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A new stochastic asset and contingent claim valuation framework with augmented Schrödinger-Sturm-Liouville EigenPrice conversion

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A New Stochastic Asset and Contingent Claim Valuation Framework with Augmented Schrödinger-Sturm-Liouville Eigen-Price Conversion

By

Adrian Euler

July 10, 2019



A thesis submitted in partial fulfilment of the University's requirements for the Degree of Doctor of Philosophy

EXECUTIVE SUMMARY

This study considers a new abstract and probabilistic stochastic finance asset price and contingent claim valuation model with an augmented Schrödinger PDE representation and Sturm-Liouville solutions, leading to additional requirements and outlooks of the probability density function and measurable price quantization effects with increased degrees of freedom. It is an analytical and valuation framework that explores existing pricing problems and models from a common point of high abstraction using a dimensionality reduction approach, under real-time trading assumptions. The context of the new model is a realistic market, made-up of a network of financial intermediaries and products whose prices are stochastic, measurable through transformable random variables, and a network of investors (individuals and firms) with rational and measurable preferences and expectations, seeking to maximize the expected utility of their final wealth in a multi-period time horizon. This study models pricing of assets and contingent claim at any time node and considers a zero-dimension reduction around each node in order to identify additional probability and price-change behavior effects, subsequently yielding new testable techniques of pricing assets and contingent claims.

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TABLE OF CONTENTS

CHAPTER 1: INTRODUCTION	7
CHAPTER 2: LITERATURE REVIEW	32
2.1 PRELIMINARIES	33
2.2 APPLIED STOCHASTIC FINANCE CONCEPTS	50
2.3 RELEVANT OUANTUM FINANCE MODELS: O-MARKETS AND PRICE	
DISTRIBUTIONS	73
2.4 LITERATURE REVIEW CLOSING REMARKS	106
CHAPTER 3: RESEARCH METHODOLOGY	110
2 1 DESEADOU DUIL OSODUV	111
$\frac{5.1 \text{ Research f hilds of h 1}}{2.2 \text{ Desearch Addom CH}}$	111
3.3 METHODOLOGICAL FOUNDATIONS	117
<u>5.5 METHODOLOGICAL FOUNDATIONS</u>	122
CHAPTER 4: AN ABSTRACT STOCHASTIC ASSET PRICING AND CONTIG	ENT
CLAIM VALUATION FRAMEWORK WITH SCHRÖDINGER PDE AUGMEN	TATION 131
4.1 INTUITION AND RATIONALE	132
4.2. THE MODEL	146
4.3 PROPOSAL PROBABILITY DENSITY AND ASSET PRICING PDES	147
4.3.1 CASE 1.0 : TIME - INDEPENDANT GSE	151
4.3.2 CASE 2.0 - CAPITAL ASSET PRICING PDE	152
4.3.2.1 SCENARIO 1.0	154
4.3.2.2 SCENARIO 2.0	157
4.3.2.3 SCENARIO 3.0	159
4.3.2.4. SCENARIO 4.0 – PART I	160
4.3.2.5. SCENARIO 4.0 – PART II	162
4.3.3 CASE 3.0 - GENERALISED OPERATOR	164
4.3.4 CASE 4.0 – TIME INDEPENDENT EIGEN-STATE FORMULATION	168
CHAPTER 5: ASSET PRICE RAPPROCHEMENT: SPLIT PDF IDENTITIES A	ND STURM-
LIOUVILLE QUANTUM FITTING	170

17

5.2. PROPOSED FINANCIAL DERIVATIVES' PARTIAL DIFFERENTIAL EQUATION	<u>s</u>
(PDEs)	179
5.2.1 CASE 5.0 – TIME-DEPENDENT ASSET PRICING PDE	179
$\frac{5.2.1.1.\text{ SCENARIO } 1.0 - \text{PART I}}{5.2.1.2.\text{ SCENARIO } 1.0 - \text{PART I}}$	180
$\frac{5.2.1.2.\text{ SCENARIO } 1.0 - PART II}{5.2. \text{ DEPENDENTIAL (IDENTITY) EUNOTION}$	183
5.3. PRELIMINARY PROPOSAL FOR THE POTENTIAL 'IDENTITY' FUNCTION	185
5.3.1 CASE 6.0 – ASSET PRICING PDE WITH A 'HARMONIC' POTENTIAL	18/
5.2.2 CASE 7.0 SUBEDSET DEDIVATIVES' DDICINIC DDE	101
<u>5.3.2 CASE 7.0 – SUPERSET DERIVATIVES PRICING PDE</u>	191
<u>5.5.2.1. SCENARIO 1.0</u> 5 4 DENGITY OF STATES AND FICEN DDICE SVSTEM	191
5.5. INFORMATION DISSIPATION. REFLECTION AND RELAY FROM 0D PRICE	195
SYSTEM	196
5.6. REBRANDING THE MASTER EXPRESSION	197
CHAPTER 6: COMPUTATION OF ASSET PRICES IN FORWARD TIME WITH EIGH	EN-
VALUE CONVERSION	201
6.1 DISCUSSIONS AND NUMERICAL ILLUSTRATIONS	206
6.1.1 GEOMETRIC BROWNIAN MOTION SIMULATION OF STOCK PRICE	206
6.1.2 MONTE-CARLO PROBABILITY DENSITY MIXING	208
6.1.3 AUGMENTED MARKET-MOTIVE POTENTIAL FUNCTION AND INFORMATION	
TUNNELING	212
6.1.4 SUITABLE POTENTIAL FUNCTIONS	214
6.1.5 NUMERICAL ANALYSIS	220
6.1.6 INFORMATION TUNNELLING AND PRICE TRANSITION	231
6.1.7 INTENSITY OF INFORMATON REFLECTION ON PRICE (IIRP)	239
6.2 STURM-LIOUVILLE LOGARITHMIC PRICE EIGEN-VALUE SYSTEM	243
CHAPTER 7: COMPARATIVE VALUATION OF FINANCIAL OPTIONS WITH EIGE	<u>N-</u>
VALUE CONVERSION AND CLASSICAL MODELS	250
7.1 FINANCIAL OPTIONS WITH EIGEN-VALUE CONVERSION	251
7.2 FINANCIAL OPTION VALUATION WITH SQUARE-WELL, GAUSSIAN, COSH, A	ND
ARCTAN ZERO-OBJECT MODELS	263
7.3 FINANCIAL OPTION VALUATION WITH BLACK-SCHOLES, CRR BINOMIAL, 3	RG
MODELS	274
7.3.1 THE BLACK-SCHOLES MODEL	274
7.3.2 THE CRR BINOMIAL MODEL	277
7.3.3 LOGNORMAL BLACK-SCHOLES MODEL	283
7.3.4 THE RECIPROCAL-GAMMA MODEL	284
7.4 EMPIRICAL TESTING AND ANALYSIS USING HISTORICAL VOLATILITY,	
CLASSICAL vs QUANTUM SW	288
7.5 EMPIRICAL TESTING AND ANALYSIS WITH LOCAL VOLATILITY AND PRICE	-
SURFACE QUANTUM SW FITTING	293
5	

7.5.1 REAL-TIME OPTION PRICING, TESTING, AND ANALYSIS	293
7.5.2 REVISED OPTION PRICING MODELS	298
7.5.3 SAMPLE HYPOTHESIS TESTING	321
CHAPTER 8: CONCLUSIONS	332
BIBLIOGRAPHY	336
APPENDIX I: BROWNIAN MOTION SIMULATION USING EULER DISC	CRETIZATION
	379
APPENDIX II: MATHEMATICAL DERIVATIONS	387

1.0 INTRODUCTION

This research study introduces a new stochastic asset pricing model that can be used to value conventional financial assets, including financial derivatives. The new model utilises a generalised Schrödinger-Sturm-Liouville Eigen-price function and algorithm¹. It stipulates a plausible probability density - price function duality phenomenon, incorporating relevant add-on price quantization effects. It allows for inclusion of increased degrees of freedom and subsequently increased accurancy in price forecasting . This is done by including effects of information dissipation and quantum 'tunnelling'². The new asset pricing model leads to a whole new set of pricing analytics that can be empirically applicable across asset classes such as equities, debt instruments, financial derivatives, etc.

The abstracted pricing framework is formed through a process of breaking-apart, rearranging, and re-assmbling previously known asset-pricing models, using a dimensionality-reduction approach, under real-time trading assumptions. It is accomplished by relaxing specific assumptions often used in obtaining and validating existing pricing models. Instead, I theorise using a general master expression that reflects

¹ A GSE-Sturm-Liouville algorithm implementing the modelled PDE in C++ with NAGS and allows for computation of Eigen price levels within an open-form quantum system. This is further elaborated in Chapters five and six. ² With the term "tunnelling" we refer to the % of information known to-day that travels to the next time point through quantum market penetration.

(i) price developments on time-series basis, and (ii) spatial probability distribution effects, carefully formulated around the probability density - price function duality. This implies an equilibrium of effects. It it possible to explain this from an efficient market hypothesis angle (Fama, 1991; Malkiel, 2003). The information, expected to be reflected in the market price, is dissipated by the quantum structure fitted at a market point. Information is fragmented and price quantized within the structure. I hypothesize that quantum and market prices develop simultaneously. Therefore by studying information and price effects in the quantum structure, one might sufficiently predict the market price. This is in-line with the broader conceptualisation around Markov's property (Markov, 1954:1971; Seneta, 1996; Gilks et al., 1996).

In order to empirically test the new pricing techniques, I consider real-life market conditions, where a financial market is made up of an integrated system of financial intermediaries and financial assets whose prices follow a stochastic process measurable through transformable³ random variables. The financial market in this study is comprised of a network of institutional and retail investors with rational and measurable preferences and expectations, in-line with the Arrow-Pratt risk aversive attribution theory (Arrow, 1971:1988; Pratt, 1964; Gollier and Schlesinger, 1996).

These investors seek to maximise the expected utility of their final wealth in a multi-

³ Changing a variable from a Cartesian system representation to spherical, but also replicative expansion of one single variable to two or more within the intended space-dimensions.

period time frame. This is in stark contrast to prior theoretical asset pricing formulations such as Black and Scholes (1973), Cox et al. (1979), Jarrow and Rudd (1983), Leisen and Reimer (1996), etc., that employ a large number of simplifications, such as oversimplified market activity and investment participation along each time-period investment horizon.

This approach enables pricing of financial assets at each point along the market line by considering a zero-dimension reduction around each point⁴. In such context, I refer to these points as zero-time objects or simply zero-objects. Zero-objects are quantum systems with attributed quantised space topologies (Barrett, 1999; Baaquie et al., 2002; Abramsky and Coecke, 2007). This is important because such zero-object representations of time points, unlike their use in traditional science disciplines, have been partially considered in a Financial domain, such as in the works of Bardou et al., (2016), Levental et al., (2016), Benaim and Raimon, (2003), Benaïm et al., (2002), Aldous et al., (1988), Nastasiuk (2015), and Casena (2007) to name a few⁵.

It makes it possible to identify additional probability and price-changing behavioural effects that could lead to better explanations of future price levels, while being consistent with the Markov property that only the present state of the price would provide relevance to any immediate future price pattern development (Spitzer, 1970; Snyder and Miller, 1991; Seneta, 1996; Parzen, 2015).

⁴ A time point as a zero-object with no space dimensions, but that can be modelled through a three-dimensional quantum system.

⁵ Treated in detail in the section "Relevant D₁-World Quantum Effects and Quantized Market Price Distribution".

Market price evolution and price pattern development are driven in part by the dissipated information and the asset price "DNA" within quantized price points. All of the dissipated information is reflected first and foremost onto price change levels, before it becomes an actualised price in the market. The quantized price point as a zero-object serves as a price repository. The price behaviour forecast depends on the degree of information dissipation and "tunnelling". We assume that information "tunnelling" effects will have stronger price reflection on near-neighbour future market prices and less so further out in time (Haven, 2002:2004:2005:2008a:2008b, Callegaro, 2018a). This is an effect directly linked to the information we may hold about the state of nature⁶ in the future (if any at all), factored in efficient market hypothesis (Malkiel, 2003).

Terms such as 'financial derivatives' and 'contingent claims' are used interchangeably here. It is a generalisation around broader types of financial derivatives, although I often refer to financial options (Jarrow and Turnbull, 1998). Both assets and contingent claims can be valued in duality; along the market line, and in a zero-dimension 'universe'⁷ i.e. a set of orthogonal Eigen-price levels within each quantum point. These patterns allow for consideration of increased degrees of freedom in price forecasting.

⁶ Future patterns of uncertainty tied in good part to future events.

⁷ Refers to a system where measurability of one or more variables (i.e. price, probability, etc.) is decided along its dimensions; typically three dimensions, but less or more than three dimensions may be used to explain the variables, depending on the nature of the variable and variable change-behaviour complexity.

Markowitz (1952), Tobin (1965), Black and Scholes (1973), Treynor and Black (1973), Stulz (1981), and many others have incorporated time as the main dimension where wealth growth can be measured. Moreover price valuation is carried out in a dual manner: at athe underlying asset base, and asset-protection levels⁸.

The time implication is of paramount importance, while computing the expectation term⁹. This is also evident in pricing models that follow a discrete time process, such as those described in the works of Cox et al. (1979), Jarrow and Rudd (1983), Leisen and Reimer (1996). They theorise pricing contingent claims through well formed multi - period binomial trees. Asset prices are evolved through multiplicative moves , following well established and testable formulae. Jarrow and Turnbull (1998) argue that a contingent claim is a random variable, defined on an underlying probability space. Subsequently it may be regarded as the payoff at a time point of some contract¹⁰ with a protection mechanism (subsequently a claim) on the underlying.

Specifically in the Jarrow and Rudd (1983) model¹¹, the individual multiplicative price moves at each time point are equally probable. Cox et al. (1979) propose an alternative choice of multiplicative price motion formulae¹² validated within a risk-neutral

⁸ Where buying or selling price is locked.

⁹ Can be seen more clearly through the Black-Scholes formula for a call in a slightly different way, as $c = e^{-tT} \left[se^{tT} N(d_1) - XN(d_2) \right]_{OT} c = e^{-tT} \left[s_T N(d_1) - XN(d_2) \right]$. The expression inside the square brackets represents the expected payoff of the option, $E^Q[]$. N(d2) is the probability that the call will be exercised in a risk-neutral world, where a share that pays no dividends has an expected return of the risk-free rate.

¹⁰ Locks the buying or selling price of the underlying.

¹¹ The Jarrow-Rudd (JR) model uses $u = \exp\left[\left(r - q - 0.5\sigma^2\right)\delta t + \sigma\sqrt{\delta t}\right]$ and $d = \exp\left[\left(r - q - 0.5\sigma^2\right)\delta t - \sigma\sqrt{\delta t}\right]$.

¹² The Cox, Ross, and Rubinstein (CRR) model uses $\ln(u) = \sigma \sqrt{\delta t}$ and $\ln(d) = -\sigma \sqrt{\delta t}$

environment with a slight upward bias.

A third set of parameters¹³ for developing the share price tree was proposed by Leisen and Reimer (1996) with two important advantages over JR and CRR parameters; firstly they suggest better and separate probability estimates, and secondly by centring the share price tree at maturity on the exercise price, the convergence oscillation reported in the Jarrow and Rudd (1983) as well as the Cox et al. (1979) trees is removed.

The contingent claim valuation models by Aase (1988) and the original work of Black and Scholes (1973) are based on continuous stochastic processes, where the expectation term has a probabilistic nature with a time parameter. Stock price changes may also be represented by a one-parameter stochastic process, whereas interest rates are naturally represented by a dual-parameter process; the first parameter is the time to maturity of the fixed income instrument, whereas the second parameter is the "real" time, such that the process modelled is a random surface from where one may obtain various implied measurable such as volatility (Merton ,1973:1974; Aase, 1988; Tanaka, 1991; Parzen, 2015).

Furthermore an equity-based contingent claim can be represented by a one parameter stochastic process, where real trading time is the only parameter. In addition, two-

¹³ The Leisen and Reisen (LR) up and down price multipliers for the share price moves in the tree are u = bp'/p where $b = exp[(r-q)\delta t]$, and d = b(1-p')/(1-p).

parameter processes serve better the purpose of price valuation of financial derivatives or contingent claims on fixed-income instruments, where the first parameter is real time, whereas the second represents time to expiration or the actual trading time. In interestrate markets, bonds are tradable assets (Cox et al., 1985; Fujihara and Park, 1990; Tanaka, 1991; Heath, et al., 1992; Fabozzi, 1995; Miltersen et al., 1997; Cuthbertson and Nitzsche, 2004; Batten et al., 2004).

The expectation term, subsequently the probability density and distribution of the underlying asset are discussed in Black and Scholes (1973), Black (1989), with well formed assumptions around normality. Likewise the classical parametrised asset and financial derivatives¹⁴ pricing models cannot fully explain price behaviour at each point along the market line.

