2} \xi^{-2/3} \ll 1$ (giving us the result of the 2D case). In the opposite case ($\gamma^2 \alpha^{-2} \xi^{-2/3} \gg 1$), the transport of angular momentum is mainly controlled by the magnetic part of the stress, i.e., by Alfvén wave turbulence. In particular, the turbulent viscosity $\nu_T = \langle \langle v_y v_z \rangle -$

$\langle b_y b_z \rangle \rangle / A$ is now positive, reminiscent of the direct cascade of energy in MHD turbulence (Kim & Dubrulle 2001). These results enable us to calculate the characteristic scale at which the transition from the Rossby wave turbulence (with negative viscosity) to the Alfvén turbulence (positive viscosity) occurs, $L_R = 2\pi(AB_0^3/\beta^3 \nu)^{1/4}$, which corresponds to the ‘‘Rhines scale’’ (referring to the scale that separates, in ordinary geostrophic turbulence, eddy turbulence from wave turbulence). Our result shows that $L_R \sim R_e^{1/4} \sim R_m^{1/4}$, as we assume a unit Prandtl number. However, it is easy to show that for $\eta \gg \nu$ (as is relevant for the tachocline), or equivalently for $P_m \ll 1$, the first correction to the magnetic potential is the same as above. Consequently, the magnetic stress remains the same, with ν being replaced by η . Thus, for low Prandtl number, the crossover scale is $L_R = 2\pi(AB_0^3/\beta^3 \eta)^{1/4} \sim R_m^{1/4}$. This scaling is the same as that of Diamond et al. (2007) obtained in a different context (MHD turbulence on a β -plane without a shear flow).

In a similar way, we can calculate the turbulent diffusivity η_T . To calculate the first two leading-order terms, we need to keep terms up to the third order in γ for the magnetic potential (not shown here for brevity). Here again, only the leading-order term survives when assuming $\phi(k_y) = \phi(-k_y)$, and thus the turbulent diffusivity remains the same as that obtained in the HD case (see eq. [8]).

4.2. Strong Magnetic Field

When $|\gamma| \gg 1$, we seek a Wentzel-Kramers-Brillouin (WKB) solution of equation (9). Up to second order in $\epsilon = 1/\gamma$, a solution for the homogeneous part can be found as

$$F(\tau) = A \frac{1}{\sqrt{1 + \tau^2}} \exp \left[i \frac{\alpha}{2} \arctan \tau + \epsilon^2 H(\tau) \right] \cos \left[\frac{1}{\epsilon} G(\tau) + \phi \right]. \quad (13)$$

Here, A and ϕ are two integration constants:

$$\begin{aligned} G(\tau) &= \tau + \frac{\epsilon^2}{4} \left(\frac{\alpha^2}{4} - 1 \right) \left(\arctan \tau + \frac{\tau}{1 + \tau^2} \right), \\ H(\tau) &= \left(1 - \frac{\alpha^2}{4} \right) \frac{1}{4(1 + \tau^2)^2}. \end{aligned} \quad (14)$$

Using the method of variation of constants, one obtains the solution for the magnetic potential and the vorticity:

$$\begin{aligned} \hat{a}(\tau) &= \int_{\tau_0}^\tau dt \frac{i \hat{f}_w(t)}{Ak_y^2 G'(t) \sqrt{1 + t^2} \sqrt{1 + \tau^2}} \\ &\quad \times \exp \left\{ \frac{i\alpha}{2} (\arctan \tau - \arctan t) \right. \\ &\quad \left. + \epsilon^2 [H(\tau) - H(t)] \right\} \sin \frac{1}{\epsilon} [G(\tau) - G(t)], \\ \hat{\omega}(\tau) &= \int_{\tau_0}^\tau dt \frac{\hat{f}_w(t) \sqrt{1 + \tau^2}}{AG'(t) \sqrt{1 + t^2}} \\ &\quad \times \exp \left\{ \frac{i\alpha}{2} (\arctan \tau - \arctan t) \right. \\ &\quad \left. + \epsilon^2 [H(\tau) - H(t)] \right\} \\ &\quad \times \left\{ G'(\tau) \cos \frac{1}{\epsilon} [G(\tau) - G(t)] \right. \\ &\quad \left. + \frac{i\alpha G'(t)^2 - 2\tau}{2(1 + \tau^2)G'(t)^2} \epsilon \sin \frac{1}{\epsilon} [G(\tau) - G(t)] \right\}. \end{aligned} \quad (15)$$

