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Original citation:
Kim, EJ & Diamond, PH 2001, 'On turbulent reconnection' Astrophysical Journal, vol. 556, no. 2 , pp. 1052-1065.
https://dx.doi.org/10.1086/321628

DOI 10.1086/321628
ISSN 0004-637X
ESSN 1538-4357

Publisher: American Astronomical Society

THE ASTROPHYSICAL JOURNAL, 556:1052–1065, 2001 August 1

ON TURBULENT RECONNECTION

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Received 2000 December 4; accepted 2001 April 9

ABSTRACT

We examine the dynamics of turbulent reconnection in two-dimensional and three-dimensional reduced MHD by calculating the effective dissipation due to coupling between small-scale fluctuations and large-scale magnetic fields. Sweet-Parker type balance relations are then used to calculate the global reconnection rate. Two approaches are employed—quasi-linear closure and an eddy-damped fluid model. Results indicate that despite the presence of turbulence, the reconnection rate remains inversely proportional to $\sqrt{\mathrm{Re}_m}$, as in the Sweet-Parker analysis. In two-dimensions, the global reconnection rate is shown to be enhanced over the Sweet-Parker result by a factor of magnetic Mach number. These results are the consequences of the constraint imposed on the global reconnection rate by the requirement of mean-square magnetic potential balance. The incompatibility of turbulent fluid-magnetic energy equipartition and stationarity of mean-square magnetic potential is demonstrated.

Subject headings: MHD — magnetic fields — turbulence

1. INTRODUCTION

Magnetic reconnection is the process whereby large-scale magnetic field energy is dissipated and magnetic topology is altered in MHD fluids and plasmas (for instance, see Vasyliunas 1975; Parker 1979; Forbes & Priest 1984; Biskamp 1993; Wang, Ma, & Bhattacharjee 1996 and references therein). Reconnection is often invoked as the explanation of large-scale magnetic energy release in space, astrophysical, and laboratory plasmas. Specifically, magnetic reconnection is thought to play an integral role in the dynamics of the magnetotail, the solar dynamo, solar coronal heating, and in the major disruption in tokamaks. For these reasons, magnetic reconnection has been extensively studied in the context of MHD, two-fluid, and kinetic models, via theory, numerical simulations, and laboratory experiments.

The basic paradigm for magnetic reconnection is the Sweet-Parker (called SP hereafter) problem (Parker 1957; Sweet 1958), in which a steady inflow velocity advects oppositely directed magnetic field lines ($\pm \boldsymbol{B}$) together, resulting in current sheet formation and, thus, reconnection (see Fig. 1). The current sheet has thickness Δ and length L, so that imposition of continuity ($v_r L = v_0 \Delta$), momentum balance ($v_0 = v_A$), and magnetic energy balance ($v_r B = \eta B/\Delta$) constrains the inflow, or "reconnection," velocity to be $v_r = v_A/\sqrt{S} \propto v_A/\sqrt{\mathrm{Re}_m}$. Here v_0 is the outflow velocity; v_A is the Alfvén speed associated with $\boldsymbol{B}$; $S \equiv v_A L/\eta$ is the Lundquist number; $\mathrm{Re}_m = ul/\eta$ is the magnetic Reynolds number, with u and l being the characteristic amplitude and length scale of the velocity—S is called the magnetic Reynolds number, Re_m, in some literatures. Note that the SP process forms strongly anisotropic current sheets since $\Delta/L = \sqrt{S}$ and $S \gg 1$. Note also the link between sheet anisotropy and the reconnection speed v_r, i.e., $v_r/v_A = \Delta/L = 1/\sqrt{S}$. Finally, it should be noted that v_r is a measure of the *global* reconnection rate, in that it parameterizes the mean inflow velocity to the layer.

The SP picture is intrinsically appealing, on account of its simplicity and dependence only upon conservation laws. Moreover, the SP prediction has been verified by laboratory experiments (Ji, Yamada, & Kulsrud 1998). However, since Re_m is extremely large in most astrophysical applications of interest (i.e.. $\mathrm{Re}_m \sim 10^{13}$ in the solar corona), the SP reconnection speed is far too slow to explain observations. Hence, there have been many attempts to develop models of *fast* reconnection. For example, in 1964 Petschek proposed a fast reconnection model involving shock formation near the reconnection layer, which predicted $v_r = v_A/\ln S$. Unfortunately, subsequent numerical (Biskamp 1986) and theoretical (Kulsrud 2000) study has indicated that Petschek's model is internally inconsistent. While research on fast, laminar reconnection continues today (i.e., Aydemir 1992; Wang & Bhattacharjee 1993; Kleva, Drake, & Waelbroeck 1995; Ma & Bhattacharjee 1996; Shay et al. 1999) in the context of two-fluid models, the failure of the Petschek scenario has sparked increased interest in turbulent reconnection (Matthaeus & Lamkin 1986) in which turbulent transport coefficients (which can be large for large Reynolds number) act as effective dissipation coefficients, and so are thought to facilitate fast reconnection (i.e., Diamond et al. 1984; Strauss 1988). Interest in turbulent reconnection has also been stimulated by the fact that many instances of reconnection occur in systems where turbulence is ubiquitous, i.e., coronal heating of turbulent accretion disks, the dynamo in the sun's convection zone, and turbulent tokamak plasmas during disruptions.

Recently, Lazarian & Vishniac (1999) (hereafter LV) presented a detailed discussion of turbulent reconnection. LV took a rather novel approach to the problem by considering the interaction of two slabs of oppositely directed, chaotic magnetic fields when advected together. LV modeled the effects of turbulence by treating the slabs' surfaces as rough, where the roughness was symptomatic of a chaotic turbulent magnetic field structure. This "rough surface" model naturally led LV to decompose the reconnection process into an ensemble of local, "microreconnection" events, which interact to form a net "global" reconnection process. LV argue that microreconnection events occur in small-scale "layers," with dimensions set by the structure of the underlying Alfvénic MHD turbulence (i.e., the $k_\perp^{-1}$ and $k_\parallel^{-1}$, as set by the Goldreich-Sridhar model). The upper bound for the microreconnection rate obtained by LV is $v_r = v_A(u/v_A)^2 = v_A(b/B_H)^2$, where B_H is the mean, reconnection field, and u and b are small-scale velocity and magnetic field.

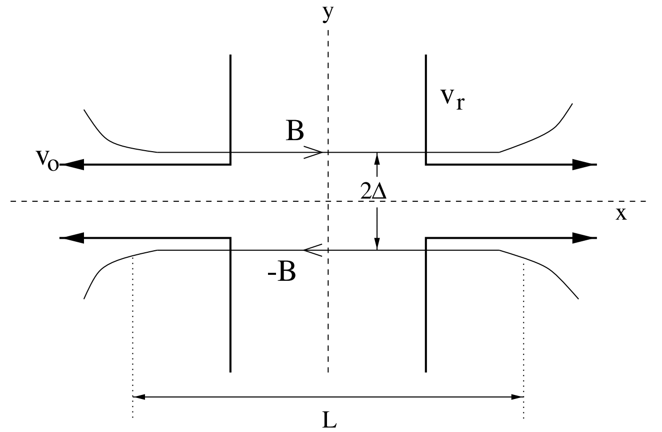

FIG. 1.—Sweet-Parker two-dimensional slab configuration. Δ and L are the thickness and length of the current sheet; $\pm \boldsymbol{B}$ are reconnecting magnetic fields; v_r and v_0 are inflow (reconnection) and outflow velocities.

While the LV arguments concerning microreconnection are at least plausible, their assertion that the global reconnection rate can be obtained by effectively superposing microreconnection events is unsubstantiated and rather dubious, in that it neglects dynamical interactions between microlayers. Such interactions are particularly important for enforcing topological conservation laws. Since the process of turbulent reconnection is intimately related to the rate of flux dissipation, and the latter is severely constrained by mean-square magnetic potential conservation, it stands to reason that such a topological conservation law will also constrain the rate of *global* magnetic reconnection. In particular, for a mean B-field with strength in excess of $B_{\rm crit} \sim (\langle u^2 \rangle / {\rm Re}_m)^{1/2}$, the flux in two-dimensions was shown to be suppressed by a factor

$$\frac{1}{1 + {\rm Re}_m \langle B \rangle^2 / \langle u^2 \rangle}\,, \tag{1}$$

where $\langle B \rangle$ is the large-scale magnetic field and $\langle u^2 \rangle$ the turbulent kinetic energy (Cattaneo & Vainshtein 1991; Gruzinov & Diamond 1994). The above expression implies that even a weak magnetic field (i.e., one far below the equipartition value $\langle b^2 \rangle \sim \langle u^2 \rangle$) is potentially important. The origin of this suppression is ultimately linked to the conservation of mean-square potential (see Das & Diamond 2000 for flux diffusion in EMHD). Hence, it is natural to investigate the effect of such constraints on reconnection, as well.

In turbulent reconnection, fluctuating magnetic fields are dynamically coupled to a large-scale magnetic field so that a similar suppression of energy transfer is expected to occur. In other words, fluctuating magnetic fields will inhibit the energy transfer from large-scale to small-scale magnetic fields (responsible for turbulent diffusion), even when the latter is far below equipartition value. This link between small and large-scale magnetic field dynamics is indeed the very feature that is missing in LV, where a global reconnection rate is considered to be a simple sum of local reconnection events, without depending on either $\langle B \rangle$ or ${\rm Re}_m$. That is, even if one local reconnection event may proceeds fast, the energy transfer from large-scale to small-scale is suppressed inversely with ${\rm Re}_m$, preventing many local reconnection events for a large ${\rm Re}_m$ and fixed large-scale field strength. Thus, the global reconnection rate is very likely to be reduced for large ${\rm Re}_m$.

The purpose of this paper is to determine the global reconnection rate by treating the dynamics of large and small-scale magnetic fields in a consistent way. The key idea is to compute the effective dissipation rate of a large-scale magnetic field (turbulent diffusivity) by taking into account small-scale field backreaction and then to use Sweek-Parker type balance relations to obtain the global reconnection rate. Since magnetic fields across current sheets are not always strictly antiparallel in real systems, we assume that only one component of the magnetic field (e.g., poloidal or horizontal field) changes its sign across the current sheet (see Fig. 2). The other component (e.g., axial field) is assumed to be very strong compared to the poloidal component. A strong axial magnetic field avoids the null point problem inherent in SP slab model, justifying the assumption of incompressibility of the flow in the poloidal (horizontal) plane. Such a magnetic configuration is ideal for the application of so-called three-dimensional reduced MHD (RMHD) (Strauss 1976). In three-dimensional RMHD, the conservation of the mean-square potential is linearly broken owing to the propagation of Alfvén waves along an axial field, but preserved by the nonlinearity. As we shall show later, the latter effect introduces additional suppression in the effective dissipation of a large-scale magnetic field compared to two-dimensional MHD. We also discuss the two-dimensional MHD case which can be recovered from our results simply by taking the limit $B_0 \to 0$, where B_0 is a axial magnetic field. Note that in this

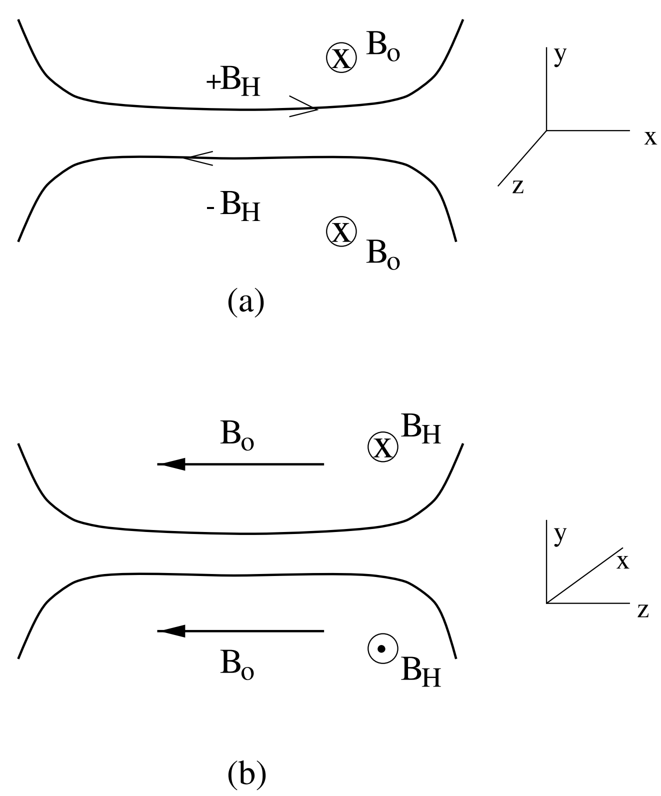

FIG. 2.—Configuration in three-dimensional RMHD. $\boldsymbol{B}_0$ is a strong axial magnetic field pointing in the z-direction, and $\pm \boldsymbol{B}_H$ are reconnecting (large-scale) magnetic fields in the x-y plane; (a) is the projection in the x-y plane, and (b) in the y-z plane.

two-dimensional limit, a neutral line appears as is the case in other works on the subject, including that of LV.