Expressed differently, existing financial pricing model can forecast price, however they fail to match its value at the end of the investment holding times¹⁵. King (1966), Elton et al. (1978), Conner and Korajczyk (1995), Bodie et al. (2009), Elton and Gruber (2011), as well as Brealey et al. (2008), Hillier et al. (2001), among, others, articulate well the use of factor-tracking based on linear indexing¹⁶ and subsequently the capital asset

¹⁴ The terms 'contingent claim' and 'financial derivative' are equivalents and are used interchangeably throughout this research study.

¹⁵ The end of buy-and-hold position within a finite interval.

¹⁶ Indexing $r_i = \alpha_i + \beta_i m + \epsilon_i$, where r_i is the rate of return of security i, beta the volatility sensitivity, m a common economic factor, and epsilon the stochastic regressor; an independent variable with a zero expectation, linked to the residual effect. The individual security variance: $\sigma^2_i = \beta^2_i \sigma^2_m + \sigma^2(\epsilon_i)$, where σ_m is the market risk and $\sigma(\epsilon_i)$, the stochastic regressor error dispersion.

pricing model, (i.e. CAPM) as an equilibrium model that underlies modern financial theory.

It is normally derived using principles of diversification with simplified assumptions stated well in the works of the above mentioned authors with original development credited to various researchers such as Markowitz, Sharpe, Lintner, and Mossin. In the model, the common economical factor tracked is usually the market return. It explains to a good degree price movements of other stock in market (Markowitz, 1952:2000; Sharpe, 1964; Lintner, 1965; Varian, 1993; Mossin, 1966).

In the single index model (Sharp, 1964) there is only a single degree of freedom that stock price return variability will depend on. According to Conner and Korajczyk (1995), as well as Hillier et al. (2001), an increase in degrees of freedom will allow for inclusion of additional macro-economic variables, firm-specific events, and statistical procedures. This would enable investors to rationalise their preferences around maximisation of their utilities. As long as factorisation is "realistic", factor loading and model maching can be achieved, subsequently allowing rational investors to identify arbitrage. Elton and Gruber (2011) provide equivalents of indexing for fixed-income assets and the argument of the price behaviour follows the same logic. The increase in the number of factors increases the degrees of freedom in the model, although the market price change behaviour is still only partially explained. I have put forward the hypothesis that a smooth and continuous development of market prices between any two points along the market line (one already observed, and the other stochastically market -implied and to be observed in the future) can only be inferred by unobserved discretised time-space "universe", where its internal orthogonal price patterns are the only quantized price clusters relevant to explain the market price at the near-neighbour point, along the market line. This signifies price perturbances due to uncertainty where factors and events fuel uncertainty and could not possibly be established only in anticipation (Callegaro et al., 2017a:2017b).

In addition, the hypothesis stipulate that orthogonal and observable discretised price jumps at any stoppage time along the market line, can be explained by an observable continuous time-space "universe" effect; often this is a combination of factors in that "universe", such as company announcements with information on internal business augmentations of sorts, dividend policy, ex-dividend dates, release of financial statements, mergers and acquisition, consolidated cash flows, and any information that would incur a time-node saturated cumulative increase or decrease in the demand or supply (Fama et al., 1969; Fama, 1970:1991; Hall, 1980; Stein and Stein, 1991; Strong, 2004).

Subsequently, the inability to fully explain future price changes is due to the inclusion of a "realistic" probability density function only in a partial consideration, and the exclusion of possible price effects that come through possible probability admixing and an "upper-ceiling" limitation in the degrees of freedom of the existing models. The contemporary

models allow, at best, forecasting on the basis of factor tracking and loading and are measured in expectation (King, 1966; Farrell, 1997; Elton et al., 1979; Hillier et al., 2011).

Price prediction requires a system with higher degrees of freedom According to Benaïm and Raimon (2003), one may theorise on the transition from factorization onto quantization with self-interacting diffusion on a compact Riemannian manifold at each time-node. It can be achieved by retaining the number of time parameters and the degrees of freedom provided by the classical model as a "classical" contributor to the price value. It also requires exploding the degrees of freedom of the pricing system around a zerodimension constrain (zero time parameter consideration). Subsequently providing a fuller time-parameterization spectrum, This would allow to better observe price behaviour at each time-node. It includes 3x additional degrees of freedom and a quantizer contribution to the price (Luschgy and Pages, 2002).

Following from the hypothesis, it is possible to consider asset pricing and contingent claim valuation models, where stochastic residual effects are intrinsically assigned an occurrence measure for unanticipated events, that can contribute to price dynamics within the quantum system (Luschgy and Pages , 2002).

Values of stochastic variables, such as price change, capital weights, etc., can be computed within the quantum system. They can be used in combination with the expectation forecast in order to match a future market value that is decided from a previously observed market value and the effect of filtration dissipation. Revus and Yor (2004), Karatzas and Shreve (1998b), Shiryaev et al. (2006), Callegaro et al. (2013), among many and in a similar manner, provide good argumentation on filtrations. They consider real-time shock effects on asset prices, observable and measurable in the market. However, we consider an additional and unrelated "orthogonal" filtration effect on asset prices, which we attribute to the strength of filtrations. This is linear to the ability of the market to dissipate information. Market's ability to dissipate information (and subsequently reflect it on price) is conceptualised in parity to quantum system's capability to recognise and process it internally. For the latter to occur, we would need to fit it with reasonable "mechanics".

I consider the "orthogonal" filtration to be responsible for the quantized price change behaviour at each p-tip¹⁷. It drives the actual p-tip spin (positive or negative price change) at the market point. Extreme price confinement is particularly of interest because the dissipated information may relay some form of "hint" or "warning" about forward-time events that are unanticipated. Information dissipated by the system may "tunnel" through to immediately adjacent future p-points as self-interacting diffusion on a compact Riemannian manifold. It reaches out to the future adjacent market points, before diminishing in the extended time horizon. This would make the forecast plausible, at least between two adjacent time-nodes (Luschgy and Pages, 2002; Benaïm and Raimon, 2003).

¹⁷ Refers to the price point or the market observed value from a quantum view..

The stochastic shock effect is often modelled through the Weiner process and has a Gaussian distribution (Stein and Stein, 1991; Shreve, 2004a:2004b). However, this research suggests that the p-tip probability distribution system contains either (i) a probability distribution with many splits, or (ii) several probability distributions with various degrees of admixing. The system itself exists in various Eigen-states where mixing or splits occur: a quantization effect that follows directly from the 3x expansion of the price-system's degrees of freedom. The Gaussian probability distribution function corresponds to the lowest and most stable Eigen-state, however at higher quantized levels (i.e. higher volatilities and irregularities in filtrations patterns), there are distribution splits, evident both mathematically and through numerical illustrations (Dirac, 1958; Callegaro, 2013:2015:2017a:2017b).

The new model considers additional price changes due to quantization effects at each market point. Subsequently factors-in the cumulative price-change (i.e. more precisely the return rate) effects through classical pricing models with additional quantization effects at each point along the market line (Chen, 2001:2003; Haven, 2002:2003:2004:2005:2008a:2008b; Callegaro, 2018a).

The new pricing framework has profound implications in financial instrument pricing, especially financial derivatives. Classical option pricing models such as Black and Scholes option pricing, among others, commonly use well-formed assumptions on normal or lognormal distributions. These assumptions need to be upgraded to include probability distribution system admixing and additional Fermi-Dirac probability distribution effects for the zero-object system fitted at each price-point along the market line. I consider a probability distribution system made up of various distributions at various stages of admixing (density of distributions); a system that possesses different levels or states (density of states), where the lowest level represents the optimum effect of admixing of distributions, as inference and reflection of the internal p-point price pattern dynamics (Dirac, 1926:1958; Black and Scholes, 1973; Merton, 1974; Black, 1989; Haven, 2002; Callegaro, 2018a).

Hence, the objective of this research is fourfold. First, I reconsider pricing problems starting from a postulated master formulation, leading to asset price and financial derivative valuation formulations in partial differential forms; secondly, I derive valuation expressions that are in-line with a rational investor's expectations, although derived as an abstract and unified formulation; thirdly, I incorporate quantization effects in the probability distribution and the price-change system, subsequently augment it to a generalised Schrödinger-Sturm-Liouville mathematical representation to obtain tangible solutions to pricing problems (Bailey, 1966; Sharpe et al., 1995). I bring the master pricing model to an application level with various new variables and concepts fitted. To that end I carry out various numerical simulations and compare results with those from various classical models.

I use a postulate-implied problem with a very abstract formulation (i.e. master expression) and solve various common cases that are in-line with contemporary pricing models. They

are further extended and empirically tested. The problem is a financial instrument valuation challenge, springing from the master expression (generalised Schrödinger equation), with Sturm-Liouville adapt-solutions, in order to observe quantization effects at a price's point in the time line (Bailey , 1966; Haven, 2002; Callegaro, 2018a).

The approach and results of this study relate to, and extend classical and contemporary works in asset pricing, among others, the works of Black et al. (1972), Black and Scholes (1973), Black (1989), Eisenberg and Jarrow (1991), Sharp et al. (1995), Cox et al. (1979), Elton and Gruber (2011), Bodie et al., (2009), Shreve, (2004), Chen (2001:2003), Haven (2002:2003:2004:2005:2008a:2008b), to name a few. I treat gaps in literature and move beyond market tracking (Hillier et al., 2011) to treat the price function along a time horizon and at each zero-dimension point.

This research work is conceptualised as a '3+1' knowledge volume. The bottom layer is constructed around a conceptual cognitive map that includes, as part of the critical literature review, concepts such as random variables, differential function preliminaries, Brownian motion fundamentals, stochastic utility, the Arrow-Pratt absolute risk aversion, stochastic portfolio, indexing, stochastic contingent claim valuation, quantum mechanics(Callegaro et al., 2017a:2017b), and the Sturm-Liouville model (Bailey, 1966; Kong and Zettl, 1996).

The middle layer represents the mathematical finance layer of the new value knowledge,

addressing the existing research gap in this domain. It infers knowledge at a deeper level, utilising quantitative and mathematical finance patterns of deductive reasoning to bring the new knowledge to the level of Finance applications. More specifically the middle layer is made up of theoretical Chapter 4, "An Abstract Stochastic Asset Pricing and Contingent Claim Valuation Framework with Schrödinger PDE Augmentation", whose main storyline is built around a postulate-implied master formulation (based on a generalised Schrödinger PDE), new pricing sub-models, the justified complete equilibrium concept, probability space continuum, and the curvature surface linked to and extending contemporary financial asset pricing models.

The middle layer includes Chapter 5 "Asset Price Rapprochement: Split PDE Identities and Sturm-Liouville Quantum Fitting", with focus on various price-cut off functions, the derivation of various option price PDEs (Black-Scholes derived from our postulateimplied master expression as a special case), as well as the acquisition of Sturm-Liouville PDEs, allowing us to fit a quantum system at each market price point i.e. the zero-object.

The top knowledge tier is the Finance and Financial trading layer organized as theoretical Chapter 6, 'Computation of Asset Prices in Forward Time with Eigen-Value Conversion" and Chapter 7, 'Comparative Valuation of Financial Options with Eigen-Value Conversion and Classical Models'. Both chapters focus the financial asset and derivatives pricing applications using the main concepts in this research with emphasis on numerical and empirical tests. I generate data algorithmically and test several cut-off price functions with clearly stated boundary conditions. More specifically testing financial instrument pricing capabilities of our model through several scenarios based on the technique of fitting a cut-off price function in each case; (i) constant, (ii) harmonic, (iii) Gaussian, (iv) Cosh, (v) inverse, (vi) Arctan, and (vii) decaying exponential.

After fitting the augmented Schrödinger-Sturm-Liouville PDE and the boundary conditions on the price zero-object¹⁸, I then compute the Eigen-price levels and generate delta- distributions. Much work is done on developing the pricing theory within a zero-object and its highly quantized space constraint. Furthermore I conceptually "connect" the © and D-worlds in one "universe", which serves to translate financial asset measures (liquidity, volatility, etc.) from the ©-world onto the D world and vice-versa. Finally I show cases the use of our model in asset and option pricing, and compare it to a range of existing pricing models.

The internal space of the zero-object is measured through the newly introduced quantum parameter, which serves as a space "converter" ¹⁹. Whereas within the zero-object space, the depth, radius, etc., are all measured relative to the quantum- translated volatility, more

¹⁸ Refers to the quantum system. Typically here it refers to quantized price points along the market line.

¹⁹ Converts values as we know them in our world to equivalents in the \mathbb{D} -world.

precisely, I fit the zero-object with a quantum volatility surface. It follows an extensive amount of work with trial and tribulations to establish the internal space dynamics of a zero-object for what it means in our financial asset pricing domain. I calculcate Eigenprices, probabilities, and translate them to a price vector in the \mathbb{C} world for further use.

All asset price attributes are represented in percentages I use Eigen-price terms as a shorthand reference to quantized logarithmic price values, differentiation of which provides us with the liquidity measure of a financial asset. Of importance here is the new element of theory added in the last theory chapter for the provision of the price spin. It allows for both positive and negative changes in price. The end results are not only the deltadistributions, Eigen-probabilities, but also price-vectors, which are translated back to the ©-world for further consideration. Both assets and financial derivatives are priced methodically within this new approach. Finally, I price options and compare the results with existing models such as the Black-Scholes (normal and lognormal versions), Binomial, RG, etc., (Black and Scholes, 1973; Cox et al., 1979; Jarrow and Rudd, 1983; Leisen and Reimer, 1996; Milevsky and Posner, 1998).

The effects of probability distribution splits are illustrated both mathematically and graphically. Although I trial a Gaussian, any other probability distribution function can be used. Moreover, financial derivatives' pricing using different Eigen-states can be achieved. The lowest Eigen-state solution matches the Black-Scholes-Merton model. The new option pricing model appears to correct the overestimation present in option prices for longer-term contracts. The testing of specific sub-models follows the hypothesis that

the abnormal spread from the market thresholds is narrowed or diminished with the new models, as compared to the existing models, thus leading to better price forecasting (Black and Scholes, 1973; Geske and Roll, 1984; Hull and White, 1987; Milevsky and Posner, 1998; Haven, 2002:2003; Chousa and González, 2016; Bustamante and Contreras, 2016).

The additional '+1' layer, dimension of this research study represents the body of research methodologies used to investigate contemporary literature, utilization of data generated streams, gaps in literature, as well as the use of analytics to reach relevant findings. This study's research methodology is rooted on deductive reasoning, on which our master-formulation and sub-models are based on. Although intrinsically non-formal as a theory, I apply strong forms of coherence and cohesion in order to investigate pricing aspects of financial assets and derivatives. This is an attempt at completeness that makes it possibleto lift the framework to an abstract level (Adams and Schvaneveldt, 1991; Cooper and Schindler, 2001;Saunders et al., 2003).

Throughout the thesis, I use an adapted form of first-order predicate logic²⁰ to put forward the ideas that allow me to strengthen the storyline throughout the thesis. It embeds firstorder language with reflection of strong forms of "identity" of items and relationships,

²⁰ A highly formal mathematical representation with quantifiable variables that uses logical description through variables rather than non-logical objects.

with variables, connectives, quantifiers, and some primitive terms represented also within logical betweenness and equidistance (Tarski and Givant, 1987; Barwise, 1977; Barwise and Etchemendy, 2000; Hazewinkel, 1997:2001; Saonov, 2001; Gamut, 1991; Haewinkel, 2001; Andrews, 2002; Rautenberg, 2010).

The simplicity, but also the expressiveness of this language consists in the fact that it allows for the quantification on individual finance-related variables and their relationships. In order to achieve measurable tests, I leverage strong forms of methodological foundations with balanced views and a clear research strategy with emphasis on correctness and sufficiency in the quantitative data. It allows me to centralise, in good part, the solutions around the postulate-implied master expression and the research hypothesis (Hazewinkel, 1997:2001; Chiribella et al., 2012).

I also consider appropriate time-horizons where I saturate the first and second tiers of this research knowledge-base and knowledge management on cross-sectional forms of research with a focus on any individual market points, with the reduced dimension effects incurring at each point; the zero object. On the third tier, I shift to a more limited longitudinal research form by investigating the behaviour of asset prices between any two or more market points (Adams and Schvaneveldt, 1991; Saunders et al., 2003; Gulati, 2009; Wilson, 2010).

It helps to conceptualise the longitudinal investment time horizon as a "medium" in which

future events, subsequent uncertainty and financial risk exist. From a present perspective, I are unable to hold certainty on any future events, subsequently observed on unexpected price dispersions. This is an intrinsic time-related limitation of effectively identifying future prices or price behaviour patterns with certainty. When I employ the use of cross-sectional research, the dimension reduction aspect helps me to identify a quantized zero object as an "environment" where price changes and patterns can exist and are investigated (Adams and Schvaneveldt, 1991; Chousa and González, 2016; Bustamante and Contreras, 2016).

I believe that cumulative flirtations dissipated by the market are fragmented with different degrees of fine granulation within the zero-object. Events are reflected in the current market price in-line with the efficient market hypothesis, whereas patterns formed out of such fragments are in parity with unexpected future price developments. This is in-line with the Markov property (Markov, 1954:1971; Seneta, 1996; Gilks et al., 1996).