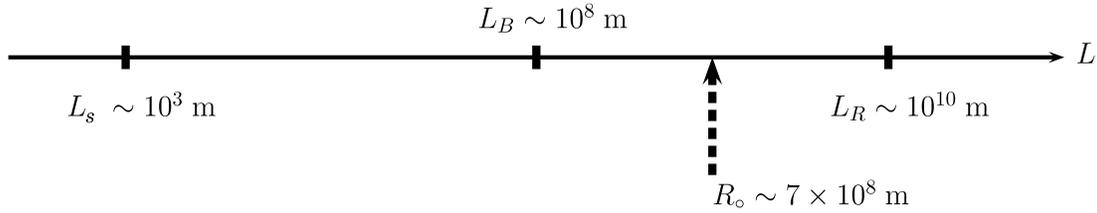


FIG. 1.—Sketch of the different relevant scales evidenced in this paper for tachoclinic values of the parameters. The parameter L_R is the scale that separates the strong magnetic field regime from the weak one, and L_B (obtained in the weak-field approximation) is the scale that separates an Alfvén-dominated regime from a Rossby-dominated one. We also included the smallest scale L_s for which the large-shear limit is valid.

We can then use equation (15) to calculate the Reynolds and Maxwell stresses and the turbulent viscosity. In the strong-shear limit ($\xi \ll 1$), we obtain the following:

$$\nu_T \sim \frac{\tau_f}{4(2\pi)^2 B_0^2} \int d^2\mathbf{k} \frac{\phi(\mathbf{k})k_y^2}{k^6} \left(1 - \frac{\alpha^2}{4}\right). \quad (16)$$

Equation (16) shows that the turbulent viscosity is the same as in 2D MHD (Kim & Dubrulle 2001) except for the additional term proportional to α^2 due to the transport reduction by Rossby waves. However, the strong magnetic field limit is only valid for $k_y \ll A/B_0$, and thus the correction term is bounded from above: $\alpha \ll \beta B_0/A^2$. Using typical solar values, $B_0 \sim 1$ T, $A \sim 6 \times 10^{-8}$ s $^{-1}$, and $\beta \sim 2.7 \times 10^{-15}$ m $^{-1}$ s $^{-1}$, we obtain $\alpha \ll 0.75$, and consequently the correction term (proportional to α^2) is very small.

The turbulent diffusivity can also be obtained in a similar way:

$$\eta_T \sim \frac{\tau_f}{6(2\pi)^2 B_0^2} \int d^2\mathbf{k} \frac{\phi(\mathbf{k})}{k^2 k_y^2} \left(\frac{3}{2}\right)^{1/3} \Gamma\left(\frac{1}{3}\right) \left(\frac{\nu k_y^2}{A}\right)^{2/3}, \quad (17)$$

which is the same as the 2D case. Therefore, both the turbulent viscosity and diffusivity are reduced by a strong large-scale magnetic field, while the effect of Rossby waves is negligible.

5. DISCUSSION

In order to elucidate the turbulent transport in the tachocline, we considered a 2D model of turbulence on a β -plane, in the presence of a large-scale latitudinal shear (with shearing rate A) and a uniform toroidal magnetic field (intensity B_0). By using the quasi-linear approximation, we have computed the turbulent viscosity and diffusivity (of magnetic fields or particle) in the two limiting cases of weak and strong magnetic field. The intensity of turbulence is not altered much by the presence of waves, whereas the turbulent transport can be severely affected due to the effect of these waves on the phase of the field.

In the case of a weak magnetic field, we have shown that there is a crossover scale $L_R = 2\pi(AB_0^3/\beta^3\eta)^{1/4}$ from an Alfvén-dominated (on small scales) to a Rossby-dominated turbulence (on large enough scales). In the Rossby-dominated regime, we found a negative turbulent viscosity, suppressed only by the shear: $\nu_T \propto -A^{-2}$. In the Alfvén-dominated regime, the turbulent viscosity is positive with the following scaling: $\nu_T \propto B_0^2(\beta A)^{-2}$. We have also shown that the turbulent diffusivity is not very much affected by a weak magnetic field and that for all scales, the turbulent diffusivity is suppressed only by Rossby waves: $\eta_T \propto \beta^{-2}$ (the same as in the HD case). In the case of a strong magnetic field, we found that the turbulent viscosity and diffusivity are positive and the same as in the 2D MHD case (except for an additional

factor in the turbulent viscosity that is very small in the tachocline context) and consequently are suppressed only by magnetic fields: $\eta_T \propto \nu_T \propto B_0^{-2}$.