To be able to obtain analytic results, we adopt the following two methods. The first is a quasi-linear closure together with τ approximation by assuming the same correlation time for fluctuating velocity and magnetic fields employing unity magnetic Prandtl number. The second is an eddy-damped fluid model, based on large viscosity (Kim 1999), which may have relevance in Galaxy where $\nu \gg \eta$. In this model, the nonlinear backreaction can be incorporated consistently, without having to invoke the presence of fully developed MHD turbulence or assumptions such as a quasi-linear closure or τ approximation. In both models, the isotropy and homogeneity of turbulence is assumed in the horizontal (poloidal) plane since the reduction in effective dissipation of a large-scale poloidal magnetic field is likely to occur when its strength is far below the equipartition value. The effect of hyperresistivity is incorporated in our analysis. This can potentially accelerate the dissipation of a large-scale poloidal magnetic field.

The paper is organized in the following way. In § 2 we set up our problem in three-dimensional RMHD and provide the quasi-linear closure using τ approximation where the flux is estimated in a stationary case. Section 3 contains a similar analysis for an eddy-damped fluid model. The global reconnection rate for both models is presented in § 4. Our main conclusion and discussion is found in § 5.

2. QUASI-LINEAR MEAN FIELD EQUATIONS

We assume that a strong constant axial magnetic field B_0 is aligned in the z-direction and that a poloidal (horizontal) magnetic field $\boldsymbol{B}_H$ lies in the horizontal x-y plane, as shown in Figure 2. The subscript H denotes horizontal direction. The total magnetic field is then expressed as $\boldsymbol{B} = B_0 \hat{z} + \boldsymbol{B}_H = B_0 \hat{z} + \nabla \times \psi \hat{z}$, in terms of a parallel component of the vector potential ψ (i.e., $\boldsymbol{B}_H = \nabla \times \psi \hat{z}$). According to the RMHD ordering, the flow in the horizontal plane $\boldsymbol{u}$ is incompressible and therefore can be written using a scalar potential ϕ as $\boldsymbol{u} = \nabla \times \phi \hat{z}$. Then, the equations governing three-dimensional RMHD are (see Strauss 1976):

$$\partial_t \psi + \boldsymbol{u} \cdot \nabla \psi = \eta \nabla^2 \psi + B_0 \partial_z \phi \, , \tag{2}$$

$$\partial_t \nabla^2 \phi + \boldsymbol{u} \cdot \nabla \nabla^2 \phi = \nu \nabla^2 \nabla^2 \phi + \boldsymbol{B} \cdot \nabla \nabla^2 \psi \, , \tag{3}$$

where η and ν are Ohmic diffusivity and viscosity, respectively, and $\nabla^2 = \partial_{xx} + \partial_{yy}$ is meant to be the two-dimensional Laplacian. For the quasi-linear closure, unity magnetic Prandtl number ($\eta = \nu$) will implicitly be assumed. In comparison with two-dimensional MHD, the equation for the vector potential contains an additional term, $B_0 \partial_z \phi$, which reflects the propagation of Alfvén wave along the axial magnetic field $B_0 \hat{z}$. Because of this additional term, the conservation of the mean-square potential is broken in three-dimensional RMHD, albeit only linearly. In other words, the nonlinear term in equation (2) conserves $\langle \psi^2 \rangle$ since $\langle \boldsymbol{u} \cdot \nabla \psi^2 \rangle = \nabla \cdot \langle \boldsymbol{u} \psi^2 \rangle = 0$, assuming that boundary terms vanish. Note that these boundary terms could bring in the additional effect on the flux diffusion, possibly relaxing its suppression (Blackman & Field 2000). Similarly, the momentum equation contains an additional term $B_0 \partial_z \nabla^2 \psi$. These additional terms are proportional to the wavenumber k_z along $B_0 \hat{z}$. Thus, the two-dimensional case can be recovered by taking $k_z \to 0$ or $B_0 \to 0$. Note that because of a strong axial field $B_0 \hat{z}$, the vertical wavenumber k_z is much smaller than horizontal wavenumber $\boldsymbol{k}_H = k_x \hat{x} + k_y \hat{z}$; specifically, the three-dimensional RMHD ordering implies that $k_z/k_H \sim B_H/B_0 \sim \epsilon \ll 1$.

We envisage a situation where large-scale magnetic fields with a horizontal component $\boldsymbol{B}_H = \langle \boldsymbol{B}_H \rangle = \nabla \times \langle \psi \rangle \hat{z}$ are embedded in a turbulent background. The turbulence can be generated by an external forcing, for instance. The horizontal component of a large-scale magnetic field $\langle \boldsymbol{B}_H \rangle$ flows to form a current sheet of thickness Δ in the horizontal plane, so $\langle \boldsymbol{B}_H \rangle$ changes sign across the current sheet. As reconnection proceeds, small-scale flows as well as magnetic fields are generated within the current sheet. It is reasonable to model the physical processes within a current sheet as well as the background turbulence by an (approximately) isotropic and homogeneous turbulence with fluctuating velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{b} = \nabla \times \psi' \hat{z}$. Here the assumption of isotropy is justified since $\langle B_H \rangle^2 \ll \langle u^2 \rangle$, i.e.. the reconnecting field is taken to be weak.

Outside the reconnection region, there are large-scale inflow and outflow in addition to the background turbulence. Thus, to obtain SP-like balance relations, small-scale flow as well as large-scale flow should be incorporated. However, since small-scale velocity is assumed to be homogeneous and isotropic, there is no net contribution from the fluctuating velocity to mass continuity. Effectively, the small-scale velocity does not appear in the momentum balance either. However, Ohm's law (magnetic energy balance) now contains an additional term due to the correlation between fluctuating fields $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle$, leading to turbulent diffusitivy and hyperresistivity. The reconnection rate is derived in § 4 and includes contributions from both turbulent diffusivity and hyperresistivity. In the limit where the turbulent diffusivity dominates over hyperresistivity, turbulence effectively changes the Ohmic diffusivity to the sum of Ohmic diffusivity and turbulent diffusivity inside current sheet. In that case, similar balance relations to the original SP hold as long as the Ohmic diffusivity is replaced by the total diffusivity.

To recapitulate, homogeneous and isotropic turbulence is assumed to be present with magnetic fields $\boldsymbol{B}_H = \langle \boldsymbol{B}_H \rangle + \boldsymbol{b}$ ($\langle \boldsymbol{b} \rangle = 0$) and small-scale velocity $\boldsymbol{u}$ ($\langle \boldsymbol{u} \rangle = \langle \phi \rangle = 0$). Once the effective dissipation rate of $\langle \boldsymbol{B}_H \rangle$ within the reconnection zone is computed, it will be used to determine the reconnection velocity v_r through SP type balance relations.

2.1. *Mean Field Equation*

The evolution equation for ψ is obtained by taking the average of the above equation as

$$\partial_t \langle \psi \rangle + \langle \boldsymbol{u} \cdot \nabla \psi' \rangle = \eta \nabla^2 \langle \psi \rangle \, . \tag{4}$$

Note that although equation (4) does not exhibit an explicit dependence on B_0, it does depend on B_0 through the flux $\Gamma_i \equiv \langle u_i \psi' \rangle$. To compute the flux Γ_i, we first do a quasi-linear closure of $\langle \boldsymbol{u} \cdot \nabla \psi' \rangle$.

The effect of the backreaction can be incorporated in the flux Γ_i by considering the change in flux Γ_i to be due to the change in the velocity as well as the fluctuating magnetic field. That is, we can rewrite the flux as

$$\Gamma_i = \epsilon_{ij3} \langle \partial_j \phi \psi' \rangle = \epsilon_{ij3} \langle \partial_j \phi \delta \psi' - \delta \phi \partial_j \psi' \rangle \, , \tag{5}$$

where unity magnetic Prandtl number is assumed for the equal splitting between $\langle \partial_j \phi \delta \psi' \rangle$ and $\langle \delta \phi \partial_j \psi' \rangle$; the latter essentially takes the backreaction to be as important as the kinematic contribution.

2.2. *Fluctuations*

From equations (2) and (3), we can write the equation for the fluctuations in the following form:

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\psi' - \langle \boldsymbol{u}\cdot\nabla\psi'\rangle = -\boldsymbol{u}\cdot\nabla\langle\psi\rangle + \eta\nabla^2\psi' + B_0\,\partial_z\phi\ ,$$

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\nabla^2\phi - \langle \boldsymbol{u}\cdot\nabla\nabla^2\phi\rangle = \nu\nabla^2\nabla^2\phi + B_0\,\partial_z\nabla^2\psi + \langle B_H\rangle\cdot\nabla\nabla^2\psi' + \boldsymbol{b}\cdot\nabla\nabla^2\langle\psi\rangle\ .$$

Here we have assumed that there is no large-scale flow in the current sheet. To estimate $\delta\phi$ and $\delta\psi'$ in equation (5), we introduce a correlation time τ that represents the overall effect of inertial and advection terms on the left-hand side of the above equations. That is, we approximate $(\partial_t + \boldsymbol{u}\cdot\nabla)\psi' - \langle\boldsymbol{u}\cdot\nabla\psi'\rangle \equiv \tau^{-1}\psi'$, and $(\partial_t + \boldsymbol{u}\cdot\nabla)\nabla^2\phi - \langle\boldsymbol{u}\cdot\nabla\nabla^2\phi\rangle \equiv \tau^{-1}\nabla^2\phi$, where the same correlation time τ is assumed for both the fluctuating flow and magnetic field due to unity magnetic Prandtl number. Then, $\delta\phi$ and $\delta\psi'$ in equation (5) can be estimated from the above equations as follows:

$$\delta\psi' = \tau[B_0\,\partial_z\phi' - \epsilon_{ij3}\,\partial_j\phi'\partial_i\langle\psi\rangle]\ , \qquad (6)$$

$$\delta\nabla^2\phi = \tau[B_0\,\partial_z\nabla^2\psi' + \epsilon_{ij3}\,\partial_j\langle\psi\rangle\partial_i\nabla^2\psi' + \epsilon_{ij3}\,\partial_j\psi'\partial_i\nabla^2\langle\psi\rangle]\ . \qquad (7)$$

In Fourier space, the above equations take the following form:

$$\delta\psi'(\boldsymbol{k}) = \tau\left[B_0\,ik_z\,\phi(\boldsymbol{k}) + \epsilon_{ij3}\int d^3k'k'_j\,\phi(\boldsymbol{k}')(k-k')_i \times \langle\psi(\boldsymbol{k}-\boldsymbol{k}')\rangle\right]\ , \qquad (8)$$

$$\delta\phi(\boldsymbol{k}) = i\tau\left\{B_0\,k_z\,\psi'(\boldsymbol{k}) + i\epsilon_{ij3}\,\frac{1}{k^2}\int d^3k' \times [(k-k')_j k'_i k'^2 + k'_j(k-k')_i(\boldsymbol{k}-\boldsymbol{k}')^2] \times \psi'(\boldsymbol{k}')\langle\psi(\boldsymbol{k}-\boldsymbol{k}')\rangle\right\}\ . \qquad (9)$$

Note that in principle, the correlation time can be a function of the spatial scale or the wavenumber, i.e., $\tau = \tau_{\boldsymbol{k}}$. Nevertheless, for the notational simplicity, we have taken τ to be a constant by assuming that the variation of $\tau_{\boldsymbol{k}}$ in $\boldsymbol{k}$ is small or that the small-scale fields possess a characteristic scale with a small spread in $\boldsymbol{k}$. Our final result will not fundamentally change when the scale dependence of τ is incorporated.