This allows me to identify different 'realities' dependent on the topological space configuration of the zero object and to contemplate fitting of appropriate price distributions. Although I put this forward as a measurable environment, it does allow for some elements of interpretation on what could appear to link to a more interpretive research philosophy that I attempt to centre around the specific internal fittings with intent to unearth logic behind such construct, which would allow me to focus on price change behaviour and pattern recognition (Remenyi et al., 1998; Haven, 2002; Busemeyer, et al., 2009; Li et al., 2017).

The fact that a new topological space construct has been fitted within a zero-object different from our own may appear as a limitation. However it comes with the advantage to deploy and use attributes of a deductive and meta-physical philosophy with the new 'universe' in which the price exists; The ^① universe is different from ours, however I unify research methodology through a functional enhancement of zero-objects and their internal topological space configurations, along the market line and orthogonal at each stoppage time. This is the inferenced enhancement in this research that successfully takes on the literature gap and the lack of function knowledge in the domain and where new value is created (Adams and Schvaneveldt, 1991; Cooper and Schindler, 2001).

The research strategy used in this study is a combination of grounded theory and experimentation with a comprehensive application of deductive reasoning in order to develop a new theory, driven by a postulated master expression, which represents the asset price unification model, whose solutions lead to existing and new asset price expressions in PDE form. The experimental research strategy helps establish the cause-effect relationship between market filtration dissipation and price evolution in forward time. It is emphasised in the use of different distributions and price cut-off "geometries" fitted in the quantized price system. Finally we deploy various pricing cases and scenarios. These are made up of various numerical simulations and empirical testing scenarios, using several sets of input data. This is aligned and integrated fully with the other two research strategies (experimental and grounded theory), for a more effective research outcome (Leedy and Ormrod, 2016; Creswell, 2013).

On the practical side, I provide a wide range of numerical illustrations through the use of various computational procedures and techniques in order to illustrate several effects and cases, such the Metropolis Algorithm described well in Gilks et al. (1996), which allows for the use of a proposal probability function and subsequent testing of various scenarios where probability distribution splits. Although the proposal distribution put forward here is a Gaussian, any other distribution could be trialled in a similar fashion (Kennedy, 1997).

This effect is demonstrated mathematically in this study. The MATLAB program implements Monte-Carlo integration and generates various distribution mixings. Various C++ programs invoke NAG routines to compute Eigen-price levels using a Schrödinger-Sturm-Liouville solution augmentation, which we then plot in Excel. Python and VBA are used to demonstrate the effect of stock price simulation using an Euler discretized Brownian motion by generating various price paths. It is along such paths that the zero-objects are fitted in our theoretical and practical considerations (Pruess et al., 1995; Karatzas and Shreve, 1998b; Gilks et al., 1996; Moller and Zettl, 1996; Agarwal and Wong, 1995; Flannery et al., 2002a:2002b; Jackson and Staunton, 2004; Pages and Printems, 2005; Hıra and Altınısık, 2014; Tharwat, 2015; Rees, 2017; Bormetti et al. 2018; Rebentrost et al., 2018).

This study adds value in the form of a new set of valuation analytics that could potentially

be used to price financial instruments, ranging from single financial securities and their derivatives, to complex portfolios, exotic option spreads, and structured products. It also inherits various limititations, such as those stemming from the statistical sampling size. In addition the multiple pricing "realities" and the subsequent interpretations may give rise to ambiguity (Daughterly, 2011; Gurajati and Porter, 2010).

The literature review, by virtue of its finite size, provides a limitation in that it is not possible to scan all literature, although a considerably broad range of published work has been investigated. Another limitation is the depth at which a topic, however related to this research, is treated by previous research and often with very limited scope without a properly established "bridge" between the zero-object with its space quantization configurations to financial asset pricing.

The quantitative method of research does allow for the application of strong forms of analytics with secondary data. An earlier limitation in generating such data was resolved, which enabled testing to be conducted. There is also a minor limitation on algorithmic simulations, in that as this is research in Finance, the time spent to construct the algorithms was significant and that took some of the focus away, such as getting access and configuring NAG routines, writing algorithms that integrate NAG functionality, etc.

Even though I provide a generalised pricing framework, only several zero-object price sub-models are tested. These are validated in partial capacity by existing literature and therefore are a safe leap forward (Haven, 2002; Nastasiuk, 2015; Luschgy and Page, 2005; Callegaro et al., 2017a:2017b:2018a). The necessary logic of obtaining contemporary pricing models from the general pricing framework is provided. This work is a conceptual framework with partial, but sufficient numerical testing. Due to the overall deductive theory approach employed here, interlinked with limited forms of interpretative philosophy, there is a possibility of bias, however we attempt to mitigate bias through sufficient numerical simulations and empirical testing.

I theorise that the zero-object is fitted with price-differential distributions at different stages of admixture, corresponding to each quantum state, This would reflect well on the levels of price definition and stability. Moreover, both the classical form of volatility and filtration-implied volatility may co-exist when we align both universes; the zerodimension topological space and the observable continuous price universe. It is also possible to convert classical measures to meaningful new quantum measures and viceversa.

Another core new value element in this research is the use of replicative function 'identities' and variable separation adaptations. It enables us to form our view of an absolute general equilibrium away from the particularism of foundational hedging effects and the likes (Black and Scholes, 1973; Merton, 1973:1974; Cox et al., 1979; Jarrow and Rudd, 1983; Foller and Schweizer, 1990; Leisen and Reimer 1996).

Such "identity" functions are polymorphic²¹. Their identities may be established by using variable separation techniques. Subsequently the "identity" of such functions work out to either be a pricing, price change distribution, or any other combination, including effects of hedging at an abstract point of consideration. It enables deeper investigation and better understanding of asset price valuation. This approach provides new value and sets this work apart, while at the same time it complements existing literature in areas of financial quantum mechanics (Bardou et al., 2016; Levental et al., 2016; Benaim and Raimon, 2003; Benaïm et al., 2002; Aldous et al., 1988; Haven, 2002; Chen, 2002; Nastasiuk, 2015; Luschgy and Page, 2005; Callegaro et al., 2017a:2017b:2018a). This study provides a testable pricing framework, using only meaningful financial variables.

The investigation of asset price behaviour within a zero-dimensional topological space ²² (Schafer, 1966; Pears, 1975; Banakh and Cauty, 1994; Banakh, 1997; Fedorchuk, 1999; Haven, 2006; Khrennikov, 2009) and a nonzero-system²³ with a functional alignment between the two, has led to various new areas of research, that could be explored beyond this study. Such is the application of the main model (and its variants) in "live" financial trading scenarios with a range of forward-time technical analysis of use in intra-day financial trading, which may rival Fibonacci²⁴ and Ichimoku²⁵ techniques. Furthermore we sought PDE representations of mathematical variable relationships, and solved at the

²¹ Variable forms.

²² The zero object or zero dimension price-point "universe".

²³ Refers to an observable continuous time-space price "universe" i.e. our own "living" world.

²⁴ https://www.ig.com/uk/trading-opportunities/a-guide-to-using-the-fibonacci-tool-to-trade-41033-171206 (accessed: 11:45am, 22/02/2018)

²⁵ https://www.ichimokutrade.com/articles/Ichimoku_Ebook.pdf (accessed: 12:57pm, 22/022018)

PDE level. Further research could be done to obtain expectation expressions through integration, while including a high degree of formalism of mathematical workings with theorems, lemmas, and corollaries, which would require a considerable amount of research time.

2.0 LITERATURE REVIEW

I consider relevant concepts of existing literature in the form of a critical literature review. Each concept has either a first or second order relevance to the research problem. I provide a review of such concepts in the context of a relevant research cognitive map, followed by the identification of the "literature gap", within which this research undertaking is justified. Shortcomings of existing theories in asset pricing are highlighted.

I consider the "universe" to be the sum effect of ©-and D-worlds. The former refers to our own world with its non-zero dimension topological space; this is the very world in which we observe prices. The latter refers to a zero-object²⁶; represents a discretised timespace world. Such object is capable of dissipating information filtrations. It also allows for the development of price behaviour. This subsequently leads to observed prices along the market line.

²⁶ Quantized or a zero-dimension topological space.

Thus the concepts I have reviewed are either D-world or C-world relevant or within an integrated C+D-view. The D-world is a relatively new construct, which I explore significantly in this study. The two "worlds" are explicitly linked to the research problem and hypothesis. A mathematically inclined reader or examiner, may wish to refer to formulae and additional descriptions provided on footers.

2.1 PRELIMINARIES

One of the essential components of this research study is the abstract and probabilistic framework, which acts as a fundamental base in asset and contingent claim valuation scenarios. All pricing and valuation work links to this framework in either explicit or implied manner. I consider here a form of probability description of either an observed or implied probability space, which may be interconnect to other implied surfaces such as the volatility surface. The probability space is structured as a probability triple²⁷. The probability triple construct is fully described and explained in existing literature (Karatzas and Shreve, 1998b; Williams, 1991; Rogers and Williams, 2000a:2000b; Shiryaev et al., 2006; Sondermann 2007; Lamberton, and Bernard 2007; Callegaro et al., 2017b).

²⁷ Represented as (Ω , F, P), consisting of a non-empty set Ω , the sample space, a δ -field F of subsets of Ω and a probability (measure) P defined on F, in a non-empty set of subsets (events) of Ω with closed under taking complements: $A \in F$ implies that $A^c = \Omega \setminus A \in F$, with countable unions: $A_i \in F$, i = 1, 2, ..., implying that $\bigcup_{i=1}^{\infty} A_i \in F$

Within a financial context, I consider probability triple effects such as the state of nature, events, and subsequently the actual probabilities. Probabilities are outcomes of events, whereas events are linked to specific states of nature²⁸. Specifically, the probability triple concept in Levental et al, (2016), is related to the construction of uncertainty with a many-dimensional standard Brownian motion over a specific time horizon. With the exception of notation variation, the concept is the same in its basic form and reported broadly in existing literature (Neveu, 1965:1975; Billingsley, 1995).

I seek to recreate the concept in a Financial asset pricing context to reflect expanding degrees of freedom. I intend to gain insight on the under-domain curvature in probabilityimplied surfaces. This is more evidently represented over a martingale and in implied volatility surfaces driven by market prices. The martingale itself is a stochastic process with a realised sequence of outcomes from uncertain events where all prior values are observed and for which, at a particular time in the sequence, the conditioned expectation of the next value in the same sequence is equal to the presently observed value (Dalang et al., 1990; Stein and Stein, 1991; Madan and Milne, 1991; Rogers and Williams, 2000a:2000b; Hazewinkel, 2001; Grimmett and Stirzaker, 2001; Gulisashvili and Stein, 2010).

I am interested in quantum asset pricing and probable patterns in implied volatility surfaces. This interest me because of price quantization at each zero-dimension object (

²⁸ Refers to any future patterns of uncertainty.

D-world) along the market-price line. Previous researchers have studied quantization within mathematical domains, and attempts have been made to link it to asset pricing, portfolio valuation, and option pricing (Pierce, 1970; Graf and Luschgy, 2000; Luschgy and Pages, 2002; Pages et al., 2003; Pages and Printems, 2005; Andersen and Piterbarg, 2007; Pages and Sagna, 2015).

Research interest in option pricing with open-form solutions (OFS) on American and exotic options has gained momentum in recent times, particularly in the works of Luschgy and Pages (2005), Levental et al. (2016), and Callegaro et al. (2018a:2018b). Their focus has been on the quantization of asset pricing with a combined treatment of underlying stochastic processes. This study complements their work by exploring zero-dimension topological configurations for asset pricing and emphasing a holistic approach to financial instrument valuation. Moreover, I attempt to unravell the Eigen-price dynamics within the quantized price space. If I was to 'observe' price developments within a zero-dimension object, what could I tell of its internal mechanics? What is its structure and composition? What are the key inputs and how do the two worlds interface in order to relay information?

Levental et al. (2016) argue that uncertainty may be represented by a finite-time²⁹ multiple-dimensional standard geometric Brownian motion. It is reflected in probability

²⁹ Refers to [0, T]
space³⁰, embedded in filtrations³¹, and observed in real time. However, Levental et al. (2016) do not suggest a possible factorisation of quantum effects, nor do they establish a link to probability surface curvatures and zero-object internal effects. I consider volatility surfaces to be implied by quantized topological spaces. This is the space where filtrations are fragmented and deposited. I consider a zero-object³² with deposited filtrations to be fully loaded. The actual price is created within its space and driven by its internal dynamics.

The work of Callegaro et al. (2017a:2017b:2018a:2018b) is particularly interesting as it appears to be concurrent and overlaps in general purpose with my own research. They include quantization effects in asset pricing. Although such effects are computed with a Fourier–quantization algorithm instead. Furthermore Callegaro et al. (2018a) do not use numerical integration required by Fourier-based approaches. They focus instead on transition probabilities for financial assets in order to price American options. Their work builds on Carr and Madan (1999), who priced options using fast Fourier transforms.

Callegaro et al. (2017a) achieved price quantization through a multinomial lattice discretization scheme, using a quantization procedure. They modelled Fourier transforms of a pure jump process at a given time. The pure jump processes are driven by

³⁰ Denoted as (Ω , F, P)

 $^{^{31} \{} F_t : t \in [0, T] \}$

³²Short for a zero-dimensional object.

compensated Poisson and Levy processes (Hull and White, 1987; Keller-Ressel, 2011). They considered an increase in problem dimensionality through the inclusion of a twofactor specification for the Heston (H) and double Heston (DH) volatility models. An indepth coverage of the H and DH models can be found on Heston (1993:1997), Christoffersen et al. (2009).

They introduced a distortion function and expressed the density of a random variable S_T in the distortion function in terms of a characteristic function through a Fourier inversion formula based in good part on Christoffersen et al. (2009). They claim to quantize all random variables (i.e. stochastic processes) at a fixed time by expressing the distortion function in terms of a Fourier representation of the price density. Furthermore it differentiates the distortion function, suggesting that its resolution tridiagonal structure is Hessian (Christoffersen et al., 2009). It builds in part on the work of Khrennikov (2007a:2007b) and his main argument of quantum randomness in financial markets. Subsequently it led them to a master equation that needed to be solved to obtain stationary quantizers.

Callegaro et al. (2013:2017a:) introduced a filtered probability space (Ω , F, (F_t) t \in [0,T], P), where filtrations satisfy the hypotheses. They considered positive asset values such that $S = (S_t) t \in [0,T]$. However, their model does not predict distribution splits, and while new functions have been introduced, there are no specific redefinitions or introduction of finance-related variables within the zero-object. What their work does well is match a quantization lattice and a d-dimensional Gaussian distribution to the Ornstein-Uhlenbeck

process and its bridge (Shephard, 1991; Stein and Stein, 1991; Gillespie, 1996; Schöbel and Zhu, 1999; Luschgy and Pages, 2002; Piotrowski et al., 2006) with quantization effects and data available from a known third party website³³.

Therefore the research by Callegaro et al. (2017a:2018a) is to a good measure a summary of known Stochastic processes with a fixed fit of a quantization grid downloaded from a third party website (Hull and White, 1987; Keller-Ressel, 2011; Madan et al., 1998; Jacobs and Li, 2008). They do provide internal topological details of the zero-object. Independent and concurrent to Callegaro et al. (2018a), my research does introduce a master expression too, however I develop new theoretical alternatives through the application of different price cut-off "geometries". This is done within a zero-object and with a generalised Schrödinger-Sturm-Liouville process (Bailey, 1966; Pruess, 1973; Kong and Zettl, 1996; Tharwat et al., 2013; Hira and Altinisik, 2014; Yang, et al., 2015). I place special emphasis on the topological constraint and its internal configuration, such that it leads to a comprehensive set of sub models that can be used to price with alternative quantization effects. Moreover I generate case and scenario specific quantized data, dependent on attributes of the zero-object. Although independent and concurrent, my research study complements the work of Callegaro et al. (2018a:2018b).

In considering uncertainty and ways it impacts on asset prices, I place emphasis on

³³ http://www.quantize.maths-fi.com

outcome measures of such uncertainty, through random variables, properties of which are of great interest and use in this work. Existing literature reasonably justifies the incorporation of a random variable in a pricing function, where the sample space is probability-implied by a non-empty set of events or filtrations, observable in real terms and whose expectations are well defined and measurable³⁴ (Hull and White, 1987; Karatzas and Shreve, 1998a:1998b; Shreve, 2004a:2004b; Shiryaev et al., 2006; Sondermann, 2007; Lamberton and Bernard, 2007; Keller-Ressel, 2011; Callegaro et al., 2013).

Levental et al. (2016) take this further and recognise that the underlying processes are progressively measurable³⁵. This is relevant in recognising filtration effects in prices. It can also be expressed in well-defined mathematical terms. However, Levental et al. (2016) do not go far enough in their investigation of the domain. It is the same domain this research is centred on. Moreover they do not provision any changes to existing pricing models. Merely including some form of mathematical representation of effects, does not change the fact that a good part of previously published research work has not led to "realistic" and comprehensive considerations of quantized effects in pricing. Although Callegaro et al. (2017a:2018a) is in a reasonable part an exception.