These findings have interesting implications for the dynamics of the tachocline. Using typical solar values, $B_0 \sim 1$ T, $A \sim 6 \times 10^{-8}$ s $^{-1}$, $\beta \sim 2.7 \times 10^{-15}$ m $^{-1}$ s $^{-1}$, $\nu \sim 10^{-2}$ m 2 s $^{-1}$, and $\eta \sim 1$ m 2 s $^{-1}$, we can calculate the boundary between the weak and the strong magnetic regime, $L_B = 2\pi B_0/A \sim 10^8$ m and the scale at which the transition from Alfvén to Rossby turbulence occurs, $L_R = 2\pi(AB_0^3/\beta^3\eta)^{1/4} \sim 10^{10}$ m. Let us recall that this scale was calculated in the weak magnetic field case ($L > L_B$), which is consistent in the tachocline case as $L_R > L_B$. Figure 1 shows the relative positions of these two scales compared to the radius of the Sun R_\odot . We also show on this figure the minimal scale $L_s = 2\pi(\nu/A)^{1/2} \sim 10^3$ m for which our strong-shear approximation holds.

As relevant motions in the tachocline are probably on scales smaller than the solar radius, the dynamics of the tachocline is likely to be insensitive to the effect of Rossby waves. In the worst case, the dynamics would be that of the weak magnetic field with Alfvén-dominated turbulence (meaning a positive turbulent viscosity). These results are mainly due to the enhancement of the back-reaction of fluctuating magnetic fields by shearing. We note that our result is directly related to Gilman & Fox (1997), who have found a similar behavior of the dominance of magnetic fields in transporting angular momentum in their numerical study of the joint instability (on a spherical shell with imposed differential rotation and magnetic field): even for weak magnetic fields, the transport of angular momentum from the mean field to the perturbation is mainly carried by the Maxwell stress rather than the Reynolds stress.

The reduction of the turbulent diffusion and transport induced by Alfvén waves may be problematic in the solar context, invalidating some of the mechanisms believed to be important for the tachocline dynamics (see also Diamond et al. 2007), e.g., the turbulent horizontal viscosity model of Spiegel & Zahn (1992). However, the important point in this particular scenario is that the horizontal viscosity can be much larger than the vertical one. We have shown (Leprovost & Kim 2006) that this is the case for a turbulence sheared more strongly in the radial direction than in the latitudinal one (as is the case in the tachocline where the radial shear is at least 1 order of magnitude greater than the latitudinal one). If the effect of the magnetic field is the same in the two directions (reduction by a factor of B_0^2), this result should not be altered by the inclusion of a magnetic field. However, a detailed analysis of this situation requires the study of a 3D model and is outside the scope of this paper.

Finally, we note that in this paper, we assumed the toroidal magnetic field to be uniform on a local Cartesian (β) plane. However, on a sphere, there is a possibility of instability in the presence of a band of toroidal magnetic field and latitudinal differential rotation (Gilman & Fox 1997). The turbulence arising from this

instability can then be considered as a source of part of the forcing in our model.

We thank D. W. Hughes and S. M. Tobias for useful comments. This work was supported by UK PPARC grant PP/B501512/1.

REFERENCES

- Charbonneau, P., Christensen-Dalsgaard, J., Henning, R., Larsen, R. M., Schou, J., Thompson, M. J., & Tomczyk, S. 1999, *ApJ*, 527, 445
- Diamond, P. H., Itoh, S.-I., Itoh, K., & Silvers, L. J. 2007, in *The Solar Tachocline*, ed. D. W. Hughes, R. Rosner, & N. O. Weiss (Cambridge: Cambridge Univ. Press), in press
- Gilman, P. A., & Fox, P. 1997, *ApJ*, 484, 439
- Gough, D. O., & McIntyre, M. E. 1998, *Nature*, 394, 755
- Kim, E. 2005, *A&A*, 441, 763
- . 2006, *Phys. Rev. Lett.*, 96, 084504
- Kim, E., & Diamond, P. H. 2003, *Phys. Rev. Lett.*, 90, 185006
- Kim, E., & Dubrulle, B. 2001, *Phys. Plasmas*, 8, 813
- Leprovoost, N., & Kim, E. 2006, *A&A*, 456, 617
- MacGregor, K. B., & Charbonneau, P. 1999, *ApJ*, 519, 911
- Moffatt, K. H. 1978, *Magnetic Field Generation in Fluids* (Cambridge: Cambridge Univ. Press)
- Parker, E. N. 1933, *ApJ*, 408, 707
- Rossby, C. G. 1939, *J. Marine Res.*, 2, 38
- Rüdiger, G., & Kichatinov, L. L. 1996, *ApJ*, 466, 1078
- Spiegel, E. A., & Zahn, J.-P. 1992, *A&A*, 265, 106
- Thompson, M. J., Christensen-Dalsgaard, J., Miesch, M. S., & Toomre, J. 2003, *ARA&A*, 41, 599
- Tobias, S. M. 2005, in *The Fluid Mechanics of Astrophysics and Geophysics*, ed. A. M. Soward et al. (Boca Raton: CRC Press), 193