The flux Γ_i can readily be computed once the statistics of small-scale magnetic field and the velocity are specified. As mentioned earlier, the statistics of both fluctuations are assumed to be homogeneous and isotropic in the x-y plane. We further assume that the former is homogeneous and reflectionally symmetric in the z-direction with no cross correlation between horizontal and vertical components, thereby eliminating a helicity term. The absence of helicity terms rules out a possibility of a mean field dynamo in our model. Note that owing to the presence of a strong axial field $B_0\,\hat{z}$, the correlation functions cannot be everywhere isotropic. Specifically, the correlation functions at equal time t are taken to have the form:

$$\langle\psi'(\boldsymbol{k}_1, t)\psi'(\boldsymbol{k}_2, t)\rangle = \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)\overline{\psi}(k_{1H}, k_{1z})\ , \qquad (10)$$

$$\langle\phi(\boldsymbol{k}_1, t)\phi(\boldsymbol{k}_2, t)\rangle = \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)\overline{\phi}(k_{1H}, k_{1z})\ , \qquad (11)$$

where $\overline{\psi}(k_{1H}, k_{1z})$ and $\overline{\phi}(k_{1H}, k_{1z})$ are the power spectra of ψ' and ϕ, respectively. These depend on only the magnitude of horizontal wavenumber $k_{1H} = (k_{1x}^2 + k_{1y}^2)^{1/2}$ and vertical wavenumber k_{1z}. Finally, we assume that $\langle\phi\psi'\rangle = 0$, which can be shown to be equivalent to excluding the generation of a large-scale flow by the Lorentz force.

Straightforward but tedious algebra using equations (8)–(11) in equation (5) leads to the following expression for the flux (the details are given in Appendix A):

$$\Gamma_i = -\frac{\tau}{2}\,[(\langle u^2\rangle - \langle b^2\rangle)\partial_i\langle\psi\rangle - \langle\psi'^2\rangle\partial_i\nabla^2\langle\psi\rangle]\ , \qquad (12)$$

where $\langle u^2\rangle = \int d^3k k^2\overline{\phi}(\boldsymbol{k})$, $\langle\psi'^2\rangle = \int d^3k\overline{\psi}(\boldsymbol{k})$, and $\langle b^2\rangle = \int d^3kk^2\overline{\psi}(\boldsymbol{k})$. The first term on the right-hand side of equation (12) represents the kinematic turbulent diffusion by fluid advection of the flux; the second represents the flux coalescence due to the backreaction of small-scale magnetic fields with the (negative) diffusion coefficient proportional to the small-scale magnetic energy $\langle b^2\rangle$. The third term is the hyperresistivity, reflecting the contribution to Γ_i due to the gradient of a large-scale current $\langle J\rangle = -\nabla^2\langle\psi\rangle$, where $J\hat{z} = \nabla\times\boldsymbol{B}_H$ (Boozer 1986; Bhattacharjee & Hameiri 1986; Bhattacharjee & Yuan 1995). Note that the value of hyperresistivity, being proportional to mean-square potential, is related to the small-scale magnetic energy as $\langle\psi'^2\rangle = L_{bH}^2\langle b^2\rangle$, where L_{bH} is the typical horizontal scale of $\boldsymbol{b}$. Thus, the negative magnetic diffusion (second) term and hyperresistivity (third) term are closely linked through the small-scale magnetic energy $\langle b^2\rangle$. Indeed, the negative diffusivity and hyperresistivity together conserve total $\langle\psi'^2\rangle$, while shuffling the $\langle\psi'^2\rangle$ spectrum toward large scales.

We now put equation (12) in the following form:

$$\langle b^2\rangle = \frac{2\Gamma_i/\tau + \langle u^2\rangle\partial_i\langle\psi\rangle}{\partial_i\langle\psi\rangle + L_{bH}^2\,\partial_i\nabla^2\langle\psi\rangle}\ , \qquad (13)$$

where no summation over the index i occurs.

2.3. *Stationary Case:* $\partial_t\langle\psi'^2\rangle = 0$

To compute the flux Γ_i, we need an additional relation between $\langle b^2\rangle$ and Γ_i besides equation (13). This can be attained by imposing a stationarity condition on $\langle\psi'^2\rangle$. The stationarity of fluctuations is achieved in a situation where the energy transfer from large-scale fields balances the dissipation of fluctuations locally, as is usually the case in the presence of an external forcing and dissipation. To obtain this relation, we multiply the equation for ψ' by ψ' and then take the average

$$\frac{1}{2}\,\partial_t\langle\psi'^2\rangle + \epsilon_{ij3}\langle\partial_j\phi\psi'\rangle\partial_i\langle\psi\rangle = -\eta\langle(\partial_i\psi')^2\rangle + B_0\langle\psi'\partial_z\phi\rangle\ . \qquad (14)$$

Here the integration by parts was used assuming that there are no boundary terms. We note that either when the stationarity condition is not satisfied or when boundary terms do not vanish, there will be a correction to our results (Blackman & Field 2000). When $\langle\psi'^2\rangle$ is stationary, the first term on the left-hand side of equation (14) vanishes, simplifying the equation that relates $\langle b^2 \rangle$ to $\Gamma_i = \langle u_i \psi' \rangle = \epsilon_{ij3} \langle \partial_i \phi \psi' \rangle$ to the form

$$\langle (\partial_i \psi')^2 \rangle = \langle b^2 \rangle = \frac{1}{\eta} [-\Gamma_i \partial_i \langle \psi \rangle + B_0 \langle \psi' \partial_z \phi \rangle] . \quad (15)$$

Note that in two-dimensional MHD ($B_0 = 0$), the flux is proportional to $\eta \langle b^2 \rangle$. This balance reflects the conservation of $\langle \psi^2 \rangle$, which is damped only by Ohmic diffusion. The second term on the right-hand side of equation (15) can be evaluated in a similar way as for Γ_i, i.e., by writing

$$\langle \psi' \partial_z \phi \rangle = \langle \delta\psi' \partial_z \phi - \partial_z \psi' \delta\phi \rangle \quad (16)$$

and then by using equations (8)–(11). Omitting the intermediate steps (see Appendix A for details), the final result is

$$\langle \psi' \partial_z \phi \rangle = \tau B_0 [\xi_v \langle u^2 \rangle - \xi_b \langle b^2 \rangle] . \quad (17)$$

Here

$$\xi_v \equiv \int d^3k k_z^2 \overline{\phi}(\boldsymbol{k}) \bigg/ \int d^3k k_H^2 \overline{\phi}(\boldsymbol{k}) , \quad (18)$$

$$\xi_b \equiv \int d^3k k_z^2 \overline{\psi}(\boldsymbol{k}) \bigg/ \int d^3k k_H^2 \overline{\psi}(\boldsymbol{k}) , \quad (19)$$

and $k_H^2 = k_x^2 + k_y^2$. If the characteristic horizontal and vertical scales of $\boldsymbol{u}$ are L_{vH} and L_{vz}, and if those of $\boldsymbol{b}$ are L_{bH} and L_{bz}, then ξ_v and ξ_b can be expressed in terms of these characteristic scales as

$$\xi_v = \frac{L_{vH}^2}{L_{vz}^2} , \quad \xi_b = \frac{L_{bH}^2}{L_{bz}^2} . \quad (20)$$

Insertion of equation (17) into (15) gives us

$$\langle b^2 \rangle = \frac{1}{\eta} [-\Gamma_i \partial_i \langle \psi \rangle + \tau \xi_v B_0^2 \langle u^2 \rangle]/[1 + (\tau \xi_b/\eta) B_0^2] . \quad (21)$$

Thus, from equations (13) and (21), we obtain

$$\Gamma_i = -\frac{\tau}{2} \langle u^2 \rangle \times \frac{[1 + \frac{\tau}{\eta} B_0^2 (\xi_b - \xi_v)] \partial_i \langle \psi \rangle - (\tau L_{bH}^2/\eta) \xi_v B_0^2 \partial_i \nabla^2 \langle \psi \rangle}{1 + \tau/\eta [(1/2) \langle B_H \rangle^2 + \xi_b B_0^2 - (L_{bH}^2/2) \langle J \rangle^2]} , \quad (22)$$

where $J\hat{z} = \nabla \times \boldsymbol{B}_H$ and the integration by part is used to express $\partial_i \langle \psi \rangle \partial_i \nabla^2 \langle \psi \rangle = -(\nabla^2 \langle \psi \rangle)^2 = -\langle J \rangle^2 < 0$. Note the last term in the numerator and denominator in equation (22) comes from the hyperresistivity (cf. Bhattacharjee & Yuan 1995). Equation (22) is the flux in three-dimensional RMHD, which generalizes the two-dimensional MHD result (Cattaneo & Vainshtein 1991; Gruzinov & Diamond 1994). Several aspects of this result are of interest. First, in the limit as $\boldsymbol{B}_0 \to 0$ and $\langle \boldsymbol{B}_H \rangle \to 0$ ($\langle J \rangle \to 0$), the flux reduces to the kinematic value $\Gamma_i = -\eta_k \partial_i \langle \psi \rangle$, with the kinematic turbulent diffusivity $\eta_k = \tau \langle u^2 \rangle/2$. This corresponds to the two-dimensional hydrodynamic result where the effect of the Lorentz force is neglected. The full two-dimensional MHD result can be obtained by taking the limit $\boldsymbol{B}_0 \to 0$ in equation (22), which will reproduce equation (1). This agrees with the well-known result on the suppression of flux diffusion in two-dimensions (Cattaneo & Vainshtein 1991; Gruzinov & Diamond 1994).

Another interesting case may be the limit $\langle \boldsymbol{B}_H \rangle \to 0$. In fact, this limit can be shown to be consistent with the ordering of three-dimensional RMHD as follows. First, note that three-dimensional RMHD ordering ($k_z/k_H \sim B_H/B_0 \sim \epsilon < 1$) requires $\xi_b B_0^2 \sim \langle B_H^2 \rangle$. Since $\langle B_H \rangle^2 \ll \langle B_H^2 \rangle \sim \langle b^2 \rangle$, we expect that $\xi_b B_0^2 \sim \langle b^2 \rangle \gg \langle B_H \rangle^2$. Furthermore, $L_{bH}^2 \langle J \rangle^2 \sim (L_{bH}/L_{BH})^2 \langle B_H \rangle^2 < \langle B_H \rangle^2$, where L_{BH} is the characteristic scale of $\langle B_H \rangle$. Thus, the dominant term in the square brackets in the denominator of equation (22) is $\xi_b B_0^2 \sim \langle b^2 \rangle$. That is, the effect of B_0 seems to be stronger than that of $\langle \boldsymbol{B}_H \rangle$ in three-dimensional RMHD.