³⁴ Random variable x in a function $x : \Omega \to R$ measurable with respect to F that is all the event $(x \le c) = (\Omega : x(\Omega) \le c) \le F$ for all real numbers $c \in R$, and the smallest δ -field with respect to which a random variable χ is measurable is the δ -field generated by x, which we denote $\delta(x)$ and the expectation for a non-negative random variable x, is defined as $E(x) = \int x(\Omega) dP$ which may be $+\infty$.

³⁵ Denotes L(V) the set of V-valued progressively measurable processes, and for any $p \in \mathbb{R}$, and further on provides an expression for the probability based L(V): $L_p(V) = \left\{ x \in L(V); E\left[\int_0^T ||x_t||^p\right] < \infty \right\}$, where $||x_t||^2 = x_t' x_t$ (respectively, trace $(x_t' x_t)$).

If the goal is to improve price predictions and develop simple-enough inner zero-object topologies with theoretical alignments across dimensional constrains, then in my view research in asset pricing and contingent claim valuation would need to integrate probable surface curvatures in a concrete and measurable manner. This is particularly relevant in this study as there needs to be clarity in terms of what price or price-related variables are stochastic and what not; should one consider the capital allocation problem to be driven by a stochastic process? In an n-asset portfolio, the weights, although constrained to 1, are stochastic variables. Furthermore, what role does information diffusion play in the capital allocation problem? The true nature of x will depend on many externalities. Previous research provides a good guide, for example Malkiel and Fama (1970), Gragg and Malkiel (1982) as supporters of efficient market hypothesis, argue that daily logarithmic stock price changes follow a random walk. Furthermore they argue that these daily events are independent of each other and move upward or downward in a random manner, and can be approximated by a normal distribution. This is was explored further in works of Stein and Stein (1991), and Malkiel (2003:2011) with focus on stock price distributions with stochastic volatility.

One can also argue that considering Black and Scholes (1973), Merton (1974), Black (1989), Shafer (2002), Terence et al. (2006) in relation to contingent claims with "rights", or more specifically the financial options, the variable measuring the pay-out of such devices is a stochastic variable indeed. More specifically these "devices" with the right to buy $c_t = max(S_t - k, 0)$ or the right to sell $p_t = max(0, k- S_t)$ are augmented random

variables. Such approach would also be implied in cases where the distribution of the underlying is not normal, such as the case of a contingent claim valuation through a binomial pricing framework where underlying follows a binomial distribution (Cox et al., 1979; Jarrow and Rudd, 1983; Leisen and Reimer, 1996).

Stochastic financial variables represent randomizer effects that are attributed to information-diffusion processes, although one would wish to identify the deterministic component of a process of this type, the non-deterministic part will be a source of contribution to the random measure (Bachelier, 1900; Kennedy, 1994; Kallenberg, 2017). Benaim and Raimon (2003) investigated convergence properties of self-interacting diffusion on a compact Riemannian manifold. They considered self-interacting diffusions to be continuous-time stochastic processes living in a Riemannian manifold and defined the process mathematically through the use of a "family" of Brownian motion, smooth vector fields, and a potential-like function³⁶.

For contingent claims with the right to buy or sell the underlying, the difference between the spot price and the exercise price (call options) or the difference between the exercise and the underlying's spot price (put options) is a non-negative random variable. Wealth is created by conditionally executing the contract if actual price rises above (calls) or

³⁶ $dx_t = \sum_a F_a(x_t) \circ dB_t^{\alpha} - \frac{1}{t} \left(\int_0^t \nabla V_{x_s}(x_t) ds \right) dt$, where $(B^{\alpha})_{\alpha}$ is a family of Brownian motions, $(F_{\alpha})_{\alpha}$ is a family of smooth vector fields on M such that $\sum_{\alpha} F_{\alpha}(F_{\alpha}f) = \Delta f$, for $f \in C^{\infty}(M)$, where Δ denotes the Laplacian on M and $V_u(x)$ a potential" function.

falls below (puts) the exercise price. Such conditioned non-negative random variables have interesting properties defined well in existing literature. For example Shephard (1991) provides a good basis of such properties, with detailed argumentations in a well-defined undercutting probability space, the conditional expectation is well defined and measurable for all filtrations, where indicators of events may be known³⁷.

This is supported by other authors (Takesaki, 1972:2001; Harrison and Pliska, 1981; Cragg and Malkiel, 1982; Cox et al., 1985; Karatzas and Shreve, 1998a:1998b; Jarrow and Turnbull, 1998; Shiryaev et al., 2006) who more specifically consider the conditional expectation E(x | g) to only be defined up to sets of probability 0 and for any random walk variable x for which the unconditioned expectation E(x) is defined. By taking $A = \Omega$, one may consider E(E(x | g)) = E(x) to be true for any x for which E(x) is defined. This is also extended in validity in cases of chained conditioning of such expectation, or the conditional form of Jenson's inequality³⁸ for convex functions operational in a real and measurable system, such as pricing and the effects of a bond's term structure.

For a random variable x and g-measurable random variable y for which both E(x) and E(xy) are defined, then E(xy | g) = E(x | g)y. This shows that when y is g-measurable, it may be treated effectively as a constant when conditioning on g and taken outside the

³⁷ that when $g \subseteq F$ is a sub δ -field of \mathcal{F} , and in cases of a non-negative random variable x, the conditional expectation of x given g can be expressed as E(x | g), with x a g-measurable random variable satisfying $E[E(x | g)I_A] = E(xI_A)$ for all events $A \in g$, where I_A is the indicator of the event A; that is, $I_A(\Omega) = 1$ or 0 according as $\Omega \in A$ or $\Omega \notin A$. ³⁸ Jensen's inequality that $f(E(x | g)) \leq E(f(x) | g)$ for a convex function $f : R \rightarrow R$ when f(x) is integrable; the inequality is reversed when f is a concave function.

conditional expectation. In other words, given g, y is known. In the case of two δ -fields g and h, with $g \subseteq h \subseteq F$, then E(E(x | h) | g) = E(x | g). This is often known as the tower property of conditional expectations (Takesaki, 1972:2001; Harrison and Pliska, 1981; Cragg and Malkiel, 1982; Cox et al., 1985; Karatzas and Shreve, 1998a:1998b; Jarrow and Turnbull, 1998; Shiryaev et al., 2006; Lamberton and Bernard, 2007)

According to Takesaki (1972), Snyder and Miller, 1991, Kennedy (1994), Kallenberg (2017), the conditional form of the random variable x is independent of the δ -field g when $\delta(x)$, the δ -generated by x, and h are independent δ -fields; in this case E(x | g) = E(x). The δ -fields g and h are independent when $P(G \cap H) = P(G)P(H)$ for all events $G \in g$ and $H \in h$.

These effects are applicable when pricing structured products (custom investment products of broad use) – both growth and yield enhancement products. In cases where fixed income instruments are combined with options (vanilla, exotic, spreads), there are two or more random variables where at least one variable is event- conditioned (Fabozzi, 1995; Sharpe et al., 1995; Huager, 2001; Strong, 2004). Although properties of random variable and financial products are well researched, I explore and re-apply stochastic concepts within a new "unified" pricing framework. In the same framework, I revisit and upgrade existing models, where possible. There is a gap in literature in terms of what is considered to be "realistic" pricing, such that counts in quantization effects (Chen, 2002; Haven, 2002:2004:2005:2008a:2008b; Khrennikov, 2009; Callegaro et al., 2016). Part

of this study represents an attempt at abstraction and unification of existing models. It is accomplished through a well-formed master expression, leading to different pricing scenarios under specific assumptions.

I am also interested in utilizing existing literature in concepts such as the rate of change of an implied probability surface or implied volatility surface. Existing literature provides a good coverage of the building blocks. Sondermann (2007), Lamberton and Bernard (2007), provide a good basis for a quantifiable probability change effect, especially as I explore diffusion-related probability changes³⁹. In the work of Shiryav et al. (2006), this is implied as a consequence of the Radon–Nikodym theorem that when Q \ll P there exists a non-negative random variable L so that the probability Q may be represented in a welldefined expression and in expectation⁴⁰. Conversely Lamberton and Bernard (2007) argue that one may start with a non-negative random variable L with EL = 1 and define Q by Q(A)= E (LIA) for all events A $\in \mathcal{F}$. It allows for Q (probability) to be absolutely continuous with respect to P and for which L = dQ / dP.

The conditioned properties of random variables can be utilized in alternative ways to derive the Black-Scholes option pricing formula (Black and Scholes, 1973). For example, Nielsen (1993) used the expected payoff expression and properties of conditioned

³⁹ It provides premises to consider the new probability (measure) Q on (Ω, \mathcal{F}) to be dominated by P (or Q is absolutely continuous with respect to P) if for any $A \in \mathcal{F}$, P(A) = 0 implies that Q(A)= 0. In the case when Q is dominated by P we write Q when Q \ll P and P \ll Q then P and Q are said to be equivalent; when Q and Pare equivalent then for an event $A \in F$, P(A) = 0 if and only if Q(A)=0

⁴⁰ Q(A) = E(LIA) for all events $A \in \mathcal{F}$, where $EL = Q(\Omega) = 1$. The random variable L is usually written as L = dQ / dP and it is known as the Radon–Nikodym derivative of Q with respect to P.

probabilities to obtain the discretised Black-Scholes option pricing expression⁴¹. He used an option's discounted expected payoff in a risk-neutral world and an expectation term with respect to a risk-neutral probability measure. I seek to use it on probability surfaces and at price diffusion points (zero-objects) along the market line.

The concept of a random variable's direction (↑↓) is argued well in Banakh (1997), Karatzas and Shreve (1998a:1998b), Shiryaev et al. (2006), and Lamberton and Bernard (2007). However there is lack of literature in terms of properties of a financial quantumworld, unlike the continuous time-space world (©-world), where randomized residual effects are exhibited and financial instruments' prices are observed.

I consider uncountable collections of random values (i.e. price behaviours in a typical case) with a supremum representation in the \mathbb{O}_{++} and \mathbb{O}_{++} -worlds. This is a new concept within the gap in existing literature. While focusing in the \mathbb{O}_{++} -world, I propose and test its internal mechanics and provide the necessary vocabulary to be used. The \mathbb{O}_{++} and \mathbb{O}_{++} constructs are well defined. In the latter case focus is placed in effects of information dissipation and relay to the \mathbb{O}_{++} system price-change density of state, thus representing an essential supremum in onstrained space. Existing literature does not treat such constructs in this way, least so on the internal dynamics of price development within the \mathbb{O}_{++}

⁴¹ ct = e-r(T-t)EQ[max(S_{T-t} - k, 0)]; pt = e-r(T-t)EQ[max(0, -S_{T-t})], where E^Q[] denotes expectation with respect to the risk-neutral probability measure Q. Using the expected payoff formula and conditioned probability attributed, we may express $c = exp(-rT)E^Q[(S_T|S_T > k)] - exp(-rT)E^Q[(k|S_T > k)]$

system, and the quantification of information dissipation by such system with a forecasting level (Haven, 2002; Khrennikov, 2009; Callegaro et al., 2016).

Typically, in modern Finance, information and its relation to pricing is theorised by efficient market hypothesis. Malkiel and Fama (1970), Malkiel (2003), Asquith, (1983), Latham (1986), Bachelier (1990), Bernard and Thomas (1990), Davis and Etheridge (2006) provide a good base for discussion on market efficiency effects with the assumption that a probability triple space event follows a stochastic process.

Therein lies the limit in terms of successful price change predictions or pre-event timehorizon measurements of liquidity in our observable world, hence one anticipates understandings of price change dynamics through the consideration of new $D_{\uparrow,\downarrow}$ -system (Shephard, 1991; Stein and Stein, 1991; Seneta, 1996; Platen, 1997; Luschgy and Pages, 2002; Parzen, 2015). To that end, I re-address existing literature gaps by further considerations on and around the price developments in relation to information filtrations within quantized space topologies.

Following from concepts in the previous paragraphs, most financial variables change over time in a way that cannot be predicted with certainty. Many of the variables in financial markets are also subject to fundamental uncertainty and are given by underlying stochastic processes. One can not be certain about their value in the future. Interest rates, exchange rates, stock prices, and other asset prices are all variables that evolve over time in an unpredictable manner. can make some probabilistic statement about the statistical distribution of their value. Those variables that are affected by uncertainty and change over time in an uncertain manner are classified as random variables, or stochastic process variables (Spitzer, 1970; Snyder and Miller, 1991; Seneta, 1996; Parzen, 2015).

It is on the basis of Makov's property that I consider the stop time and the ©th system, within which information not only is dissipated, but also some degree of information 'tunnelling' occurs. This is something that existing literature does not include in finance or related domains in abundance. The information tunnelling effect is quantified well in literature, related to applied mathematics, but no feasible adaptation of it in finance has been reported.

Harrison and Pliska (1983), Karatzas and Shreve (1998b), Shiryaev et al., (2006), Sondermann (2007), Lamberton and Bernard (2007), among many provide a similar description of a stochastic process in continuous time that takes real values and is a Brownian motion (or a Wiener process). Te set of values calculated as the difference of two time dependent adjacent random values demonstrate a normal distribution with a zero mean and a time dependent variance. They make their assumptions and the values are occurring in real time, and specify the distribution of the displacement of the process between two time points, with the classical statistical concept of the variance referred to as the variance parameter of the process itself; in the finance context the square root of that variance represents the volatility of the process. Lamberton and Bernard (2007) explain that by the scaling property of the normal distribution, we obtain a Brownian motion with modified process variance, a modification that will depend on the a constant parameter, and could be standardized in scaling to obtain a standard Brownian motion with a zero mean and volatility of one. The adaptation of such process (standard Brownian motion) has been of widespread use in Finance with Levental et al. (2016) adapting such processes also in the investigation of optimization problems involving linked recursive preferences in a continuous Brownian motion setting and subsequently to infer that preferences depend on the volatility of wealth realization.

In addition to the scaling property, Sondermann (2007) argues that a Brownian motion dependents not only on the present price value, but also the entire history up to the stopping time (at present), which itself implies a stronger than Markov property. Moreover, the price displacement between two time points is independent of the position at the first point (the first of any two time points), which Jacob and Shiryaev (2003) explain allows us to obtain a well-defined expression for the transition probabilities, the probability density function of the normal distribution, and the joint probability density function⁴².

 $^{^{42}} P(W_{t_n} \le x_n \mid W_{t_i} = x_{i,} \ 0 \le i \le n-1) = P(W_{t_n} - W_{t_{n-1}} \le x_n - x_{n-1}) = \int_{-\infty}^{x_n - x_{n-1}} \phi(u, t_n - t_{n-1}) \, du \,, \quad \text{where}$ $\varphi(x,t) = e^{-x^2/2t}/\sqrt{2\pi t}$ is the probability density function of the normal distribution with mean 0 and variance t, and joint probability density function of $W_{t1, ..., W}$ to as $f(x_1, ..., x_n) = \prod_{i=1}^{n} \varphi(x_i - x_{i-1}, t_i - t_{i-1})$

Lamberton and Bernard (2007) argue that by the property of the normal distribution, it follows that the sum of independent random variables each having a normal distribution again has a normal distribution. It is well reported (Aase, 1988; Miltersen et al., 1997; Sondermann, 2007; Lamberton and Bernard, 2007; Shiryaev et al., 2006; Kallenberg, 2017) that the process is spatially homogeneous so that the distribution of the price displacement or increment between any two time stops is dependent on the position at the first time stop, which is due to the zero convergence of expectation⁴³. For Lamberton and Bernard (2007) the multivariate normal distribution is determined by its means and covariances, and normally-distributed random variables are independent if and only if their covariances are zero and the joint distribution is normal with zero means and covariances are well define mathematically; Jacob and Shiryaev (2003) describe them as the finite-dimensional distributions of the process in good detail.

In accordance with Revuz and Yor (2004), the most important terminal-value claims are the European call and put options with strike price c and expiry time t_0 . These are contracts that entitle (but do not require) the holder to buy/sell one unit of stock at the fixed strike price at the fixed expiry time t_0 ; on the other hand, an American call/put option entitles the holder to buy/sell one unit of stock at the fixed strike price at or before the fixed expiry time and it is not a terminal-value claim. The European call pays $(S_t - c)_+$ at the expiry time, whereas the put pays $(c - S_t)_+$ at the expiry time t; their prices at time t

⁴³Increment $W_{t+s} - W_s$ does not depend on the position, W_s , at time s for s, t > 0. For any t > s > 0, since $EW_s = EW_t = 0$ and $W_t - W_s$ is independent of W_s , it follows that the covariance of W_s and W_t is $Cov(W_s, W_t) = E(W_sW_t) = E[W_s(W_t - W_s + W_s)] = E[W_s(W_t - W_s)] + E(W_s^2) - E(W_s)E(W_t - W_s) + s = s$, and for any s, t > 0 that the covariance is given by Cov $(W_s, W_t) = s \land t$, (14) where $s \land t = min (s, t)$.

are related through what is known as the call-put parity.

2.2 APPLIED STOCHASTIC FINANCE CONCEPTS

A review of literature on various stochastic finance concepts is provided here, starting with attitudes of individual investors in relation to subjective investment decisions and risk, more specifically the notion of an investor's individual utility function.

Shiryaev et al. (2006) argue that in a deterministic model an investor will seek to maximise his or her wealth by making rational investment choices. Under a stochastic model, the investor's final wealth is typically a random variable, w, and it would no longer make sense for the investor to make investment decisions seeking to maximise a random quantity, instead, the investor may wish to maximise the expected value of his or her final wealth, E(x), so that the investor achieves the largest wealth on average, or more generally it is often postulated that the investor will seek to maximise Eu(x) for some appropriate function u; this function is referred to as the investor's utility function (Klinger and Levy (2009). Furthermore Lamberton and Bernard (2007) imply that any investor who orders his or her preferences of random outcomes in a suitably consistent way possesses an essentially unique utility function and that properties of this function may characterise his or her attitude towards risk.

In an *n*-asset portfolio, the capital weights are considered to be in range (zero to one) stochastic variables. A Monte-Carlo approach may be deployed to facilitate mean-variance analysis and the construction of the efficient frontier. Porter and Gaumnitz (1972) articulate a stochastic approach to mean-variance analysis, where, among portfolios giving a fixed mean return, an investor chooses the portfolio with smallest variance of the return. Sondermann (2007) reinforces the argument that the model is subjective both in its choice of optimality criterion, but also in its dependence on the investors' beliefs about the means of the returns of the various available assets as well as the covariances between those returns. Lamberton and Bernard (2007) consider the implications for the whole market of the actions of individual investors within the context of capital-asset pricing equilibrium.

Revuz and Yor (2004) define the term "equilibrium" in a general economic scale as the balance in equal measure between the overall demand and supply. Following from Revuz and Yor (2004) in the context here, the equilibrium in the market portfolio coincides with the tangency portfolio. It allows the pricing equation to be rewritten⁴⁴. Lamberton and Bernard (2007) rewrite the capital asset pricing model⁴⁵, where μ_m denotes the expected return on the market portfolio, and $\beta_m = (\beta_{1,m}, ..., \beta_{s,m})^T$, where $\beta_{i,m}$ represents the market beta for the market portfolio⁴⁶.

⁴⁴ To have supply equal to demand would require that $(x_m)_i = \frac{(\sum_{j \in J} w_j(1-x_{0,j}))(x_t)_i}{\sum_{j \in J} \sum_{k=1}^s w_j x_{k,j}} = \frac{(\sum_{j \in J} w_j(1-x_{0,j}))(x_t)_i}{(\sum_{j \in J} w_j(1-x_{0,j})) \sum_{k=1}^s (x_t)_k}$. Since $x_t^T e = 1$, in equilibrium the market portfolio coincides with the tangency portfolio. 45 $r = r_0 e + (\mu_m - r_0)\beta_m$ 46 $\beta_{i,m} = Cov(R_I, x_m^T R)/Var(x_m^T R)$

Lamberton and Bernard (2007) note that while mean-variance analysis provides a useful framework for thinking about the issues of portfolio choice, its usefulness in applications depends on the availability of good estimates of mean returns of assets and of the covariance between those returns, which may not be easy to obtain. Revuz and Yor (2004) point out that problems of similar nature arise with the capital-asset pricing model, in such cases when it is viewed in a dynamic setting, where changes over time in estimates of parameters in the model from market data may lead to instability in the market beta estimates.

Sondermann (2007) defines a contingent claim, f, as a random variable on an underlying probability space, and regards it as the payoff at time t+h of some contract; the value of f is not observed until time t+h. Shiryaev et al. (2006) elaborate on the archetypal example of a contingent claim, that of a call option at some strike price k on one of the risky assets (for example asset 1), which would pay $(S_{1,1}-k)_+$ at time 1, where contingent claims with a finite second moment are denoted $f = \{f: Ef^2 < \infty\}$. In the simplest form in a time horizon h (t to t+h), it would take two values: $f(\omega_1)$ in the case when ω_1 is the true state of nature (in which case the stock price becomes uS_0 at time t+h) and $f(\omega_2)$ when ω_2 is the governing state of nature (and then the stock price takes the value dS₀ at time t+h). The holder of the claim, who has bought it at time t with no knowledge of which states of nature, ω_1 or ω_2

would be the prevailing one, receives the random payoff f at time t+h.

Shiryaev et al. (2006) follow this from the assumptions that one would wish to establish the amount that an investor would pay at time t to hold the claim f, or equivalently the amount charged by the seller of the claim who will have the liability to pay out the amount f at time t+h. Cox et al. (1979) consider the problem from the point of view of an investor who has sold the claim and wishes to 'hedge' against his or her liability to pay f at time t+h by forming a portfolio at time t consisting of an amount y in the bank account, $-\infty < y < \infty$, and x units of stock, $-\infty < x < \infty$.

It is argued in Cox et al. (1979) that a negative value of y would correspond to borrowing from the bank while a negative value of x corresponds to holding a short position in the stock; that is, |x| units of stock are borrowed at time t (and they must be paid back at time t+h).

According to Lamberton and Bernard (2007), these form the basis of a one time-period (denoted here as h) model. Cox et al. (1979) use the one-time period model as an option pricing building block of a larger binomial pricing framework, which can be used to determine the price of an option (i.e. contingent claim with a 'right' buy or sell the underlying), given the characteristics of the stock or other underlying asset, and under the assumption that the price of the underlying asset can move up or down by a specified

amount, and with the price following a binomial distribution.

In Cox et al. (1979), the one-time period model specifically considers one time horizon (h=1), between t and t+h and the existence of two assets, a riskless asset such a bank account (or treasury bill/government bond) in that 1 unit of wealth at time t held in the bank account becomes (1+r) with certainty at time t+h, where $r \ge 0$ is a constant and may be interpreted as the interest rate on the bank account, and a risky asset, typically stock where St is price at time t, where St > 0 is constant, and its price at time t+h is a random variable St+h.

Hull (2014) discusses a similar model, with assumptions that at S_{t+h} , takes just two possible values uS_t and dS_t where u and d are given constants, the quantities ω_1 and ω_2 represent a probability sample set, interpreted as the two states of nature - two outcomes of the future uncertainty, leading to two states of economy, or discrete market-level movements. The u and d are proportional changes to the stock price; u reflects an 'up' jump of the stock price, and d the 'down' price correction, such that 0 < u > 1 and 0 < d < 1.

Single-step binomial trees for the stock price S, the contingent claim f, and a riskless portfolio can be enstruided. The latter replicates the payoff of the contigent claim and consists of Δ shares of stock and an amount B in a bank account (borrowed or a short

position on a bond). An expression for Δ can be derived⁴⁷. Hull (2014) shows that when combining m shares with n claims, the portfolio becomes riskless⁴⁸. It is referred here as the portfolio (x, y) that replicates (or hedges) the claim f, thus eliminating the exposure to risk on the part of the seller of the claim independent of the prevailing state of nature. At time 1 the portfolio provides exactly the amount required to pay the claim.

Cox et al. (1979) compute the initial cost of setting up such portfolio, $\alpha[q_1f(\omega_1) + q_2f(\omega_2)]$, and determine the minimum price at which the seller would be prepared to sell the claim. It follows from the arguments in Cox et al. (1979) and confirmed among others by Hull (2014) that any amount more than that sum would yield a riskless profit to the seller, and the amount used is the maximum amount that the buyer would be prepared to pay to hold the claim; if the claim was priced at an amount below that in Cox et al. (1979), one could sell the claim from the portfolio and take in the riskless profit. Cox et al. (1979) argue that in a single-step binomial model, the discounted probability average is the "fair" claim. This is important because the model represents the building block of expanded binomial trees, where the price development follows a pattern of pricing regularity.

Cox et al. (1979), Jarrow and Rudd (1983), Leisen and Reimer (1996) provide multi-

 $^{{}^{47}}c_{t+h}(\omega) = \Delta s_{t+h}(\omega) - b(1+r)$, which in the case of the one period binomial becomes the set of two equations: $c_{t+h}(\omega_1) = \Delta s_{t+h}(\omega_1) - b(1+r) \rightarrow b(1+r) = \Delta s_{t+h}(\omega_1) - c_{t+h}(\omega_1),$ $c_{t+h}(\omega_2) = \Delta s_{t+h}(\omega_2) - b(1+r) \rightarrow b(1+r) = \Delta S_{t+h}(\omega_2) - P_{t+h}(\omega_2).$

Subsequently after subtracting the two expressions above we find

 $[\]Delta = \frac{c_{t+h}(\omega_1) - c_{t+h}(\omega_2)}{s_{t+h}(\omega_1) - S_{t+h}(\omega_2)}$

 $^{^{48} \}hspace{0.1in} \pi_{t+h}(\omega_1) \!=\! \pi_{t+h}(\omega_2) \!\equiv\! \pi_{t+h}$

period binomial models where the stock price at time r is represented by $S_r = S_0 \prod_{i=1}^r Z_i$. The random variables (Z_i) are assumed to be independent and identically distributed (iid); much of the discussion of pricing and hedging claims may be extended to the case where both of these assumptions are relaxed in the probability space Ω , as given in earlier considerations, with stock price at time r depending only on ($\omega_1, ..., \omega_r$); that is, $S_r = S_r$ ($\omega_1, ..., \omega_r$).

The proportional change in the stock price between times r and r + 1, for $1 \le r < n$, can be computed⁴⁹; $Z_1(\omega_1) = S_1(\omega_1) / S_0$ where the initial stock price S_0 is a constant with $u_0 = S_1(1) / S_0$ and $d_0 = S_1(0) / S$, u_r and d_r are random variables with values determined at time r, $1 \le r < n$, such that that $u_r > d_r$. Here, u_0 and d_0 will be constants with $u_0 > d_0$. The interest rate on the bank account between times r and r + 1 may be considered to be a random variable $\rho_r = \rho_r(\omega_1, ..., \omega_r)$.

The information available at time r will be $\mathcal{F}_r = \sigma(Z_1, ..., Z)$. It is equivalent to knowing the values of $\omega_1, ..., \omega_r$. The interest rate on the bank account for the period r to r + 1 is then known at time r when one has observed \mathcal{F}_r , so investment in the bank account for that period is riskless; equivalently, if one sets $\alpha_r = 1 / (1 + \rho_r)$, then α_r is the price of a bond bought at time r yielding 1 unit with certainty at time r + 1. Assume that on each

 $^{{}^{49}} Z_{r+1}(\omega_1, ..., \omega_{r+1}) = \frac{S_{r+1}(\omega_1, ..., \omega_{r+1})}{S_r(\omega_1, ..., \omega_r)} \quad \text{and } Z_{r+1} \text{ will be assumed to take just two values } Z_{r+1}(\omega_1, ..., \omega_r, 1) = u_r(\omega_1, ..., \omega_r) \\ \dots, \omega_r) \quad \text{and} \quad Z_{r+1}(\omega_1, ..., \omega_r, 0) = d_r(\omega_1, ..., \omega_r) \quad \text{corresponding to an up jump, } \omega_{r+1} = 1, \text{ and a down jump, } \omega_{r+1} = 0, \\ \text{respectively.}$

branch of the binary tree there is no arbitrage⁵⁰. To specify the underlying probability P on the sample space Ω , I first assume that for $1 \le r < n$, $p_r = p_r(\omega_1, ..., \omega_r)$, with $0 < p_r < 1$, denotes the conditional probability of an "up" jump between r and r + 1 given the outcomes $\omega_1, ..., \omega_r$, with 1 - p_r being the conditional probability of a "down" jump⁵¹ (Cox et al., 1979; Shiryaev et al., 2006; Lamberton and Bernard, 2007).

In accordance with Cox et al. (1979), Jarrow and Rudd (1983), Leisen and Reimer (1996), I consider an economy operating over one period from time t to time t+h. Suppose that there are s risky assets, i = 1,..., s; the prices of these at time 0 are given by a deterministic vector $S_0 = (S_{1, 0}, ..., S_{s, 0})^{\mathsf{T}} \in \mathbb{R}^s$ and the prices at time t+h are determined by a random vector $S_1 = (S_{1, t+h}, ..., S_{s, t+h})^{\mathsf{T}}$ taking values in \mathbb{R}^s .

In addition, there is a riskless asset, 0, which provides a deterministic return $r_1 > 0$ between time t and time t+h; the initial price of the riskless asset may be taken as 1 and here $r_1 - 1$ is the fixed interest rate, with the price of the riskless asset at time 1 being r_1 . Underlying the model is a probability space (Ω , \mathcal{F} , P) on which the random vector S₁ is defined. The set Ω , which represents the set of possible states of nature $\omega \in \Omega$, is equipped with a σ -

⁵¹ that is $P(\omega_{r+1} = 1 | \mathcal{F}_r) = p_r$ and $P(\omega_{r+1} = 0 | \mathcal{F}_r) = 1 - p_r$ a constant p_0 , with $0 < p_0 < 1$, will be the (unconditional) probability of an up jump between times 0 and 1. For $1 \le r < n$, the conditional probability at the r^{th} step is then, $P(\omega_{r+1}|\omega_1, ..., \omega_r) = p_r^{\omega_{r+1}}(1 - p_r)^{1 - \omega_{r+1}}$

field \mathcal{F} of measurable events, or subsets, of Ω and P is a given probability. In this section assume that $E \parallel S_1 \parallel^2 < \infty$, that is $E(S_{i,1})^2 < \infty$ for each i = 1, ..., s (Shiryaev et al., 2006).

Shiryaev et al. (2006) consider a market at times 0, 1, ..., n and s risky assets for which the prices are specified by S₀, S₁,...,S_n. The random vector $S_j = (S_{1, j}, ..., S_{s, j})^T$, which is defined on some underlying probability space Ω , is such that $S_{i,j}$ is the price of asset i (i = 1, ..., s) at time j (j = 0, 1, ..., n). In line with Shiryaev et al. (2006), I consider a market evolving in time with the information available to investors at each time point. Sondermann (2007) treats this mathematically by specifying an expanding sequence of σ -fields $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_n$ in Ω such a sequence is known as a filtration, literature coverage is also provided in the previous section of the literature review.

In Shiryaev et al. (2006), the σ -field F_j represents the information available at time j and intuitively it may be thought of as being specified by a partition of events (subsets) of Ω , and when Ω is finite this is precisely the situation, but for more general Ω the intuition that this provides will be adequate normally. Sondermann (2007) considers that the evolution of the system is governed by the actual state of nature $\omega \in \Omega$, which is not observed, assuming that at time j one has gathered information by observing events that have occurred (or that have not occurred), that itself which allows for narrowing down of the reigning $\omega \in \Omega$. In line with Shiryaev et al. (2006), the collection of events (denoted F_j) is the event set whose occurrence or non-occurrence is known at time j, therefore by having the access to F_j one may infer the actual partition in which ω lies. Shiryaev et al. (2006) argue that at time j + 1, one may have acquired more information than at time j, so that can think of F_{j+1} as being a finer subdivision or partition of Ω , and so forth, which allows us to conceptualize the information within a cascaded structure. Lamberton and Bernard (2007) articulate that an investor observes the prices S_j at time j so the components of the random vector must be random variables which are effectively constant on the events in the partition determining F_j (for, if not, they would be giving further information about the 'true' ω).

Further on, they state that a random variable is an F_j random variable if its value is known after observing F_j and when the sequence $\{X_j\}$ is such that X_j is an F_j random vector, for each j, such that the sequence is adapted to the filtration $\{\mathcal{F}_j\}$. Moreover Revuz and Yor (2004) note that F_j will often be determined by $\{S_0, ..., S_j, R_0, ..., R_{j+1}\}$ (in this situation, formally say F_j is generated by $\{S_0, ..., S_j, R_0, ..., R_{j+1}\}$ and write $\mathcal{F}_j = \partial \{S_0, ..., S_j, R_0,$..., $R_{j+1}\}$, and argue that no restriction need to be placed to that case as investors may be able to observe other random variables (other than the asset prices), which help provide important information about the underlying $\omega \in \Omega$.

Further onto trading strategies and in relative links to concepts reviewed above, a trading strategy or portfolio is said to be self-financing when no money is injected or withdrawn between setting up the strategy and the terminal time according to Lamberton and Bernard (2007), who consider a self-financing trading strategy to replicate the claim when its value

matches that of the claim at the payoff time of the claim, which in the most typified case is the terminal time n and articulate well that for any claim f a replicating trading strategy exists and show how it is calculated.

In such cases and under the same theme, Hull (2014) makes various assumptions in terms of the number of units of stock held between time r and r+1 for the trading strategy by X_r and the holding in the bank account (or riskless bond) from r to r+1 by Y_r , where X_r and Y_r are considered random variables with attributions consistent with those reviewed in the previous section, where the claim dependency on stock price S_r at time r is well established.