Finally, to determine whether $\boldsymbol{B}_0$ enhances the flux or not, we note that $\xi_v - \xi_b$ in equation (22) can be taken to be zero, since the scales for $\boldsymbol{b}$ and $\boldsymbol{u}$ are likely to be comparable in this model, which employs unity magnetic Prandtl number. Then, we estimate the last term in the numerator, due to hyperresistivity, to be $\tau \langle b^2 \rangle L_{bH}^2/(\eta L_{BH}^2) \sim (L_{bH}/L_{BH})^2 \mathrm{Re}_m$ where $\xi_b B_0^2 \sim \langle b^2 \rangle$ and $\langle b^2 \rangle \sim \langle u^2 \rangle$ are used. If $(L_{bH}/L_{BH})^2 \sim \mathrm{Re}_m^{-\chi}$, this term will be of order unity for $\chi \sim 1/2$. When $\chi \gtrsim \frac{1}{2}$, the hyperresistivity can be neglected compared to the other term in the numerator (i.e., the turbulent diffusivity). More discussion on this is provided in § 4.1. Note $\mathrm{Re}_m = ul/\eta$ is the magnetic Reynolds number, with u and l being the characteristic amplitude and length scale of the velocity. Therefore, equation (22) indicates that the flux is reduced on account of the strong axial magnetic field B_0 as well as the horizontal reconnecting field $\langle \boldsymbol{B}_H \rangle$. The above analyses will be used in § 4.1 in order to estimate global reconnection rate.

3. EDDY-DAMPED FLUID MODEL

The analysis performed in the previous section introduced an arbitrary correlation time τ that is assumed to be the same for both small-scale velocity and small-scale magnetic fields. Moreover, the quasi-linear closure is valid strictly only when the small-scale fields remain weaker than the large-scale fields. In order to compensate for these shortcomings, we now consider an eddy-damped fluid model which is based on a large viscosity (Kim 1999). In this model, the fluid motion is self-consistently generated by a forcing with a prescribed statistics as well as by the Lorentz force, without having to assume the presence of fully developed MHD turbulence, to invoke a quasi-linear closure, or to introduce an arbitrary correlation time for the fluctuating fields. This is the simplest model within which the nonlinear effect of the back-reaction can rigorously be treated. Even though this model has limited applicability to a system with a large viscosity, it could be quite relevant to small-scale fields in the Galaxy where $\nu \gg \eta$. As shall be shown later, this model gives rise to an effective correlation time for the fluctuating magnetic fields that is given by the viscous time $\tau_v = l_{bH}^2/\nu$, where l_{bH} is the typical scale of the magnetic fluctuations in the horizontal plane (cf. eqs. [22] and [32]). Thus, in comparison with the τ approximation in the previous section, this model is equivalent to replacing τ by τ_v despite the fact that some of detailed results for the two models are not the same.

3.1. *Splitting of Velocity*

In a high viscosity limit with the fluid kinetic Reynolds number $\mathrm{Re} = ul/\nu < 1$, the nonlinear advection term as well as inertial term in the momentum equation can be neglected. Then, the linearity of the remaining terms in the momentum equation enables us to split the velocity into two components; the first—random velocity—is solely governed by the random forcing, and the second—induced velocity—is governed by the Lorentz force only. Specifically, we express the total velocity $\boldsymbol{u}$ as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}'$, where $\boldsymbol{v}$ and $\boldsymbol{v}'$ are the random and induced velocity, respectively, and introduce velocity potential ϕ_0 and ϕ_I as $\boldsymbol{v} = \nabla \times \phi_0 \hat{z}$ and $\boldsymbol{v}' = \nabla \times \phi_I \hat{z}$. Then, the equations for these potentials are

$$0 = \nu\nabla^2\nabla^2\phi_0 + F , \tag{23}$$

$$0 = \nu\nabla^2\nabla^2\phi_I + \boldsymbol{B} \cdot \nabla\nabla^2\psi , \tag{24}$$

where the nonlinear advection term as well as the inertial term is neglected since Re < 1 is assumed. In equation (23), F is a prescribed forcing with known statistics. Instead of solving equation (23) for ϕ_0, we can equivalently prescribe the statistics of the random velocity ϕ_0 (or $\boldsymbol{v}$). Therefore, we assume that the statistics of random component satisfies homogeneity and isotropy in the horizontal plane and homogeneity and reflectional symmetry in the z-direction, respectively. Furthermore, we assume that it is delta correlated in time. The correlation function is then given by

$$\langle \phi_0(\boldsymbol{k}_1, t_1)\phi_0(\boldsymbol{k}_2, t_2)\rangle = \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)\delta(t_1 - t_2)\overline{\phi}_0(k_{1H}, k_{1z}) , \tag{25}$$

where $\overline{\phi}_0(k_{1H}, k_{1z})$ is the power spectrum of ϕ_0. Note that $\tau_0\langle\phi_0^2\rangle = \int d^3k\overline{\phi}(\boldsymbol{k})$ and $\tau_0\langle v^2\rangle = \int d^3k k^2\overline{\phi}(\boldsymbol{k})$, where τ_0 is the correlation time of $\boldsymbol{v}$ that is assumed to be short.

On the other hand, the induced velocity can be constructed by solving equation (24) for ϕ_I in terms of $\boldsymbol{B}$. This can easily be done in Fourier space as

$$\phi_I(\boldsymbol{k}) = \frac{i}{\nu k_H^{\psi}}\left[B_0 k^2 k_H^2 + i\epsilon_{ij3} \times \int d^3k'(k - k')_j k'_{Hi} k_H'^2 \psi(\boldsymbol{k} - \boldsymbol{k}')\psi(\boldsymbol{k}')\right], \tag{26}$$

where $B_{Hi}(\boldsymbol{k}) = i\epsilon_{ij3} k_j \psi(\boldsymbol{k})$ is used. Note that the ψ in the above equation contains both mean and fluctuating parts.

3.2. *Magnetic Field*

Both random and induced velocities are to be substituted in equation (2) to solve for the magnetic field. Notice that equation (2) then has a cubic nonlinearity, since the induced velocity is quadratic in $\boldsymbol{B}$. We again assume that the magnetic field in the horizontal plane consists of mean and fluctuating components, i.e., $\psi = \langle\psi\rangle + \psi'$ and that the fluctuation is homogeneous and isotropic in the x-y plane and homogeneous and reflectionally symmetric in the z-direction, satisfying the same correlation function as equation (10).

To obtain equations for $\langle\psi\rangle$ and $\langle\psi^2\rangle$, we utilize the delta-correlation in time of $\boldsymbol{v}$ and iterate equation (2) for small time intervals δt. Specifically, we use $\langle v_i(t_1)B(t)_j\rangle = 0$ for $t_1 > t$ and $v \sim O[(\delta t)^{-1/2}]$ since $\langle v_i(t_1)v_j(t_2)\rangle \propto \delta(t_1 - t_2) \sim 1/\delta t$, where $\delta t = t_1 - t_2$. Then, for $\delta t \ll 1$, equation (2) can be iterated up to order $O(\delta t)$ as

$$\begin{aligned}\psi(t + \delta t) = {}& \psi(t) + \delta t\eta\nabla^2\psi(t) \\ &+ \int_t^{t+\delta t} dt_1[\epsilon_{ij3}\,\partial_j\psi(t)\partial_i\phi(t_1) + B_0\,\partial_z\psi(t_1)] \\ &+ \frac{1}{2}\,\epsilon_{ij3}\int_t^{t+\delta t} dt_1\,dt_2\{\epsilon_{lm3}\,\partial_i\phi(t_1) \\ &\times \partial_j[\partial_m\psi(t)\partial_l\phi(t_2)] \\ &+ B_0\,\partial_i\phi(t_1)\partial_{jz}\phi(t_2)\} + O(\delta t^{3/2}) ,\end{aligned} \tag{27}$$

where ψ and ϕ are to be evaluated at the same spatial position $\boldsymbol{x}$.

The mean field equation is obtained by substituting equation (26) in (27), by taking the average with the help of equations (10) and (25), and then by taking the limit $\delta t \to 0$. The derivation is tedious and is outlined in Appendix B. Here we give the final result:

$$\begin{aligned}\partial_t\langle\psi\rangle &= \eta\nabla^2\langle\psi\rangle + \left(\frac{\tau_0}{4}\,\langle v^2\rangle - \frac{1}{2\nu}\,G\right)\nabla^2\langle\psi\rangle \\ &\quad - \frac{F}{\nu}\,\nabla^2\nabla^2\langle\psi\rangle \\ &= (\eta + \eta_M)\nabla^2\langle\psi\rangle - \mu\nabla^2\nabla^2\langle\psi\rangle .\end{aligned} \tag{28}$$

Here τ_0 is the short correlation time of random velocity $\boldsymbol{v}$ and

$$\eta_M \equiv \frac{\tau_0}{4}\,\langle v^2\rangle - \frac{1}{2\nu}\,G \equiv \eta_k - \frac{1}{2\nu}\,G ,$$

$$\mu \equiv \frac{F}{\nu} ,$$

$$G \equiv \int d^3k\overline{\psi}(\boldsymbol{k}) = \langle\psi'^2\rangle \equiv \kappa\langle b^2\rangle ,$$

$$F \equiv \int d^3k\,\frac{k_z^2}{k_H^4}\,\overline{\psi}(\boldsymbol{k}) \simeq \frac{L_{bH}^4}{L_{bz}^2}\,G \equiv \gamma G ,$$

where $\eta_k = \tau_0\langle v^2\rangle/4$ is the kinematic diffusivity; $\kappa \equiv L_{bH}^2$ and $\gamma \equiv L_{bH}^4/L_{bz}^2 = \kappa\xi_b$. The above equation implies that the flux $\Gamma_i = \langle u_i\psi'\rangle$ is given by

$$\Gamma_i = -\eta_M\,\partial_i\langle\psi\rangle + \mu\partial_i\nabla^2\langle\psi\rangle . \tag{29}$$

Again, the two terms in η_M are due to the kinematic turbulent diffusivity and backreaction. Note that the kinematic diffusivity $\eta_k = \tau_0\langle v^2\rangle/4$ now comes only from the random velocity, with τ_0 being its correlation time that can be prescribed. The backreaction term is proportional to $\langle\psi'^2\rangle$, not $\langle b^2\rangle$ (see eq. [12]) and inversely proportional to the viscosity ν. It is because the cutoff scale of the magnetic field l_η is smaller than that of the velocity l_ν in this model, so that for a larger ν, there are magnetic modes over a larger interval of scale l between l_η and l_ν (i.e., $l_\eta < l < l_\nu$) where the velocity is absent owing to viscous damping. That is, the induced velocity (Lorentz force) cannot be generated on this scale ($l_\eta < l < l_\nu$) owing to viscous damping, thereby weakening the overall effect of backreaction (see eq. [48]). Now, the last term in equation (29) is the contribution from the hyperresistivity μ. It is interesting to see that μ is inversely proportional to L_{bz}^2 and thus vanishes as $L_{bz} \to \infty$ (or $\gamma \to 0$). which corresponds to the two-dimensional limit. Therefore, in this eddy-damped fluid model, the hyperresistivity term vanishes in two dimensions. It should be contrasted to the case considered in the previous section where the hyperresistivity, being proportional $\langle\psi'^2\rangle$, survives in two-dimensional MHD limit (see eq. [12]).

For use later, we solve equation (29) for $\langle b^2\rangle$, yielding

$$\langle b^2\rangle = \frac{\Gamma_i + \eta_k\,\partial_i\langle\psi\rangle}{\frac{\kappa}{2\nu}\partial_i\langle\psi\rangle + (\kappa\gamma/\nu)\partial_i\nabla^2\langle\psi\rangle} , \tag{30}$$

where again the summation over the index i is not implied.