Similarly, Cox et al. (1979) determine the appropriate values of these random variables by using dynamic programming, which is the technique of backward induction, to calculate the values (X_r, Y_r) successively for r = n - 1, n - 2, ..., 0, by calculating first the values that the pair (X_{n-1}, Y_{n-1}) should take at all possible nodes of the form (i, n - 1), $0 \le i \le n - 1$, at time n-1, that is for all possible values of the stock price S_{n-1} at time n-1, then move on to calculate the values (X_{n-2}, Y_{n-2}) , and so forth, as the step backward induction goes through the binary tree.

Revuz and Yor (2004), Shiryaev et al. (2006), model a trading strategy $T = ((X_0, Y_0), (X_1, Y_1), ..., (X_n, Y_n))$, as an adapted sequence of random vectors $X = \{X_j\}$ and an adapted

sequence of random variables $Y = \{Y_j\}$; here $X_j = (X_{1,j}, ..., X_{s,j})^T$, with $X_{i,j}$ representing the amount of asset i and Y_j the amount of asset 0 held from time j to time j + 1 using the strategy T, under the assumption that $X_n \equiv 0$ and $Y_n \equiv 0$, so that the model terminates at time n. The requirement that the strategy is adapted means that an investor may wait until he or she has observed the prices S_j and R_{j+1} at time j before assembling the portfolio to hold for the period from j to j + 1 and onto its application of the trading strategy T = (X,Y)and its dividend sequence realisation⁵². The discount price process is given by $\{B_jS_j, \mathcal{F}_j :$ $0 \le j \le n\}$, where j = 1, ..., n are time nodes, with non occurring one–period arbitrage, and an equivalent probability Q such that the discounted price process⁵³ is a martingale under Q.

This is consistent with both continuous and discrete underlying process. Under a continuous process, investor will be selective in his or her choice of observation times and the frequency (preferable at some regularity). In discretised models, the observations match the time nodes (at possible stoppage times), but the consideration of the filtration and its effect on the price follows a similar pattern; it provides a clear framework in the terms of pricing under the effects of filtration, although no insight in terms of prediction on the future price is included (Cox et al., 1979; Jarrow and Rudd, 1983; Leisen and Reimer, 1996; Shiryaev et al., 2006; Klinger and Levy, 2009).

⁵² $D^T = (D_1^T, ..., D_n^T)$, given by $D_j^T = (X_{j-1} - X_j)^T S_j + (Y_{j-1} - Y_j) R_j$ for j = 1, ..., n. The F_j-random variable D_j^T is the amount 'consumed' at time j using the strategy T and it is the difference between the amount the portfolio is worth at time j through investing from time j - 1 and the amount re-invested to be carried over to time j + 1. ⁵³ that is $E_Q \parallel B_j S_j \parallel < \infty$ and $B_{j-1} S_{j-1} = E_Q$ ($B_j S_j \mid \mathcal{F}_{j-1}$) for $1 \le j \le n$

Independent of the nature of the underlying process, literature does not provide clear provision of the underworkings of the filtrations within the market "medium", nor does it provide a system within which one may try to investigate the dissipation effects of information and provide, at least in part, better price change prediction to at least the nearest future time point.

An understanding of dimensionality in important to understand better the underworkings of information at each dissipation point along the market line, one of which (and a very important one) is time. I consider here a "black-box medium" where events are vectorised along a pattern of uncertainty that subsequently affect price change projections. It triggers market line residual (stochastic) deviations in prediction. In both continuous and discrete processes, market price observations at well-articulated intervals, and the sense of betweenness and equidistance in unobserved intervals in my logic follow Halmos (1957), Tarski and Givant (1987).

In the previous consideration of a simple single-step stock price development at time t+h (one time horizon h=1), the stock price at time t+h will depend on the state of nature. In a generic model, the elemental condition is 0 < d < 1 < u, however Jarrow and Rudd (1983), referred to as JR here, provide a traceable set of the price multipliers as well as a well justified set of probabilities. The share price multipliers u and d, are chosen such that the mean and variance over extended number of steps match those required for share

price returns⁵⁴. The probabilities are said to be equiprobable.

The share price process in the JR tree superimposes a risk-neutral drift term and a second term based on volatility. The drift term ensures that the share has an expected rate of return of (r-q) when valued in a risk-neutral world. The expected value of the price multiplier is $\exp[(r-q)\delta t]$. Since the annual volatility measure σ is a standard deviation, the volatility term $\sigma\sqrt{\delta t}$ incorporates the square root of the step size (Jarrow and Rudd, 1983).

Variance at an individual binomial step is given by $\sigma^2 \delta t$. After n steps or over time T, the variance of log share price becomes $\sigma^2 T$. Cox et al. (1979) underlines that dividends will affect only the probabilities in the CRR model, not the share price values, whereas in the JR tree the opposite is true. In addition, Cox et al. (1979) serve a generalised option pricing expression, similar in "shape" to the Black-Scholes model, where the continuous normal distribution functions, N(d), in the Black – Scholes result is replaced by discrete binomial distribution functions (Black and Scholes, 1989; Jarrow and Rudd, 1983).

These binomial approximations hold for European options. They involve the so called "complimentary" binomial distribution function Φ , which is one minus the distribution

⁵⁴ Expressed algebraically, the JR tree parameters are: $\ln(u) = \left(r - q - \frac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t}$, $\ln(d) = \left(r - q - \frac{1}{2}\sigma^2\right)\delta t - \sigma\sqrt{\delta t}$

function. Each of the N(d) terms in the Black-Scholes formula can be replaced with a term based on the complimentary binomial distribution function Φ . The quantity a represents the minimum number of "up" moves required for the call to end "in-themoney", that is for the terminal value to exceed X. The other new parameter, p* is a modified probability⁵⁵. Cox et al. (1979) provide a concise CRR binomial option pricing formula where distribution functions are generally defined in terms of the probabilities in the left-hand tail of the distribution, whereas the complimentary distribution function refers to the right-hand tail⁵⁶.

The third set of parameters is that proposed by Leisen and Reimer (1996). Their choice has two important advantages over JR and CRR parameters. First, they suggest better and separate estimates for the $N(d_1)$ and $N(d_2)$ values in the Black-Scholes formula. Then by centring the share price tree at maturity around the exercise price, the oscillation in convergence seen with JR and CRR trees is removed. In the LR model, the parameters are chosen in reverse order to the JR and CRR models. Probabilities are decided first and then the share price moves. Expressions for probabilities are derived using an inversion formula that provides accurate binomial estimates for the normal distribution function. Probability p relates to d_2 , whereas probability p^* relates to d_1 found in option pricing expressions by Black and Scholes (1973), Nielsen (1993), Shafer (2002).

⁵⁵ given by the formula: $p' = p \frac{u}{e^{(r-q)\delta t}}$ ⁵⁶ $c = se^{-q(t^*-t)}\Phi(a:n,p') - ke^{-r(t^*-t)}N(\Phi(a:n,p))$

The accuracy of the term equivalent to $N(d_2)$ (i.e. the $\Phi(a:n,p)$) can be seen by comparing it to Black-Scholes $N(d_2)$. Note that parameter *a* takes the value (n+1)/2 in the LR tree to ensure that the share price tree is centred around the exercise price. Similar to the two models reviewed above, Leisen and Reimer (1996) describe their own price multipliers for the share price moves in the binomial tree⁵⁷.

The option pricing model published by Black and Scholes (1973) has gained popularity and use over the years, despite initial resistance to accept it. It is now the most widely used option valuation model, typified in the pricing of European options, but agile enough to respond to adaptations for other types of financial derivatives and with various types of underlying securities.

The Black-Scholes option pricing model takes its name from the two financial theorists that came up with it in the early 70's. Both Black and Scholes recognised that their work on option pricing was preceded by research and publications in derivatives pricing, especially in the 60's, with initial focus on warrants. The works of Sprenckle (1961), Ayres (1963), Bones (1964), Samuelson (1965), Baumol et al. (1966), Chen (1970), all produced valuation formulas in a similar general and incomplete form. Their models included one or more arbitrary parameters. Particularly the Sprenckle's derivative pricing

 $^{57} u = e^{(r-q)\delta t} \frac{p^{'}}{p}; \ d = e^{(r-q)\delta t} \frac{1-p^{'}}{1-p}$

expression appears to be close to the Black and Scholes option formula (Black, 1989).

I treat the Black-Scholes option pricing model here as a special case under the new abstracted and unified financial instrument valuation framework. In first appearance, the differences between Black and Scholes (1973) option pricing model and earlier Sprenckle's (1961) formula⁵⁸ appear to be minor. They are similar in their general mathematical form and the underlying stochastic process. In Sprenckle (1961), x is defined to be the stock price, c the exercise price, t* the maturity date, t the current date, v² the variance rate of the return on the stock, *ln* the natural logarithm, N(b) the cumulative normal density function. However in the model, k and k* are unknown parameters or parameters that were not fully justified theoretically or empirically. Although Sprenkle (1961) defined k as the ratio of the expected value of the stock price at the time the warrant matures to the current stock price, and k* as a discount factor that depends on the risk of the stock. Furthermore Sprenkle (1961) tried to estimate the values of k and k* empirically to no avail (Black, 1989).

Samuelson (1965) published his pricing model with parameters α and β , where α is the

 ${}^{58}kxN(d_1) - k^*cN(b_2); b_1 = \frac{lnkx/c + \frac{1}{2}v^2(t^* - t)}{v\sqrt{(t^* - t)}} \ ; \ b_2 = \frac{lnkx/c - \frac{1}{2}v^2(t^* - t)}{v\sqrt{(t^* - t)}}$

rate of expected return on the stock, and β is the rate of expected return on the warrant or the discount rate to be applied to the warrant. Samuelson (1965) makes the assumption that a log-normal distribution prevails for possible stock prices at warrant's maturity and takes the expected value of this distribution, with a cut-off at the exercise price, which is then discounted at present value, under the discounting rate β . The pricing process in Samuelson (1965) is not supported by literature, nor justified under any implied conditions of capital market equilibrium for security pricing. It was not regarded to be an appropriate procedure to price a warrant either, nor could empirical testing justify it. In their subsequent paper, Samuelson and Merton (1969) recognised the fact that discounting the expected value of the distribution of possible values of the warrant when it is exercised is not an appropriate procedure and advanced the theory by treating the option price as a function of the stock price, which is fully justified.

Furthermore Samuelson and Merton (1969) recognised that the discount rates are determined in part by the requirement that investors would be willing to hold all of the outstanding amounts of both the stock and the option. Their view on this is limited because they do not make use of the fact that investors must hold other assets as well or that investors are mean-variance efficient with no justification of why they should take idiosyncratic risk, and the risk of an option or stock that affects its discount rate is only that part of the risk that cannot be diversified away. Their pricing model depends on the general form of the utility function assumed by them for a typical investor.

Others had tried to come up with a warrant pricing model, using an empirical testing and

fitting procedure. Thorp and Kassouf (1967) obtained an empirical valuation formula for warrants by fitting a curve to actual warrant prices, a concept that was taken into consideration by Black and Scholes (1973) in their work. Furthermore Thorp and Kassouf (1967) use the empirically fitted formula to calculate the ratio of shares of stock to options needed to create a hedged position by holding a long on one security and short on the other.

However, Thorp and Kassouf (1967) failed to pick up on the fact that in equilibrium such hedged position would lead to a riskless portfolio with an expected return matching that of a riskless asset, something that Black and Scholes (1973) fully utilised in their option pricing model. It is exactly this equilibrium condition that can be used to derive a theoretical option pricing formula. Black and Scholes (1973) initial purpose was to come up with an option pricing formula that could be used to price European options with no probability of an early exercise, recognise sensitivities of the option price on the stock price, exercise price, stock volatility, and interest rate, although the dividend yield sensitivity was factored in the initial model by Merton (1974), Rubinstein (1976), and Cox et al. (1979).

Merton (1974) had considered the case of a constant dividend yield without success in finding a closed form solution. Geske (1978) extended the option pricing model to include the stochastic effect of the dividend yield on stock, which he based on assumption of a log-normal dividend yield as a reasonable stochastic process describing the uncertainty around dividends, which seems to address in part successfully the under-

pricing effects for deep-out-money options and prevalent for longer exercise periods for options and warrants.

Merton (1974) did not change the initial Black and Scholes (1973) essential argument on the expected correlation between a traded underlying stock and the derivative contract hedging a trading position by going long on the stock and short on the derivative by a well-defined volume of trade, thus creating a riskless trading portfolio. He recognised that in order to maintain an effective and continuous hedged position in the trade, one would need to maintain changing the position on the underlying, the option, or both. Masoliver and Perello articulate this well in their 2003 work. Merton (1974) did not find a closedform solution to the constant dividend for options with a finite life. He did however find a reasonable solution for a perpetual option when there is no probability of early exercise, which seems to align well to the initial model by Black and Scholes (1973).

Once a riskless portfolio is set up, Black and Scholes' (1973) intuition drove the process forward, in that such portfolio can only yield a rate of return equivalent to that of a riskless investment such as T-Bills, LIBOR rate, etc. Further on Black (1989) explain that one can obtain the same option pricing formula by considering a reverse hedge basis, or even a neutral spread position, under the assumption that a neutral spread must earn the interest rate, that is when shorting one option and longing another option on the same stock in the right ratio, one obtains a neutral spread. Furthermore this is plausible even for a spread where you take in money, because one may be in a position to invest the proceeds of a sale of options for our own benefit. Black (1989) in the paper 'How we came up with the formula' further argue that the option pricing formula can be obtained without assuming any hedging or spreading at all.

Black and Scholes option pricing model is based on a broad list of initial assumptions, such as that (i) the short term interest rate is known and constant at all times, (ii) stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price; thus the distribution of possible stock prices at the end of the finite interval is log-normal, the variance on stock is constant or homoscedastic, (iii) stock pays no dividend, or other distributions, (iv) the option is "European", that is it can only be exercised at maturity, (v) there are no transaction costs in buying or selling the stock or the option, (vi) it is possible to borrow any fraction of the price of a security to buy it or to hold it at the short term interest rate, (vii) there are no penalties to short selling; a seller that does not own a security will simply accept the price of the security from a buy, and will agree to settle with the buyer on some future date by passing him an amount equal to the price of the security on that date (Black and Scholes, 1973; Black, 1976:1989).

Black (1989) reported that not all assumptions were necessary and the model itself failed empirical testing on possible matches to the option market prices, but subsequent testing showed its validity in the inverted manner, using the Black and Scholes formula with market prices for the option to compute the implied volatility. Merton (1974) contributed the dividend effect to the initial Black and Scholes option pricing model, thus updating it to what is known as the Black, Scholes and Merton formula, still flowing from the same assumption that log returns of the share underlying the option are normally distributed with the share price assumed to follow a stochastic process with a multiplicative sequence of moves of variable size or, more so, a process known as geometric Brownian motion, with the specificity of the Ito process and subsequently the use of the Ito Lemma in the derivation of the Black and Scholes differential equation (Black and Scholes, 1973).

Itó (1951) and Goldstein (1969) cover well the broad basis of the Itó process. The well known Black and Scholes option pricing formula was derived on the basis of the Itó process and Itó Lemma (Black and Scholes, 1973). Curtain and Falb (1970) had previously extended the Itó lemma to a Hilbert space context for real random variables. It allowed for the formulation of a juxtaposition of the Weiner process and stochastic differential processes in a Hilbert space. This was then extended with focus to local behaviour of Hilbert space (Hilbert and Ackermann, 1950; Kotelenez and Curtain, 1982; Aerts and Gabora, 2005).

Without deriving the Black-Scholes formula here, the values for a European call option, c, and a European put, p, on a share that pays dividends is given through well defined and applicable expressions. The call price c (s, k, r, q, t, σ) dependents on the current share
price s, the exercise price k, the continuously compounded risk-free interest rate of return, stock's dividend yield q, option life t, and the stock price volatility. The expression uses exp(-rT) to measure the risk-free discount factor over period T. It further makes use of state of nature quantities N (d₁), and N (d₂). The notation N (d) is used to denote the cumulative standard probability distribution for value d. Here d₁ and d₂ are well defined expressions. The expression of the put can also be obtained, alternatively, by using the put-call parity formula. This is achieved by linking the value of the put and call options on the same share with current value s, exercise price k, and time to maturity t (Black and Scholes, 1973; Black, 1989).

It represents a self-replicating portfolio (well-formed long position on stock and borrowing), with further characterisation of the distribution through its moments about the mean of a normal distribution or equivalently through its moments about zero of a log normal distribution. Moreover it recognises the implications of normal returns (log-normal distribution, moments with M_1 , M_2) versus lognormal returns (normal distribution with moments M, V) in the option pricing formula (Black and Scholes, 1973; Geske and Roll, 1984; Hull and White, 1987; Hull, 2014).

The first moment of any distribution is its mean (denoted M_1 or M), while the second moment (about the mean) is the variance, V as opposed to the second moment about zero (Lipster and Shiryaev, 2000; Shiryaev et al., 2006), which allows one to instrument properly the information on moments in using correctly either the normal Black-Scholes or the log-normal Black-Scholes option pricing formula as described by many contemporary authors, (Hull and White, 1990; Batten et al., 2004; Blümke, 2009; Hull, 2014). The Black-Scholes discretised option pricing formula may be derived using alternative ways. Nielsen (1993) used the conditional probability concept, coupled with the payoff at expiration time T of an option to derive an expression that matches the Black-Scholes (1973).