3.3. *Stationary Case:* $\partial_t\langle\psi'^2\rangle = 0$

The additional relation between the flux Γ_i and magnetic energy $\langle b^2\rangle$ is obtained for the case of stationary $\langle\psi'^2\rangle$. To derive an equation for $\langle\psi^2\rangle$, we multiply equation (27) by itself, take the average, and then take the limit of $\delta t \to 0$. After considerable algebra (see Appendix B), we obtain the following equation

$$\partial_t\langle\psi'^2\rangle + \partial_t\langle\psi\rangle^2 - 2\eta[-\langle(\partial_i\psi)^2\rangle + \langle\psi\rangle\nabla^2\langle\psi\rangle] = B_0^2\left[\xi_v\langle v^2\rangle - \frac{2}{v}\overline{G}\right], \quad (31)$$

where

$$\overline{G} \equiv \int d^3k\,\frac{k_z^2}{k^2}\,\overline{\psi}(\boldsymbol{k}) \sim \frac{L_{bz}^2}{L_{bH}^2}\,G = \xi_b G\,,$$

In a stationary case, equations (28), (30), and (31) lead us to the following expression for the flux:

$$\Gamma_i \cong -\frac{\tau_0}{4}\langle v^2\rangle \times \frac{[1 + (\kappa/\eta v)B_0^2(\xi_b - \xi_v)]\partial_i\langle\psi\rangle - (2\kappa\gamma/\eta v)\xi_v B_0^2\,\partial_i\nabla^2\langle\psi\rangle}{1 + (\kappa/\eta v)(\xi_b B_0^2 + \frac{1}{2}\langle B_H\rangle^2 - \gamma\langle J\rangle^2)}\,, \quad (32)$$

where $J\hat{z} = \nabla \times \boldsymbol{B}_H$, and $\partial_i\nabla^2\langle\psi\rangle\partial_i\langle\psi\rangle = -(\nabla^2\langle\psi\rangle)^2 = -\langle J\rangle^2 < 0$ is used. When the characteristic scales of fluctuating velocity and magnetic field are comparable, or when only the ratios of vertical to horizontal scales of the fluctuating velocity and magnetic fields are comparable, ξ_v can be taken to be equal to ξ_b, simplifying the above expression.

It is worth considering a few interesting limits of equation (32). First, in the limit $B_0 \to 0$ and $B_H \to 0$, equation (32) again recovers the two-dimensional hydrodynamic result with the kinematic diffusivity $\eta_k = \tau_0\langle v^2\rangle/4$. The limit $B_0 \to 0$ leads to two-dimensional MHD case where the suppression of the turbulent diffusion arises from $\langle\boldsymbol{B}_H\rangle$. In three-dimensional RMHD, the dominant suppression in the flux comes from B_0 when $\xi_v = \xi_b$, as discussed in § 2.3.

We note that the last term in the numerator and denominator is due to the hyperresistivity, which comes with a multiplicative factor $\gamma = L_{bH}^2\xi_b$ where $\xi_b = L_{bH}^2/L_{bz}^2 \ll 1$. Therefore, the effect of hyperresistivity can be neglected as compared to other terms in equation (32). Since $\gamma \to 0$ in two-dimensional MHD, there is no contribution from the hyperresistivity to the flux in two-dimensional in this model. The estimate of the effective dissipation in this model is provided in § 4.2.

It is very interesting to compare equation (32) with (22). We recall that in order to derive equation (22), the same correlation time τ was assumed for both fluctuating magnetic field and velocity, which appears in front of the mean magnetic fields B_0 and $\langle\psi\rangle$ in equation (22). In contrast, τ_0 in equation (32) is the correlation time of the random component of the velocity, which can be arbitrarily prescribed. Moreover, τ in front of mean magnetic fields in equation (22) is now replaced by viscous timescale $\tau_v = \kappa/v = L_{bH}^2/v$ in equation (32). The latter represents the viscous timescale across the typical horizontal scale of fluctuating magnetic fields. Thus, as noted at the beginning of this section, this viscous time τ_v replaces τ in the quasi-linear closure, which was assumed to be a parameter.

4. RECONNECTION RATE

In previous sections, the flux Γ_i was derived by using a quasi-linear closure and an eddy-damped fluid model. Since the flux Γ_i involves two terms proportional to $\partial_i\langle\psi\rangle$ and $\partial_i\nabla\langle\psi\rangle$ in both cases (see eqs. [22] and [32]), it can be expressed as a sum of turbulent diffusivity $\eta_{\rm eff}$ and hyperresistivity D_H as follows:

$$\Gamma_i = -\eta_{\rm eff}\,\partial_i\langle\psi\rangle + D_H\,\partial_i\nabla\langle\psi\rangle\,. \quad (33)$$

Upon using equation (33), the mean field equation (4) then becomes

$$\partial_t\langle\psi\rangle = \eta_T\nabla^2\langle\psi\rangle - D_H\nabla^2\nabla^2\langle\psi\rangle\,, \quad (34)$$

where $\eta_T \equiv \eta + \eta_{\rm eff}$ is the total dissipation rate of the mean field and D_H is the hyperresistivity. η_T and D_H represent the overall decay rate of a large-scale magnetic field due to both small-scale motions and magnetic fluctuations. That is, the dynamical system consisting of both small and large-scale fields can be represented by the evolution of a large-scale field only when the effect of small-scale fields is absorbed in these turbulent coefficients.

In order to determine a global reconnection rate, we now invoke the original SP type balance equations for mass continuity ($v_r L = v_0\Delta$), momentum balance ($v_0 = v_{\rm A}$), and the magnetic energy balance

$$v_r\langle B_H\rangle \simeq \eta_T\frac{\langle B_H\rangle}{\Delta} + D_H\frac{\langle B_H\rangle}{\Delta^3}\,.$$

by keeping the hyperresistivity. Note that $v_{\rm A}$ is the Alfvén speed associated with $\langle\boldsymbol{B}_H\rangle$, but not B_0. From these, the reconnection speed follows as

$$\left(\frac{v_r}{v_{\rm A}}\right)^2 \simeq \frac{1}{2}\left[\frac{\eta_T}{Lv_{\rm A}} \pm \sqrt{\left(\frac{\eta_T}{Lv_{\rm A}}\right)^2 + 4\frac{D_H}{v_{\rm A}L^3}}\right]. \quad (35)$$

In the limit of small hyperresistivity, the above equation recovers the usual SP relation mentioned in the the Introduction. In the opposite limit, it reduces to $v_r/v_{\rm A} \propto D_H^{1/4}$ scaling (for instance, see, Biskamp 1993).

In the following subsections, we assume $\xi_v = \xi_b$ for simplicity and estimate the reconnection rate via equation (35). Then, we briefly comment on the implication for reconnection assuming "Alfvénic turbulence," as Lazarian & Vishniac (1999) did.

4.1. *Using the Quasi-Linear Result*

By using $\xi_v = \xi_b$ and noting that $\xi_b B_0^2 \sim \langle b^2\rangle$ is the dominant term in the square brackets in the denominator of equation (22) (see § 2.3), we approximate the two turbulent transport coefficients as

$$\eta_{\rm eff} \sim \eta_k\frac{1}{1 + \tau\langle b^2\rangle/\eta} \sim \eta_k\frac{1}{1 + 2{\rm Re}_m\langle b^2\rangle/\langle u^2\rangle}\,, \quad (36)$$

$$D_H \sim \eta_k\frac{2{\rm Re}_m L_{bH}^2\langle b^2\rangle/\langle u^2\rangle}{1 + 2{\rm Re}_m\langle b^2\rangle/\langle u^2\rangle}\,, \quad (37)$$

where $\eta_k = \tau\langle u^2\rangle/2$ is the kinematic value of turbulent diffusivity in two-dimensional and ${\rm Re}_m = \eta_k/\eta$. Note that in contrast to the two-dimensional MHD result (eq. [1]), the equation (36) reveals that the turbulent diffusivity in three-dimensional RMHD is more severely reduced as $\langle b^2\rangle \gg \langle B_H\rangle^2$ $(=\langle B\rangle^2)$. It is owing to the radiative loss of Alfvén waves along B_0^2. The reconnection rate can be obtained upon substituting equations (36) and (37) in equation (35). A simpler scaling relation can however be obtained by noting that the effect of D_H can be neglected compared to that of $\eta_{\rm eff}$. It is because the ratio of the former to the latter, $(L_{bH}/L_{BH})^2 {\rm Re}_m \propto {\rm Re}_m^{1-2\chi}$ with $(L_{bH}/L_{BH}) \sim {\rm Re}_m^{-\chi}$ (see § 2.3), is likely to be of unity in two-dimensions with $\chi \sim 1/2$ and less than unity in three-dimensions with $\chi \gtrsim 1/2$ (as suggested by Vainshtein & Cattaneo 1992). Note that $\chi \gtrsim 1/2$ in three-dimensions follows from the observation that in three-dimensions, small-scale magnetic fields tend to have smaller scales with stronger fluctuations compared to two-dimensions. For this reason, D_H will be neglected in the following analysis. The scaling relation for $\chi < 1/2$, i.e., the case when D_H dominates over $\eta_{\rm eff}$, is discussed in Appendix C.

To determine the leading order contribution in equation (36), we need to estimate $\langle b^2 \rangle$. To do so, we substitute equations (33) and (36) into equation (13) and use $L_{bH} < L_{BH}$ to obtain

$$\langle b^2 \rangle \sim \langle u^2 \rangle - \frac{\eta}{\tau} \sim \langle u^2 \rangle \left(1 - \frac{1}{2{\rm Re}_m}\right) , \tag{38}$$

where ${\rm Re}_m = \eta_k/\eta = \tau \langle u^2 \rangle / 2\eta$ is used. We note that $\langle b^2 \rangle > 0$ is guaranteed since $\langle b^2 \rangle > \langle B_H \rangle^2$ (implying ${\rm Re}_m > 1$) was assumed to derive the above equation. Thus,

$$\frac{\tau \langle b^2 \rangle}{\eta} \sim 2{\rm Re}_m - 1 .$$

That is, for ${\rm Re}_m \gg 1$, $\tau\langle b^2 \rangle/\eta \gg 1$. Insertion of the above equation in (36) then gives us

$$\eta_{\rm eff} \sim \eta_k \frac{1}{2{\rm Re}_m} \sim \frac{\eta}{2} . \tag{39}$$

In other words, to leading order, the effective dissipation rate is just that given by Ohmic diffusivity! Therefore, by inserting equation (39) into (35) with $\eta_T = \eta + \eta_{\rm eff}$, the reconnection rate is found to have the original SP scaling with η, i.e.,

$$v_r \sim \frac{v_{\rm A}}{\sqrt{v_{\rm A} L/\eta}} . \tag{40}$$

It is interesting to contrast this result to the two-dimension case where $B_0 = 0$. In that case, the dominant term in equation (22) is $\langle B_H \rangle^2$, with $\eta_{\rm eff} \sim \eta_k \langle u^2 \rangle / {\rm Re}_m \langle B_H \rangle^2 \sim \eta \langle u^2 \rangle / \langle B_H \rangle^2 \sim \eta u^2 / v_{\rm A}^2 > \eta$, where u is the typical velocity. Therefore, in two-dimensions, the global reconnection rate becomes

$$v_r \sim \frac{v_{\rm A}}{\sqrt{v_{\rm A} L/\eta}} \frac{u}{v_{\rm A}} , \tag{41}$$

which is larger than SP by a factor of magnetic Mach number $M_{\rm A} = u/v_{\rm A}$. Note that the reduction in the effective dissipation of a large-scale magnetic field is more severe in three-dimensional than in two-dimensional MHD by a factor of $\langle u^2 \rangle / \langle B_H \rangle^2 \sim \langle u^2 \rangle / v_{\rm A}^2$.

4.2. *Using the Eddy-Damped Fluid Model Result*

For an eddy-damped fluid model, equations (32) and (33) yield

$$\eta_{\rm eff} \sim \eta_k \frac{1}{1 + (\kappa/\nu\eta)\langle b^2 \rangle} , \tag{42}$$

$$D_H \sim \eta_k \frac{(2\kappa\gamma/\eta\nu)\langle b^2 \rangle}{1 + (\kappa/\nu\eta)\langle b^2 \rangle} , \tag{43}$$

where we assumed $\xi_v = \xi_b$ and kept the leading order term $\xi_b B_0^2 \sim \langle b^2 \rangle$ in the square brackets in the denominator of equation (32) (see § 3.3); $\eta_k \equiv \tau_0 \langle v^2 \rangle / 4$ is the kinematic value of the turbulent diffusivity in two-dimensional and $\kappa = L_{bH}^2$. As mentioned in § 3.3, hyperresistivity D_H, involving a multiplicative factor $\gamma = \kappa \xi_b$ with $\xi_b = L_{bH}^2 / L_{bz}^2 \ll 1$, is very small compared to turbulent diffusivity $\eta_{\rm eff}$, and therefore will be neglected in the following. To obtain the leading order behavior of equation (42), we estimate $\langle b^2 \rangle$ with the help of equation (30) to be

$$\langle b^2 \rangle \sim \frac{\eta\nu}{\kappa} (2{\rm Re}_m - 1) , \tag{44}$$

where ${\rm Re}_m = \eta_k/\eta$. By inserting equation (44) in (43), we obtain

$$\eta_{\rm eff} \sim \frac{\eta_k}{2{\rm Re}_m} \sim \frac{\eta}{2} . \tag{45}$$

Thus, the reconnection rate is again given by

$$v_r \sim \frac{v_{\rm A}}{\sqrt{v_{\rm A} L/\eta}} , \tag{46}$$

i.e., SP scaling with η persists!

It is interesting to estimate $\langle b^2 \rangle$ in equation (44) by using

$$\frac{\eta\nu}{\kappa} = \langle v^2 \rangle \frac{\eta}{\sqrt{\langle v^2 \rangle} L_{bH}} \frac{\nu}{\sqrt{\langle v^2 \rangle} L_{bH}} \sim \langle v^2 \rangle \frac{1}{{\rm Re}_m {\rm Re}} , \tag{47}$$

where ${\rm Re} = \sqrt{\langle v^2 \rangle} L_{bH}/\nu$ is the fluid Reynolds number. Thus, equation (44) becomes

$$\langle b^2 \rangle \sim \langle v^2 \rangle \frac{1}{\rm Re} \left(2 - \frac{1}{{\rm Re}_m}\right) . \tag{48}$$

The above equation clearly demonstrates that $\langle b^2 \rangle > \langle v^2 \rangle$ for our model (${\rm Re} < 1$) when ${\rm Re}_m > 1$, as pointed out near the end of § 3.2. Note that we have neglected a multiplicative correction factor to the reconnection rate in the eddy-damped model since its dependence on ν is weak with $\frac{1}{4}$ power (for instance, see, Biskamp 1993).

Finally, we note that in two-dimensional limit with $B_0 \to 0$, the dominant term in the square brackets in the denominator of equation (32) is $\langle B_H \rangle^2$. Thus, $\eta_{\rm eff} \sim \eta_k \langle v^2 \rangle / {\rm Re}{\rm Re}_m \langle B_H \rangle^2 \sim \eta \langle v^2 \rangle / {\rm Re} \langle B_H \rangle^2 \sim \eta u^2 / {\rm Re} v_{\rm A}^2 > \eta$, where u is the typical velocity. Therefore, in two-dimensional, the global reconnection rate becomes

$$v_r \sim \frac{1}{\sqrt{\rm Re}} \frac{v_{\rm A}}{\sqrt{v_{\rm A} L/\eta}} \frac{u}{v_{\rm A}} , \tag{49}$$

where $u/v_{\rm A} = M_{\rm A}$ is the magnetic Mach number. In comparison with equation (41), the global reconnection rate in this model is thus larger in the two-dimensional limit (recall ${\rm Re} < 1$).

4.3. *Alfvénic Turbulence*

In Alfvénic turbulence (Goldreich & Sridhar 1994; 1995; 1997), the equipartition between $\langle b^2 \rangle$ and $\langle u^2 \rangle$ is assumed from the start. It is to be contrasted to the present analysis in which the relation between $\langle b^2 \rangle$ and $\langle u^2 \rangle$ i.e., equations (38) and (49), follows from the condition of stationarity of $\langle \psi'^2 \rangle$ in the presence of $\boldsymbol{B}_0$ and $\langle \boldsymbol{B}_H \rangle$. As can be seen from equation (38), in the quasi-linear closure with unity magnetic Prandtl number, exact equipartition is possible only for $\eta = 0$. In the eddy-damped fluid model, exact equipartition can never be satisfied since the assumption $\mathrm{Re} < 1$ implies $\langle b^2 \rangle > \langle v^2 \rangle$ when $\mathrm{Re}_m > 1$ (see eq. [48])! Therefore, in general, stationarity of $\langle \psi'^2 \rangle$ and exact Alfvénic equipartition cannot be simultaneously achieved. In other words, if Alfvénic turbulence is assumed, $\langle \psi'^2 \rangle$ cannot be stationary; if $\langle \psi'^2 \rangle$ is stationary, the turbulence cannot be in a state of Alfvénic equipartition.

We easily confirm this in two-dimensional MHD by quasi-linear closure. The exact equipartition ($\langle u^2 - b^2 \rangle = 0$) implies that the flux Γ_i in equation (12) is given by hyperresistivity only: $\Gamma_i = -\tau \langle \psi'^2 \rangle \partial_i \nabla^2 \langle \psi \rangle / 2$. Then, if we were to impose the stationarity of $\langle \psi'^2 \rangle$, equation (15) would indicate $\langle \psi'^2 \rangle \tau \partial \langle J \rangle \langle B_H \rangle = \eta \langle b^2 \rangle$. Thus,

$$\frac{\langle B_H \rangle^2}{\langle b^2 \rangle} \mathrm{Re}_m \sim \left(\frac{l_B}{l_b} \right)^2 , \tag{50}$$

where l_B and l_b are the characteristic scales of $\langle \boldsymbol{B}_H \rangle$ and $\boldsymbol{b}$, respectively. Since $\langle B_H \rangle^2 / \langle u^2 \rangle \sim 1/\mathrm{Re}_m$ (with $\langle b^2 \rangle \sim \langle u^2 \rangle$) and $(l_B/l_b)^2 \sim 1/\mathrm{Re}_m$ in two-dimensional MHD, the relation (50) (for stationarity) cannot be satisfied.

5. CONCLUSION AND DISCUSSION

In view of the ubiquity of turbulence in space and astrophysical plasmas, magnetic reconnection will likely occur in an environments with turbulence. On the other hand, the reconnection itself generates small-scale fluctuation, feeding back the turbulence. Thus, it is important to treat these two processes consistently, accounting for the back reaction. Although LV argued that the local reconnection rate can be fast, they basically neglected the dynamic coupling between small and large-scale fields, therefore leaving the issue of the global reconnection rate unresolved. The coupling between global and local reconnection rates should be treated self consistently. The aim of the present work was to shed some light on this issue by taking the simplest approach that is analytically tractable.

Our main strategy was to self-consistently compute turbulent diffusitivy and hyperresistivity within the current sheet, by using stationarity of $\langle \psi'^2 \rangle$ and by exploiting the "linearly broken" mean-square magnetic potential conservation. These turbulent coefficients are then used in SP type balance relations to obtain the global reconnection rate. To avoid the null point problem associated with a two-dimensional slab model, we considered three-dimensional RMHD, within which we can solidly justify the incompressibility of the fluid in the horizontal plane. To facilitate analysis, two models (methods) were employed, one being a quasi-linear closure with τ approximation and the other eddy-damped fluid model. In each model, we computed turbulent diffusivity and hyperresistivity, indicating that the former is likely to be more important than the latter.

The turbulent diffusivity η_{eff} that we obtained generalizes the two-dimensional MHD result (Cattaneo & Vainshtein 1991; Gruzinov & Diamond 1994). The quasi-linear closure predicted $\eta_{\mathrm{eff}} \sim \eta_k / (1 + 2\mathrm{Re}_m \langle b^2 \rangle / \langle u^2 \rangle) \sim \eta/2$ (see eqs. [37]–[39]). A similar result was obtained in the eddy-damped fluid model with $\eta_{\mathrm{eff}} \sim \eta_k / (1 + \mathrm{Re}_m \mathrm{Re} \langle b^2 \rangle / \langle u^2 \rangle) \sim \eta/2$ (see eqs. [42]–[45] and [47]).

The two-dimensional result can simply be recovered from our results on the flux by taking the limit $B_0 \to 0$. In that limit, $\eta_{\mathrm{eff}} \sim \eta_k / (1 + \mathrm{Re}_m \langle B_H \rangle^2 / \langle u^2 \rangle)$ according to the quasi-linear closure, consistent with previous work. In the eddy-damped fluid model, $\eta_{\mathrm{eff}} \sim \eta_k / (1 + \mathrm{Re}_m \mathrm{Re} \langle B_H \rangle^2 / \langle u^2 \rangle)$.

Since the turbulent diffusivity η_{eff} was found to be the same in both models (in three-dimensional RMHD), the global reconnection, obtained by invoking SP balance relations, was also the same with the value $v_r \sim v_A / (v_A L/\eta)^{1/2}$ in both models. This result indicates that the global reconnection rate is suppressed for large Re_m as an inverse power of $\mathrm{Re}_m^{1/2}$ such that the original SP scaling with η persists. Again, this persistent η scaling results from the reduction in the turbulent diffusivity η_{eff} for large Re_m mainly due to a strong axial magnetic field, with $\eta_{\mathrm{eff}} \sim \eta$.

Furthermore, in the two-dimensional limit, the quasi-linear closure yielded the global reconnection rate $v_r \sim [v_A / (v_A L/\eta)^{1/2}](u/v_A)$, which is enhanced over SP by a factor of $M_A = u/v_A$ (note that M_A can be large). In contrast, the eddy-damped fluid model gave $v_r \sim \sqrt{\mathrm{Re}^{-1}} [v_A / (v_A L/\eta)^{1/2}](u/v_A)$.

The implication of these results for the LV scenario is that no matter how fast local reconnection events proceed, there may not be enough energy transfer from large-scale to small-scale magnetic fields to allow fast global reconnection. Therefore, global reconnection cannot be given by a simple sum of the local reconnection events as LV suggested. We emphasize again that the $\langle \psi'^2 \rangle$ balance, followed from the stationarity and mean-square potential conservation, played the crucial role in determining the global reconnection rate consistently. Alternatively, an accurate calculation of the global reconnection rates requires that (global) topological conservation laws be enforced.