2.3 *RELEVANT* QUANTUM FINANCE MODELS: Q-MARKETS AND PRICE DISTRIBUTIONS

This section includes a review of relevant literature with focus on effects of quantization in markets and asset pricing.

Chen (2001) proposes a non-arbitrage quantum model for a binomial financial market. He argues that its risk-neutral world exhibits a disk-like structure in the unit ball of R3 whose radius is a function of the risk-free interest rate with two states. He proves that the Cox et al. (1979) binomial option pricing expression can be re-established by considering Maxwell-Boltzmann statistics of the system of N distinguishable particles. Meanwhile, Haven (2002) considers the Black–Scholes option pricing model (Black and Scholes, 1973), within a quantum physical context. The option price is considered to be a state function, which he then uses to establish a "potential" function. Thus allowing for the option price to satisfy the Schrodinger's partial differential equation (PDE). Haven (2002) interprets the "potential" function as an arbitrage lever and argues that when the arbitrage is established, the existence of a "Financial" state may be determined. He favours the existence of an arbitrage-free price when the "potential" function converges to one. The existence of arbitrage hinges on the non-zero value of the Planck constant. He then links this constant to a parameter which regulates the probability of occurrence of strategy paths, brands it the "belief" parameter, and interprets it as a proxy arbitrage. Haven's (2002) work moves from the Black–Scholes option pricing model (Black and Scholes, 1973) to a quantum version of it, thus providing the first step in the inclusion of arbitrage in an otherwise arbitrage free model.

Meanwhile, Baaquie et al (2002) use quantum theory techniques with emphasis on the connection between quantum mechanics and quantum field theory to illustrate some of the methods of lattice simulations of path integrals for option pricing. The ideas are sketched out for simple option pricing models, such as the Black-Scholes (1973) model, where analytical and numerical results are then compared. Applications of the method to nonlinear systems are also briefly overviewed in Baaquie et al (2002). More general models, for exotic or path-dependent options are discussed in the same work. These models are partially based on the work of Rubinstein and Reiner (1995).

In quantum pricing models, stock is assumed to behave according to Maxwell-Boltzmann classical statistics (Meyer, 1999; Chen, 2001). In his 2003 research work, Chen

considered a many-particle system satisfying Bose-Einstein statistics as a model of the multi-period binomial markets, yielding an alternative binomial option pricing expression. By replacing Maxwell–Boltzmann statistics with quantum Bose–Einstein statistics, he was able to produce prices that are different from those obtained by the Cox-Ross-Rubinstein option pricing model (Cox et al., 1979). This is because he treated stock like a quantum boson particle instead of a classical particle.

In his 2002 and 2003 work, Haven argues that the quantum physical Hamiltonian may be derived from the classical Hamiltonian, through the use of an operator. He makes the case that once a quantum physical Hamiltonian operator is found, the time-dependent Schrodinger can then be obtained. Haven consideres the original physical parameters present in the Schrodinger PDE, such as mass, complex number, Planck's coefficient, and potential function. He then provides interpretations within a financial context for these parameters. He reproduces the equation in its short form, using the definition of the Hamiltonian as the sum of potential and kinetic energy.

Haven (2002:2003) is of the view that arbitrage is ill-modelled in Finance. He proposes the arbitrage potential to address explicit mispricing and subsequently capture arbitrage. It is this arbitrage potential that allowes Haven to define the existence of a probability amplitude function, and then solve the Schrodinger equation. He further addresses mispricing modalities, and finds that the arbitrage parameter is a call option with hedging parameter of '1' or else a fully hedged call option. He discusses the arbitrage opportunities in both cases and comes to conclusion that in the case of a fully hedged call option, the Planck constant for no-arbitrage must be non-zero.

Haven (2002:2003) provides an intuitive discussion on the Financial aspects of the Planck constant too. In essence he looks at the difference between the potential and energy as a potential arbitrage proxy in a financial context. The issue of establishing a relationship between the number of paths and the value of the Planck constant is of key importance, since multiple paths, when those paths are strategies, must indicate there is limited rationality (Abramowitz and Stegun, 1972; Glimm and Jaffe, 1981; Baaquie, 2004).

In his 2003 paper "A Black-Scholes-Schrödinger option price: bit versus qubit", Haven re-focused on the Black-Scholes differential equation in order to further explore pricing of financial derivatives. He argues that the uncertainty environment of an option price can be described by the classical "bit": a system with two possible states. He introduces an uncertainty environment, characterised by a "qubit" (quantum "bit"), to obtain an information-based option price. He then discusses the differences between this option price and the classical option price obtained through the Black and Scholes (1973) model.

Haven (2002:2003:2004) work utilises the Brownian motion as the process on which the Black-Scholes (1973) option pricing model is based on. A detailed coverage of the Brownian motion can be found in Nelson (1967), Karatzas and Shreve (1998b), Revuz and Yor (2004), Shreve (2004b), etc. In Finance literature there exists a close connection between the binomial option pricing model and the Black-Scholes model (Cox et al., 1979; Nielsen, 1993; Hull, 2014). The Black-Scholes (1973) model can be derived from the binomial model when the number of binomial steps is increased sufficiently to allow for full convergence of the binomial distribution to a normal distribution.

In binomial models (Cox et al., 1979; Jarrow and Rudd, 1983), Leisen and Reimer, 996), stock price can take on two different positions at each time step. In analogy with information theory (Cover et al. 1989; Mackay, 2003), the binomial model represents a bit: a system with two possible states (at each time step). The "bit" notion is also implicitly used in the so called "Arrow-Debreu" paradigm, where future payments are a function of both time and the states of the world (Föllmer and Sondermann, 1986; Föllmer and Schweizer, 1990).

According to William Sharpe, a well recognised finance academic, this paradigm is part of what he brands 'nuclear financial economics'. The "qubit" model is the most uncertain, since the information carried per price step in the next period of time is quite less than in the "bit" model. The Black-Scholes portfolio is not even risk free in that case (Sharpe , 1964; Varian, 1993; Sharpe et al., 1995).

Baaquie et al. (2003) considered a broader focus with pricing of options, warrants and other derivative securities. He articulates that these financial products can be modeled and simulated using quantum mechanical instruments based on a Hamiltonian formulation. This paved the way for inter-disciplinary research, such as the work of Vidya and Shivakumar (2007), who constructed quantum algorithms in order to speed up solutions of Hamiltonian cycle problems. Baaquie et al. (2003) also demonstrate some applications of these methods for various potentials, via lattice Langevin and Monte Carlo algorithms. on barrier or path dependent options, showing in some detail the computational strategies involved. Further work of relevance on Monte-Carlo simulations within this context can also be found on Gilks et al. (1996), Robert and Casella (2004), Pages and Printems (2005), Rebentrost et al. (2018).

Haven (2004) proposed an interpretation of the wave-equivalent of the Black–Scholes option price. He considered Nelson's version of the Brownian motion (Nelson, 1967) and uses this specific motion as an input to produce a Black–Scholes PDE with a risk premium. Previously, Cox et al. (1985) in their seminar paper had also demonstrated that the price of any contingent claim satisfies a particular PDE in the Black–Scholes world⁵⁹. Haven (2004) provides argumentation as to why pilot-wave theory could be of use in financial economics through the adaptation of the notion of information wave. This allows for the introduction of a stochastic guidance equation where part of the "drift" term in that equation refers to the phase of the wave. He argues that in order to embed information in financial option pricing, one may use such a drift.

 $^{}_{59} \quad \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$

Baaquie et al. (2003) believes that his path integral formulation will turn out to be very effective for the simulation of complex portfolios, as well as for the inclusion of constraints in the price evolution of derivatives. Haven (2005) on the other hand shows that that if a portfolio of a financial option and a stock is placed in an environment where the value of an asset (besides its price) is formalised as a superposition of price states, such portfolio may not be risk free and fuzzy preferences for risk premiums may exist. He argues for a modification of the classical Brownian motion process (Nelson, 1967) in order to price financial options. He further argues that such modification can be shown to have a connection with the quantum physical Schrödinger equation.

Piotrowski et al. (2005) proposed an option pricing model based on the Ornstein-Uhlenbeck process. It represents a new way of looking at the Black-Scholes formula from the foundations of quantum game theory. They articulate the differences between a classical model, where price evolution is driven by a Wiener process, and price development supported by a quantum model. They argue that these differences are visible for very liquid financial instruments. Furthermore, they provide an alternative description of the time evolution of market price, making corrections in the Wiener-Bachelier model, which follows itself from the Ornstein-Uhlenback process (Gillespie, 1996; Lamberton and Lapevre, 2008; Kennedy, 2010; Parzen, 2015). This process has been explored previously by Vasicek (1977) for modeling short time interest rates. There has been an increase in the use of game theory with the inclusion of quantum formalism. This is particularly exemplified in Piotrowski and Sładkowski (2004), based on earlier work by Eiser et al. (1999). It has qualitatively broaden the capabilities of this discipline describing the strategy which can not be realised in classical models. Game theory (von Neumann and Morgenstern, 1947; Rasmusen, 1989; Anjan, 1991), describes conflict scenarios between a number of individuals or groups who try to maximise their own profit, or minimise profits made by their opponents. However, by adopting quantum trading strategies it seems that players can make more sophisticated decisions, which may lead to better profit opportunities.

The success of quantum information theory (quantum algorithm or quantum cryptography) could make these futuristic-sounding quantum trading systems a reality, due to quantum computer development it will be possible to better model the market and price derivative instruments with relative ease (Preskill, 1988; Baaquie et. al., 2002; Piotrowski and Sładkowski, 2004).

Rebentrost et al. (2018) argue that existing algorithms allow quantum computers to price financial derivatives with a square root advantage over classical methods. It marks a shift from using quantum mechanics to gain insight into computational finance, to using quantum systems - quantum computers, to perform those calculations. Finance community is always looking for ways to overcome the performance issues that arise when pricing options. This has led to research on alternative financial computing techniques such as quantum computing (Deutsch and Jozsa, 1992; Cleve et al., 1998;

Chen et al., 2000; Raj and Shivakumar, 2007).

Quantum computers have shown to outperform classical computers when it comes to simulating quantum mechanics (Boghosian and Washington,, 1998), as well as several other algorithms such as Shor's algorithm for factorisation and Grover's algorithm for quantum search, making them an attractive area to research for solving computational finance problems (Deutsch and Jozsa, 1992; Shor, 1997; Grover, 1996; Cleve et al., 1998; Chen et al., 2000; Raj and Shivakumar, 2007; Nielsen and Chuang, 2010).

Accardi and Boukas (2007) were motivated by the work of Segal and Segal (1998) on the Black-Scholes pricing formula in the quantum context. They studied a quantum extension of the Black-Scholes equation within the context of Hudson-Parthasarathy quantum stochastic calculus (Parthasarathy, 1992). They use a quantum Brownian motion and a Poisson process to describe stock markets.

Pricing options quickly and accurately is a well known problem in finance. Quantum computing is being researched with the hope that quantum computers will be able to price options more efficiently than classical computers. Research work of Meyer (2009) extends the quantum binomial option pricing model proposed by Chen (2001:2003:2004) to European put options and to Barrier options and develops a quantum algorithm to price them. It produced three key results. First, when Maxwell-Boltzmann statistics are assumed, the quantum binomial model option prices are equivalent to the classical

binomial model. Second, options can be priced efficiently on a quantum computer after the circuit has been built. The time complexity falls under the quantum computational complexity class. Finally, challenges extending the quantum binomial model to American, Asian and Bermudan options still persist as the quantum binomial model does not take early exercise into account.

The quantum probability theory was developed to address the paradoxical findings from classical theory that can not be explained. Recent findings in cognitive psychology have revealed that quantum probability can describe human decisions in an elegant way. Moreira and Wichert (2014) explain that human thoughts are seen as superposed waves that can interfere with each other, influencing decisions, whenever a decision is to be made. Bianchi (2013a:2013b) argues that most of quantum mechanics' predictions are irreducibly statistical, therefore quantum mechanics must itself be a probabilistic theory.

The first attempts to clarify quantum mchanics' content made use of the concept of statistical ensembles, describing identical abstract copies of the system under consideration, each of which would represent a different state in which the system might be found to be in. This statistical ensemble interpretation of quantum systems was originally put forward by Albert Einstein (Einstein, 1958), and subsequently supported by others, such as Ballentine (1970).

Pearl (1988) argues that Bayesian Networks are structures that integrate data from

multiple sources of evidence and enable the generation of a coherent interpretation of that data through a reasoning process. The fusion of all these multiple data sources can be done using Bayes theorem. When a data source is unknown, then the Bayes rule is extended in order to sum out all possible values of the probability distribution representing the unknown data source. Quantum Bayesian Network takes advantage of these uncertainties by representing them in a superposition state (Tucci, 1995; Gutiérrez and Hadi, 1997; Gal, 2007; Fenton and Neil, 2007).

Their results in Moreira and Wichert (2014:2016) revealed that the quantum-like Bayesian Network can affect drastically the probabilistic inferences, specially when the levels of uncertainty of the network are very high. The proposed quantum-like network collapses to its classical counterpart under low levels of uncertainty. This validates partially the work on Quantum Bayesian networks introduced in Tucci (1995).

The overall results of Moreira and Wichert, (2014) suggest that when the classical probability of some variable is already high, then the quantum probability tends to increase it even more. When the classical probability is very low, then the proposed model tends to lower it. When there are many unobserved nodes in the network then the levels of uncertainty are very high. But, in the opposite scenario, when there are very few unobserved nodes, then the proposed quantum model tends to collapse into its classical counterpart, since the uncertainty levels are very low.

Moreira and Wichert (2014) were motivated by preliminary experiments of Tversky and Kahneman (1992) about violations of the classical probability theory on the sure-thing principle. This principle states that if one chooses action A over B in a state of the world X, and if one also chooses action A over B under the complementary state of the world X, then one should always choose action A over B, even when the state of the world in unspecified. When humans need to make decisions under risk, several heuristics are used, since humans cannot process large amounts of data. These decisions coupled with heuristics lead to violations on the law of total probability.

Research in quantum probability theory has revealed new techniques of computing probabilities by expanding existing classical probabilities through the inclusion of an interference effect. Such effect is linked to variable beliefs that are constantly updated, when making a decision. The classical probability theory assumes that all beliefs have a definitive value prior to a decision is made, and this value is the outcome of the decision. Therefore, quantum probability theory includes the classical probabilities as a special case when the interference term is zero. This is particularly useful in modeling cognitive systems of decision making (Hoerrniuo, 1963; Khrennikov and Haven, 2009; Aerts et al., 2010; Khrennikov, 2012; Haven and Khrennikov, 2013; Moreira and Wichert, 2014:2016; Haven and Khrennikov, 2016).

There are two ongoing areas of research that are particularly of interest in relation to this study; (i) the development of a better stochastic model describing the main features encountered in empirical analysis with the non-Gaussian shape of price returns PDF as

one of common themes, (ii) the development of a theoretical model that is able to encompass the essential features of real financial system which is characterised by such PDF. Existing research combines a statistical-mechanical description with a quantum mechanical representation as a way to construct a financial quantum theory. This is often done through a direct postulation of Schrödinger's equation with pre-set boundry conditions (Nastasiuk's, 2015; Moreira and Wichert, 2016; Haven and Khrennikov, 2016).

Nastasiuk (2015) argues that it is possible to derive a comprehensive quantum mechanical framework by extermizing Fisher information. It can be applied to finance with a statistical characterisation of financial markets through the inclusion of PDF evaluation., where the probability distribution function (PDF) for financial market prices can be obtained. Nastasiuk (2015) uses a common Fisher-technique to replace the system entropy. This leads to a quantum-like description of financial markets where different models map out to quantum mechanical equivalents. While the maximum entropy problem has a solution of a fixed exponential form, the minimizing (extremizing) Fisher information function leads to a second order differential equation of Schrödinger type, whose solutions exhibit a variety of mathematical forms with the flexibility of varying the potential function.

Arbitrage is a very important concept in the theory of asset pricing and is key in financial decision making and behavioural economics. The presence of arbitrage possibilities has an observable effect on the psychology of financial market agents. It implies that when

taking a trading position in an asset which entails no financial risk, a positive financial return can be realised, which is in excess of the risk free rate of interest. One may argue that, after disregarding the cost that may incur for finding arbitrage opportunities, an arbitrage opportunity is akin to earning what is often referred in common parlance as a "free lunch" (Ross, 1976; Harrison and Kreps, 1979; Ross and Roll, 1980; Dalang et al., 1990; Delbaen and Schachermayer, 2006; Haven and Khrennikov, 2016).