The reduction in the turbulent diffusivity in two-dimensions is closely linked to the conservation of mean-square magnetic potential. In three-dimensional RMHD, the mean-square of parallel component of potential is no longer an ideal invariant owing to the propagation of Alfvén waves along a strong axial magnetic field. Nevertheless, the conservation of mean magnetic potential is broken only linearly, which turned out to introduce additional suppression factors, as compared to two-dimensions. The interesting question is then how relevant these results would be in three-dimensions (for reconnection in three-dimensions, see, for instance, Greene 1989; Lau & Finn 1990). The mean-square potential is not an invariant of three-dimensional MHD. However, its conservation is broken nonlinearly, unlike three-dimensional RMHD. Therefore, the effective dissipation rate of a large-scale magnetic field in three-dimensional MHD may be very different from that in three-dimensional RMHD, with the possibility that the former may not be reduced, at least, in the weak magnetic field limit (Gruzinov & Diamond 1994; Kim 1999). Moreover, in three-dimensions, there is a possibility of a dynamo, which brings in an additional transport coefficient (the α effect) into the problem. Some insights into the problem of effective dissipation of a large-scale field in the presence of a dynamo process might be obtained by considering a simple extension of the present three-dimensional RMHD model by allowing a large-scale dynamo in the horizontal plane. Recall that this possibility was ruled out in the present paper by assuming isotropy in the horizontal plane and reflectional symmetry in the axial direction, with no helicity term (i.e., no correlation between horizontal and vertical component of fluctuations).

Considering some of limitations of the two models that were analyzed in the paper, such as the τ approximation, quasi-linear closure, low kinetic Reynolds number limit, etc, it will be very interesting to investigate our predictions via numerical computation. The stationarity of $\langle \psi'^2 \rangle$ can be maintained as long as there is an energy source in the system, such as an external forcing. By incorporating the proper ordering required for three-dimensional RMHD, one can measure the decay rate of $\langle \boldsymbol{B}_H \rangle$ to check our predictions for $\eta_{\rm eff} \sim \eta$ (see eqs. [40] and [46]). Ultimately, a numerical simulation with a simple reconnection configuration should be performed to measure a global reconnection rate as a function of Re_m as well as B_0 and $\langle B_H \rangle$. It will also be interesting to investigate nonstationary states such as plasmoid formation (Forbes & Priest 1983; Priest 1984; Matthaeus & Lamkin 1986).

We note that the nonstationarity of small-scale fields, such as the aforementioned bursty "plasmoid ejection" events of Matthaeus & Lamkin (1986), modifies the relation between $\langle \psi'^2 \rangle$, $\langle b^2 \rangle$, and Γ_i. This modification can potentially relax the conservation law constraints on reconnection. A key question is then whether the plasmoid ejection process is periodic (as in a limit cycle) or temporally chaotic and intermittent. Even if detailed calculations must await a future paper, we suspect that some analytical progress can be made in the case of a limit cycle.

We thank E. Zweibel for bringing this problem to our attention and for many interesting discussions. We also thank E. T. Vishniac and A. S. Ware for stimulating conversations. This research was supported by US DOE FG03-88ER 53275. P. H. Diamond also acknowledges partial support from the National Science Foundation under grant PHY99-07949 to the Institute for Theoretical Physics at U.C.S.B., where part of this work was performed. E. Kim acknowledges partial support from HAO/NCAR where part of this work was completed.

APPENDIX A

In this appendix, we provide some of steps leading to equations (12) and (17). First, to derive equation (12), we let $\Gamma_i = \Gamma_i^{(1)} - \Gamma_i^{(2)}$, where $\Gamma_i^{(1)} = \epsilon_{ij3}\langle \partial_j \phi \delta\psi' \rangle$ and $\Gamma_i^{(2)} = \epsilon_{ij3}\langle \phi \partial_i \delta\psi' \rangle$, and begin with $\Gamma_i^{(1)}$.

$$\begin{aligned}\Gamma_i^{(1)} &= \epsilon_{ij3}\langle \partial_j \phi \delta\psi' \rangle \\ &= \epsilon_{ij3}\int d^3k_1\, d^3k_2\, ik_{1j}\langle \phi(\boldsymbol{k}_1)\delta\psi'(\boldsymbol{k}_2)\rangle \exp\{i(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot \boldsymbol{x}\}\ . \end{aligned} \tag{A1}$$

After inserting equation (8) in (A1) and using equation (11), we can easily obtain

$$\begin{aligned}\Gamma_i^{(1)} &= -i\tau\epsilon_{ij3}\,\epsilon_{lm3}\int d^3k_1\, d^3k k_{1j}\, k_{1m}\, k_l\, \overline{\phi}(\boldsymbol{k}_1)\langle \psi(\boldsymbol{k})\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \tau\epsilon_{ij3}\int d^3k k_{1j}\, k_{1z}\, B_0\, \overline{\phi}(\boldsymbol{k}_1) \\ &= -\frac{\tau}{2}\,\partial_l\langle\psi\rangle\delta_{il}\int d^3k_1\, k_1^2\,\overline{\phi}(\boldsymbol{k}_1) = -\frac{\tau}{2}\,\langle u^2\rangle\partial_i\langle\psi\rangle\ , \end{aligned} \tag{A2}$$

where $\langle u^2 \rangle = \int d^3k_1\, k_1^2\, \overline{\phi}(\boldsymbol{k}_1)$. To obtain the last line in equation (A2), we use the following relations

$$\begin{aligned}&\int d^3k k_j\, k_m\, \overline{\phi}(\boldsymbol{k}) = \frac{1}{2}\,\delta_{jm}\int d^3k k^2 \overline{\phi}(\boldsymbol{k})\ , \\ &\int d^3k k_j\, k_z\, \overline{\phi}(\boldsymbol{k}) = 0\ , \end{aligned} \tag{A3}$$

which follows from the isotropy of ϕ in the x-y plane, and reflectional symmetry in the z-direction.

The second part, $\Gamma_i^{(2)}$, is calculated in a similar way.

$$\begin{aligned}\Gamma_i^{(2)} &= \epsilon_{ij3}\langle \delta\phi \partial_j \psi' \rangle \\ &= \epsilon_{ij3}\int d^3k_1\, d^3k_2\, ik_{2j}\langle \delta\phi(\boldsymbol{k}_1)\psi'(\boldsymbol{k}_2)\rangle \exp\{i(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot \boldsymbol{x}\}\ . \end{aligned} \tag{A4}$$

We insert equation (9) in (A4) and use equation (10) to obtain

$$\begin{aligned}\Gamma_i^{(2)} = i\tau\epsilon_{ij3}\Big[&-iB_0\int d^3k_1\, k_{1z}\, k_{1j}\,\overline{\psi}(\boldsymbol{k}_1) \\ &+ \epsilon_{lm3}\int d^3k_2\, d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,\frac{1}{(\boldsymbol{k}+\boldsymbol{k}_2)^2}\,(k_m\, k_{2l}\, k_2^2 + k_{2m}\, k_l\, k^2)k_{2j}\,\overline{\psi}(\boldsymbol{k}_2)\langle\psi(\boldsymbol{k})\rangle\Big]\ . \end{aligned} \tag{A5}$$

Since $\langle \psi \rangle$ has a scale much larger than ψ', $k_2 \gg k$ in the second integral on the right-hand side. We thus expand the integrand of this second term and use the following isotropy relations:

$$\int d^3k k_j k_m \overline{\psi}(\boldsymbol{k}) = \frac{1}{2}\,\delta_{jm} \int d^3k k^2 \overline{\psi}(\boldsymbol{k}) ,$$
$$\int d^3k k_i k_j k_l k_m \overline{\psi}(\boldsymbol{k}) = \frac{1}{8}\,(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) \int d^3k k^4 \overline{\psi}(k) ,$$
$$\int d^3k k_i k_z \overline{\psi}(\boldsymbol{k}) = 0 . \tag{A6}$$

A bit of algebra then gives us

$$\Gamma_i^{(2)} = \frac{\tau}{2}\,[-\langle b^2\rangle \partial_i \langle \psi(\boldsymbol{x})\rangle - \langle \psi'^2\rangle \partial_i \nabla^2 \langle \psi(\boldsymbol{x})\rangle] . \tag{A7}$$

Thus, from equations (A3) and (A7), we obtain equation (12) in the main text.

Second, to derive equation (17), we again compute the correlation function on the right-hand side of equation (16) in Fourier space. The first term can be rewritten as

$$\langle \delta\psi' \partial_z \phi\rangle = \int d^3k_1\, d^3k_2\, ik_{1z} \langle \phi(\boldsymbol{k}_1)\delta\psi'(\boldsymbol{k}_2)\rangle \exp\{i(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot \boldsymbol{x}\} . \tag{A8}$$

Then, inserting equation (8) in (A8) and using equation (11) gives us

$$\langle \delta\psi' \partial_z \phi\rangle = \tau\left[\int d^3k_1\, k_{1z} k_{1z} B_0 \overline{\phi}(\boldsymbol{k}_1) - \epsilon_{lm3} \int d^3k_1\, d^3k k_{1z} k_{im} k_l \overline{\phi}(\boldsymbol{k}_1) \langle \psi(\boldsymbol{k})\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right]$$
$$= \tau B_0 \int d^3k_1\, k_{1z}^2 \overline{\phi}(\boldsymbol{k}_1) = \tau B_0\, \xi_v \langle u^2\rangle , \tag{A9}$$

where the isotropy and equation (18) were used to obtain the last line. Similarly, the second term on the right-hand side of equation (16) is easily calculated (in Fourier space) by using the isotropy condition. The result is

$$\langle \partial_z \psi' \delta\phi\rangle = \tau B_0 \int d^3k_1\, k_{1z}^2 \overline{\psi}(\boldsymbol{k}_1) = \tau B_0\, \xi_b \langle b^2\rangle . \tag{A10}$$

Thus, equations (16), (A9), and (A10) yield equation (17), in the main text.

APPENDIX B

In this appendix we provide some of intermediate steps used to obtain equations (28) and (31). For the mean field equation (28), we first take the average of equation (27)

$$\langle \psi(t + \delta t)\rangle - \langle \psi(t)\rangle - \delta t \eta \nabla^2 \langle \psi(t)\rangle = I_1 + I_2 + I_3 , \tag{B1}$$

where

$$I_1 = \int_t^{t+\delta t} dt_1 [\epsilon_{ij3}\, \partial_j \psi(t) \partial_i \phi_I(t_1)] \simeq \delta t \epsilon_{ij3}\, \partial_i \langle \partial_j \psi(t) \phi_I(t)\rangle \equiv \delta t \partial_i \Delta_i ,$$
$$I_2 = \int_t^{t+\delta t} dt_1\, B_0\, \partial_z \langle \phi_I(t_1)\rangle \simeq \delta t B_0\, \partial_z \langle \phi_I(t)\rangle ,$$
$$I_3 = \frac{1}{2}\,\epsilon_{ij3} \int_t^{t+\delta t} dt_1\, dt_2 \langle \epsilon_{lm3}\, \partial_i \phi_0(t_1)[\partial_{jm} \psi(t) \partial_l \phi_0(t_2) + \partial_m \psi(t) \partial_{jl} \phi_0(t_2)]$$
$$+ B_0\, \partial_i \phi_0(t_1) \partial_{jz} \phi_0(t_2)\rangle , \tag{B2}$$

where $\Delta_i \equiv \epsilon_{ij3} \langle \partial_j \psi(t) \phi_I(t)\rangle$ and the smooth variation of the induced velocity ϕ_I in time was used to approximate the time integrals in I_1 and I_2. To compute the averages, it is convenient to express the correlation function (24) in terms of $\boldsymbol{v}$ in real space as

$$\langle v_i(\boldsymbol{x}, t_1) v_j(\boldsymbol{y}, t_2)\rangle = \delta(t_1 - t_2)\left[T_L(r_H, r_z)\delta_{ij} + r_H\, \frac{\partial T_L}{\partial r_H}\left(\delta_{ij} - \frac{r_{Hi} r_{Hj}}{r_H^2}\right)\right], \tag{B3}$$

where $\boldsymbol{r} \equiv \boldsymbol{y} - \boldsymbol{x}$ and $\boldsymbol{r}_H$ is the horizontal component. Note that the above relation implies that at $\boldsymbol{r} = 0$, $\langle v_i(\boldsymbol{x}, t_1) v_j(\boldsymbol{x}, t_2)\rangle = \delta(t_1 - t_2)\delta_{ij} T_L(r = 0)$ so that $T_L(0) = \tau_0\langle v^2\rangle/2 = 2\eta_k$. Here τ_0 is the short correlation time of $\boldsymbol{v}$ and $\eta_k = \tau_0\langle v^2\rangle/4$ is the kinematic diffusivity. $\langle v_i(\boldsymbol{x}) v_j(\boldsymbol{x})\rangle$ is obviously related to ϕ_0 by $\langle \partial_i \phi_0(\boldsymbol{x}, t_1)\partial_l \phi_0(\boldsymbol{y}, t_2)\rangle = \delta_{il}\langle v_j(\boldsymbol{x}, t_1) v_j(\boldsymbol{y}, t_2)\rangle + \langle v_i(\boldsymbol{x}, t_1) v_l(\boldsymbol{y}, t_2)\rangle$. By using $\boldsymbol{v}_j = -\epsilon_{ij3}\partial_i \phi_0$ and $\langle \phi_0(t_1)\psi(t)\rangle = 0$ for $t > t_1$, I_3 is determined to be

$$I_3 = \frac{1}{2}\,\delta t T_L(0)\nabla^2\langle\psi\rangle\ . \tag{B4}$$

I_3 represents the kinematic turbulent diffusivity. Next, to compute I_2, we take the inverse Fourier transform of equation (26) and then take the average. Upon neglecting $\partial_z\langle\psi\rangle \sim 0$, one can easily show that $I_2 = 0$. Finally, I_1 contains the backreaction as well as hyperresistivity. To evaluate this term, we insert equation (26) in Δ_i to obtain

$$\begin{aligned}\Delta_i &= \epsilon_{ij3}\langle \partial_j \psi(t)\phi_I(t)\rangle \\ &= -\frac{i}{\nu}\,\epsilon_{ij3}\,\epsilon_{lm3}\int d^3k_2\, d^3k' e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\,\frac{1}{(\boldsymbol{k}_H + \boldsymbol{k}'_H)^4}\,\overline{\psi}(-\boldsymbol{k})P_{jlm}\langle\phi(\boldsymbol{k}')\rangle\ ,\end{aligned} \tag{B5}$$

where

$$P_{jlm} \equiv -k_j[k_m k'_l k'^2_H + k_l k'_m k^2_H]\ .$$

For notational convenience, we introduce $\boldsymbol{q} = \boldsymbol{k}_H$ so that $q_3 = 0$. Since the characteristic scale of $\langle\psi\rangle$ is much larger than that of ψ', $k' \ll k$ in equation (B5). Thus, we expand the integrand of equation (B5) to second order in (k'/k) and exploit the isotropy and homogeneity of ψ' in the $x - y$ plane. The latter implies equation (A5) (recall $\boldsymbol{q} = \boldsymbol{k}_H$) and also the following relations

$$\begin{aligned}\int d^3k q_j q_l q_r k_n &= \int d^3k q_j q_l q_r q_n\ ,\\ \int d^3k q_j q_l k_z k_z &= \frac{1}{2}\,\delta_{jl}\int d^3k q^2 k_z^2\ .\end{aligned} \tag{B6}$$

Then, a fair amount of algebra reduces equation (B5) to

$$\begin{aligned}\Delta_i &= -\frac{1}{2\nu}\,\partial_i\langle\psi\rangle\int d^3k\overline{\psi}(\boldsymbol{k}) - \frac{1}{\nu}\,\partial_i\nabla^2\langle\psi\rangle\int d^3k\,\frac{k_H^2 k_z^2}{k_H^6}\,\overline{\psi}(\boldsymbol{k})\\ &= -\frac{G}{2\nu}\,\partial_i\langle\psi\rangle - \frac{F}{\nu}\,\partial_i\nabla^2\langle\psi\rangle\ .\end{aligned} \tag{B7}$$

Note that there is no contribution from the first-order term. By inserting equation (B7) into (B1), by dividing both sides by δt, and then by taking the limit of $\delta t \to 0$, we obtain equation (28).

Next, to derive equation (31), we multiply equation (27) by ψ and then take average to obtain the following equation:

$$\langle\psi^2(t + \delta t)\rangle - \langle\psi^2(t)\rangle - 2\eta\delta t\langle\psi(t)\nabla^2\psi(t)\rangle = J_1 + J_2 + 2J_3\ , \tag{B8}$$

where

$$\begin{aligned}J_1 &\equiv \int_t^{t+\delta t} dt_1\, dt_2\{\epsilon_{ij3}\,\epsilon_{lm3}\langle\partial_j\psi(t)\partial_m\psi(t)\partial_i\phi_I(t_1)\partial_l\phi_I(t_2)\rangle + 2B_0\,\epsilon_{ij3}\langle\partial_j\psi(t)\partial_i\phi_I(t_1)\partial_z\phi_I(t_2)\rangle B_0^2[\partial_z\phi_0(t_1)\partial_z\phi_0(t_2)]\}\ ,\\ J_2 &= \epsilon_{ij3}\int_t^{t+\delta t} dt_1\, dt_2\langle\psi(t)[\partial_i\phi_0(t_1)\epsilon_{lm3}\,\partial_j[\partial_m\psi(t)\partial_l\phi_0(t_2)] + B_0\,\partial_i\phi_0(t_1)\partial_{jz}\phi_0(t_2)]\rangle\ ,\\ J_3 &= \int_t^{t+\delta t} dt_1\langle\psi(t)[\epsilon_{ij3}\,\partial_j\psi(t)\partial_i\phi_I(t_1) + B_0\,\partial_z\phi_I(t_1)]\rangle \equiv \delta t(J_{31} + J_{32})\ ,\end{aligned} \tag{B9}$$

where $J_{31} \equiv \epsilon_{ij3}\langle\psi(t)\partial_j\psi(t)\partial_i\phi_I(t)\rangle$ and $J_{32} \equiv B_0\langle\psi(t)\partial_z\phi_I(t)\rangle$.

First, J_1 can easily be computed by using the correlation functions as

$$J_1 = \delta t\left[T_L(0)(\langle b^2\rangle + \langle B_H\rangle^2) + B_0^2\int d^3k_z^2\,\overline{\phi}(\boldsymbol{k})\right]. \tag{B10}$$

Next, J_2 can be computed upon substituting equation (26) and then splitting average by using $\langle\psi(t)\phi(t_1)\rangle = 0$, with the result

$$J_2 = \delta t T_L(0)[-\langle b^2\rangle + \langle\psi\rangle\nabla^2\langle\psi\rangle]\ . \tag{B11}$$

For J_3, one can first show $J_{31} = 0$ due to isotropy. To compute J_{32}, we substitute equation (26) and use $\langle \phi_I \rangle = 0$ to obtain

$$J_{32} = -B_0 \int d^3k_1\, d^3k \,\exp\,\{i(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot \boldsymbol{x}\}\, \frac{k_z}{\nu k_H^2 k^2} \times \left\langle \psi'(\boldsymbol{k}_1)\left[B_0\, k_z\, k_H^2\, \psi'(\boldsymbol{k}) + i\epsilon_{ij3}\int d^3k' \psi(\boldsymbol{k}-\boldsymbol{k}')(k-k')_j k'_i k'^2_H\, \psi(\boldsymbol{k}')\right]\right\rangle$$

$$= -\frac{B_0}{\nu}\left[B_0 \int d^3k_1\, \frac{k_{1z}^2}{k_1^2}\, \overline{\psi}(\boldsymbol{k}_1) + i\epsilon_{ij3}\int d^3k d^3k_1\, \exp\,\{i(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x}\}\, \frac{k_z}{k_H^4}\, Q_{ij}\, \overline{\psi}(\boldsymbol{k}_1)\langle \psi(\boldsymbol{k}+\boldsymbol{k}_1)\rangle\right], \tag{B12}$$

where $Q_{ij} \equiv -k_{1j}(k+k_1)_i(\boldsymbol{k}_H + \boldsymbol{k}_{1H})^2 - k_{1i}(k+k_1)_j k_{1H}^2$. By using the definition of $\overline{G}$ (see immediately after eq. [31]) and $\epsilon_{ij3} Q_{ij} = -\epsilon_{ij3}(k_H^2 + 2\boldsymbol{k}_H \cdot \boldsymbol{k}_H) k_i k_{1j}$, equation (B12) becomes

$$J_{32} = -\frac{B_0}{\nu}\left[B_0\,\overline{G} - i\epsilon_{ij3}\int d^3k' d^3k e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\, \frac{k_z}{k_H^4}\,[k_H^2 + 2k_l(k'-k)_l] k_i (k'-k)_j\, \overline{\psi}(-\boldsymbol{k}+\boldsymbol{k}')\langle\psi(\boldsymbol{k}')\rangle\right]$$

$$= -\frac{B_0}{\nu}\left[B_0\,\overline{G} - \epsilon_{ij3}\,\partial_j \int d^3k' d^3k e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\, \frac{k_z k_i}{k_H^2}\left[-1 + \frac{2k_l k'_l}{k_H^2}\right]\overline{\psi}(-\boldsymbol{k}+\boldsymbol{k}')\langle\psi(\boldsymbol{k}')\rangle\right]. \tag{B13}$$

Now, since $k' \ll k$, we expand the integrand of equation (B13) to second order in k'/k, in order to show that there is no contribution from the second term in equation (B13) (to this order). Therefore, $J_{32} = -B_0^2\,\overline{G}/\nu$. Inserting J_1, J_2, and J_3 in equation (B8), dividing by δt, and then taking the limit $\delta t \to 0$ finally yields equation (31).

APPENDIX C

This appendix discusses the scaling relation for the reconnection rate when the effect of hyperresistivity dominates over turbulent diffusivity in quasi-linear closure model in § 2. As mentioned in § 2.3, this is the case when $\chi < 1/2$ if $(L_{bH}/L_{BH}) \propto \mathrm{Re}_m^{-\chi}$, where L_{bH} and L_{BH} are the characteristic horizontal scales of fluctuating and large-scale magnetic fields, respectively. By using $\langle u^2 \rangle \sim \langle b^2 \rangle$ and by assuming $\mathrm{Re}_m > 1$, equation (37) is simplified as

$$D_H \sim \eta_k L_{bH}^2\,. \tag{C1}$$

Upon using $L_{bH}/\Delta \sim L_{bH}/L_{BH} \sim \mathrm{Re}_m^{-\chi}$ and $\Delta/L = v_r/v_{\mathrm{A}}$, the reconnection rate, equation (35), becomes

$$\frac{v_r}{v_{\mathrm{A}}} \sim \left(\frac{\eta_k L_{bH}^2}{v_{\mathrm{A}} L^3}\right)^{1/4}. \tag{C2}$$

Upon using $L_{bH}/\Delta \sim L_{bH}/L_{BH} \sim \mathrm{Re}_m^{-\chi}$ and $\Delta/L = v_r/v_{\mathrm{A}}$, we now express L_{bH}/L as

$$\frac{L_{bH}}{L} = \frac{L_{bH}}{\Delta}\frac{\Delta}{L} \sim \mathrm{Re}_m^{-\chi}\,\frac{v_r}{v_{\mathrm{A}}}\,. \tag{C3}$$

Therefore, the substitution of equation (C3) into (C2) gives us

$$\frac{v_r}{v_{\mathrm{A}}} \sim \sqrt{\frac{\eta_k}{v_{\mathrm{A}} L}}\,\mathrm{Re}_m^{-\chi}\,. \tag{C4}$$

As $\chi < 1/2$ in this case, the reconnection speed has a weaker dependence on Re_m compared to the case where η_{eff} dominates over D_H.

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