Haven and Khrennikov (2016) make good use of Fourier transforms to introduce quantum-like ideas in economics. They argue that superposed quantum values should match the unobserved, agent based prices. The collapse of a wave function could then yield the per-capita based price. They articulate that arbitrage/non arbitrage can be well defined within a quantum-like paradigm and briefly theorise on its behavioural dimension. Callegaro et al. (2018a) modelled Fourier transforms of a pure jump process at a given time to obtain price quantization through a multinomial lattice discretization scheme.

Haven and Khrennikov (2016) characterise arbitrage by a curvature measure derived from the theory of "fibre bundles", explained in good detail in Steenrod (1951). They argue that the curvature parameter can be entered into an "action" function. A Fourier transform can then be used to show that a PDF^{60} on an amplitude function of wave

⁶⁰ Probability distribution function.

numbers yields another PDF from which one can source the arbitrage based risk neutral probabilities. They then connect the probabilities emerging from a (non) zero linear action with those risk neutral probabilities and show that the Fourier transform cannot be sensibly used in the non-arbitrage case as the PDF is not useable.

It is intuitive that the underlying process of pricing financial assets is often a martingale. However, this is not necessarily the case when one considers a continuous parameter process. Asset pricing hinges on the concept of an equivalent martingale which in essence refers to the use of a probability which converts a semi-martingale into a martingale. (Williams, 1991; Rodgers and Williams, 2000a:200b; Revus and Yor, 2004; Lamberton and Lapevre, 2008; Kennedy, 2010; Parzen, 2015). Karatzas and Schreve (1998b) argue that this type of equivalent measure bears a striking similarity to de Finetti's (1937:1974) theory of coherent subjective probabilities and inferences. Nau (2001) agrees with De Finetti's argument that probability does not exist in an objective sense. Instead, it exits only subjectively within an individual's mind and across individuals in their beliefs.

De Finetti (1937:1974) defines subjective probabilities in terms of the rates at which one would be willing to bet money on events, even though, in principle, these betting rates may very well depend on individual preferences for money as well as well-formed individual beliefs at a specific time. Haven and Khrennikov (2016) more recently discuss subjective interpretations of probability within a quantum-like environment. In this research study, I take de Finetti's view on probability, however I view it dependent to state-dependent marginal utility (preference for money), but also on the Eigen-states

within the fitted quantum well, in accordance with my own beliefs (refer to chapter 7.0 in this work for actual calculations and use in pricing options).

The utilisation of wave functions, or the use of analogies with an uncertainty principle can (but does not have to) invoke quantum features. Wave functions are found in classical mechanics (Goldstein et al., 2002; Thornton and Marion, 2003; Kibble et al., 2004; Morin, 2008). The concept of quantum probability (in decision making) is deployed when a context with a "quantum feature" is needed, such as the case in this research work, and can be difficult to study. Quantum features occur also in the use of Fisher information in economics and finance. Fisher information is narrowly linked to a particular "potential" function which emerges from quantum mechanics (Hoerrniuo, 1963; Meyer, 1993; Haven, 2002:2005; Choustova, 2006:2009; Khrennikov, 2010; Hawkins and Frieden, 2010; Nastasiuk, 2015).

For Khrennikov and Haven (2016) it is the preponderance of the field of quantum information, which puts to the fore the interpretation that the wave function, a central object in quantum mechanics, is informational in nature. They further explain, the formalism of quantum mechanics can be used to describe information processing of any system, whether social or physical, with the caveat that there must be some sort of quantum feature to the system under study.

Bustamante and Contreras (2016) developed an interacting model for option pricing that

generalises the usual Black–Scholes formulation to include a more general case of quantum interactions, defined by arbitrage possibilities, and triggered by market imperfections, such as such as transaction cost, asymmetric information issues, short-term volatility, extreme discontinuities, or serial correlations. Simialr to Haven (2002), they interpret the Black–Scholes PDE as the free particle's imaginary time Schrödinger equation. It can be used to represent a financial system in a state of equilibrium. However, market inperfections will cause deviations from the state of equilibrium, subsequently violating the classical non-arbitrage assumption of the Black–Scholes model, implying a non-risk-free portfolio.

The new Black–Scholes–Schrödinger model in Bustamante and Contreras (2016) is based in the endogenous arbitrage option pricing formulation introduced by Contreras et al. (2010). Arbitrage is incorporated as an external time dependent force with an associated potential related to the random dynamic of the underlying asset price. The model can be interpreted as a Schrödinger equation in imaginary time for a particle of mass $1/\sigma^2$ with a wave function in an external field force generated by the arbitrage potential. They applied semi-classical methods to find an approximate analytical solution of the Black–Scholes equation in the presence of market imperfections.

The tests in Bustamante and Contreras (2016) include a suitable potential of the analogous Schrödinger equation for different arbitrage bubble forms (step, linear and parabolic). Contreras et al. (2010) modelled the arbitrage bubble as an inverse scattering problem, that is, from the real financial data, the "nuclear potential" V(S,t) and the

arbitrage bubble f(S,t), can be obtained. Still, they assumed an evolutionary dynamic problem, where the arbitrage bubble should be determined by maximizing or minimizing a certain functional in each time step. Bustamante and Contreras (2016) argue that, during empirical testing, the quantum model could be calibrated on a case by case basis. They concluded that for the different time dependent arbitrage bubble forms, the semi-classical solution is a good approximation, when the bubble amplitudes remain below 40% of the market's volatility.

The analysis of classical Hamiltonian and Lagrangian of a quantized Black–Scholes system of many-assets, reveal that the canonical momentums can at times be written in terms of quantities other than the presumed velocities. The Black–Scholes equation for a portfolio of assets can be seen as a multi-dimensional Schrödinger equation of one partcile.(Vinogradov and Kupershmidt, 1981; Baaquie et al., 2003; Baaquie, 2004: Vidya and Shivakumar, 2007; Bustamante and Contreras, 2016). This feature is a typical characteristic of the constrained system that appears in the high-energy physics. To study this model in the proper form, one must apply Dirac's method for constrained systems. The results of the Dirac's analysis indicate that in the correlation parameters space of the multi-assets model, there exists a surface (called the Kummer surface ΣK , where the determinant of the correlation matrix is null) on which the constraint number can vary (Dirac, 1958; Godstein and Poole, 2002; Tharwat, 2015).

Bustamante and Contreras (2016) argue that a financial multi-assets Black–Scholes model is a variable constrained system. The structure of the constraints in the space-phase is completely determined by the geometry of the Kummer surface ΣK . Pestov's (2016)

studied high-dimensional spaces with an exact nearest-neighbour search focus and how this is affected at a fundamental level. He relied on nearest-neighbour learning algorithms, such as the *NN*-classifier. He proved that the performance of the classifier is unstable in very high dimensions and inconsistent at each reduced level of dimensionality. Existing models for statistical learning are oblivious of dimension of the domain, therefore every learning problem admits a universally consistent deterministic reduction to the onedimensional case by means of Boral isomorphism (Berberian, 1988; Kechris, 1995; Dudley, 2002; Kallenberg, 2017).

In this research study, I assume that price-surfaces and subsequently implied volatility surfaces can traced and fitted with quantum grids. The latter are assumed to have "memory" about the forward nearest-neighbour market points. It extends to a degree the work of Bustamante and Contreras (2016), by identifying and testing various "geometries" for the fitted quantum system. Moreover, I rely on a comprehensive solution of the generalised Schrödinger-Sturm-Liouville PDE. This complements and exceeds existing research in the domain (Meyer, 1993; Goldstein et al., 2002; Thornton and Marion, 2003; Kibble et al., 2004; Morin, 2008; Choustova, 2006:2009; Khrennikov, 2010; Hawkins and Frieden, 2010; Nastasiuk, 2015).

Rebentrost et al. (2018) researched the interface between quantum computing and finance. They explored relevant probability distributions in quantum superposition and created a quantum Monte-Carlo algorithm for pricing purposes. Key to their argument

was the use of quantum circuits to implement payoff functions, and quantum measurements to price financial derivatives. They applied the amplitude estimation algorithm, explained in detail in Lomonaco (2002), and Nielsen and Chuang (2010), to achieve a quadratic quantum speedup in the number of steps needed to compute the price with high confidence. Their work represents a starting point for further research in the domain.

Duwell (2018) examined the theoretical foundations of quantum mechanics and quantum phenomena. He extended Le Bihan's (2017) view on the understanding of phenomena by focusing on the rich nexus of internal and external understandings in intricate theories. In his analysis, he used a modal approach to argue that theory understandings are not mutually exclusive, instead they aid further exploration. The modal view of understanding is capable of explaining how significantly different activities are when formulating theories in terms of what one can or cannot do. But also when developing new axiomatizations of existing theories, or investigating plausible interfaces and correlations of multiple-worlds. Thus aiding in the development of theories that can be seen to facilitate understanding of phenomena in a quantized financial pricing system (Barnum et al., 2012; Abramsky and Coecke, 2007; Acin et al., 2006; Barrett et al., 2005; Piotrowski and Sladkowski, 2001:2002).

Duwell's (2018) extended modal-view of understanding provides a unified approach in

quantum domains and is relevant to the understanding of constrained topologies such as the zero-objects considered in this research study. This is so because of the pragmatic approach to understanding adequacy conditions when dealing with both the \bigcirc and \bigcirc worlds in a unified "universe". Adequacy conditions do shift across researchers and over time. Quite clearly, any condition that works must be open to further development and new interpretations and is not an adequacy condition that all researchers have to adopt.

Khrennikov (2018) explored randomness in classical and quantum-like models by analyzing complex financial and general economic processes. His analysis and hypothesis) imply that a quantum-like probabilistic description is more natural for financial markets than the classical one. He looked at the possibility of applications of the quantum probabilistic models to agents of financial markets. Furthermore, Khrennikov (2018) argues that although the direct quantum (physical) reduction, based on using the scales of quantum mechanics, is meaningless, one may still apply quantum-like models. He considered quantum-like probabilistic behaviour to be a consequence of the context of statistical data in finance and economics in general. However, the hypothesis on the "quantumness" of financial data needed to be tested experimentally.

Khrennikov (2018) presented a new statistical test based on a generalization of the well known quantum physics Bell's inequality (Khrennikov, 2002; Acin et al.,2006). The financial market is a complex dynamical system. However there have been a number of studies devoted to various aspects of the random description of financial processes over a significant time period, such as the works of Bachelier (1890), Davis and Etheridge

(2006), etc. The conventional quantum interpretation of superposition induces a rather special viewpoint on randomness i.e. individual randomness (von Neumann, 1955). It is commonly assumed that quantum randomness described by a complex wave function $\psi(x)$) may not be reduced to the classical ensemble randomness. The latter is induced by a variety of properties of elements of a statistical ensemble. It is described by the classical measure-theoretical approach based on the axiomatics of Kolmogorov (Kolmogorov, 1956; Camerer and Ho, 1994; Cover et al., 1989).

Einstein, Schrodinger, De Broglie, and Bohm (Einstein et al., 1935; Bohm, 1951; Einstein, 1948:1958; Bohm and Hiley, 1993) strongly opposed views that see randomness as a quantum attribute. They were convinced that quantum randomness could be reduced to classical ensemble randomness. Khrennikov (2018) recognizes that such views have been persistent, yet maintains that in combining quantum probability with classical ensemble probability, the main problem is to find a reasonable explanation of the interference of probabilities, instead of just the ordinary addition of probabilities of alternatives. Therefore Khrennikov's (2018) proposed test aimed to confirm the pre-set hypothesis about quantum-like probabilistic behaviour of financial markets. This would have interesting consequences for foundations of finance. In a quantum-like approach, the fundamental assumption of modern finance, namely, the efficient market hypothesis may be questioned (Fama, 1970; Black and Scholes, 1972; Rubinstein, 1975; Piotrowski and Sladkowski, 2002).

In Khrennikov (2018), the financial context (the financial market, expectations,

prognoses, political situation, social opinion) is represented by a complex probability amplitude, financial wave function $\psi(t, q)$, where t is time, and q is a vector of stock prices (it has a very large dimension). He describes the evolution of $\psi(t, q)$ by using the Schrodinger equation. His hopes are that the evolution of the financial context could be predicted, at least in principle. He argues that at the moment one could not even dream about the possibility to solve the problem analytically or numerically. His main argument is based on the ambiguity on how the "financial Hamiltonian" should be constructed, that is the quantum-like operator representing the "energy" of the financial market. Khrennikov (2018) alleges that another problem is the huge dimension of the problem. This is based in the earlier work by Haven (2004:2006), Haven and Khrennikov (2016), etc.

The quantum-like model in this research study here may provide the necessary impliedqualitative and quantitative prediction that given enough time, for further developments of financial technologies, one may improve and apply to induce perpetually exploitable profit opportunities. This would be complementary to the conventional model based on the efficient market hypothesis, rather than in opposition as argued by Khrennikov (2018). My view is congruent to Einstein, Schrodinger, De Broglie, and Bohm. However it also validates some aspects of Khrennikov (2018). That is: quantum randomness may be mapped out to classical ensemble randomness or there may be parity between the two, where both may be co-varied. The quantum randomness is what classical randomness would look like in a dimensionally reduced "universe", such as the D-world. Then the question is whether the \bigcirc and \bigcirc worlds are correlated and with a clear interface. I argue here that they may be, hence randomness may be in parity in the two worlds. But if the topological constrain "suppresses" the classical randomness, then the quantized randomness is expected to be further fragmented and shuffled, thus leading to better random attribution that in extreme cases may be alterned extensively. Although this is tackled here from a different angle than Khrennikov, it does confirm part of his view. Could the two be independent? Does the existence of quantum randomness cancel out classical randomness as stipulated by Khrennikov (2018)? The answer to the first question is linked to very nature of the fitted quantum wells (depth, radius, price cut-off potential function, etc.). On the second, one needs to reflect in terms of the very nature of the mathematical model used, which I address in detail in the theory/empirical chapters of this study (Einstein et al., 1935; Bohm, 1951; Einstein, 1948:1958; Bohm and Hiley, 1993; Raedt et al., 2012; Khrennikov, 2018).

I emphasize that the creation of quantum-like financial concepts does not imply lower complexity in the model, compared with the conventional one (Morin, 2008; Khrennikov and Haven, 2009). The latter implies that financial processes can be represented by a special class of classical stochastic processes, martingales (Revus and Yor, 2004; Lamberton and Lapevre, 2008; Kennedy, 2010; Parzen, 2015). For any such process, I may construct a single Kolmogorov probability space for all realizations of this process. This is the essence of the famous Kolmogorov theorem (Kolmogorov, 1956; Cover et al., 1989). In contrast to a single space description, in quantum asset pricing models, I can not assume that the quantum-like process, based on the evolution of the financial context,

could be embedded into a single Kolmogorov probability space.

In classical financial mathematics, there have been numerous fundamental investigations to find an adequate stochastic process that could match real financial data: Brownian, geometric Brownian, general Levy processes (Nelson, 1967; Madan and Milne, 1991; Gobet, 2000; Kuchler and Tappe, 2008). My own view is that when using a quantum-like approach, the problem cannot even be formulated in such a way. There is no classical stochastic process that matches real financial data, because a single Kolmogorov space is not efficient to describe the whole financial market. Financial data can only be represented by a quantum-like financial process (Doob, 1953; Harrison and Pliska, 1981, Karlin and Taylor, 1981; Malliaris, 1982; Karatzas and Shreve, 1998b; Shiryaev, 1998, Oksendal, 2000; Shiryaev et al., 2006; Lamberton and Lapevre, 2008; Keller-Ressel, 2011; Kijima, 2013).

Bardou et al. (2016) focused their work on the process of risk minimization as a way to hedge various forms of risk on financial and energy markets. They articulated an optimal portfolio strategy by virtue of dynamic minimization of conditional value-at-risk. They used a stochastic approximation algorithm, obtaining the optimal quantization through the application of variance reduction techniques, where filtration oddities were linked to highly impacting, but infrequent, probability triple space events. They assumed that the process describing the source of risk and financial asset prices is a Markovian process (Garczynski, 1969; Spitzer, 1970; Markov, 1971; Hull and White, 1987; Stein and Stein, 1991; Seneta, 1996; Mura and Swiatczak, 2007).

Furthermore, Bardou et al. (2016) considered a complete financial market, under which an investor, faced with a contingent claim, has the choice to perfectly hedge the underlying across a finite horizon time, considering realistic financial and energy market conditions, which are intrinsically incomplete due to the effects of stochastic volatility, jumps, and other externalities with impact on the market - for example the impact of temperature in price of a commodity (Stein and Stein, 1991; Keller-Ressel, 2011). This is a limitation across energy markets. Bardot et al. (2016) argue that imperfect conditions on a complete market complicate the pricing process through replication. Such pricing limitations can be bypassed with the use of alternative pricing frameworks. It is relatively safe to argue that all current pricing models carry deficiencies in the provision of a consistent and effective pricing tool across markets.

Pages et al. (2003) used a stochastic approximation algorithm with a detailed treatment of numerical performances of quadratic functional quantization with limited applications in Finance. They concentrated their effort on product quantizers and the Karhunen-Loeve expansion of Gaussian processes with special interest in the Brownian motion. This is also in line with research work from Luschgy and Pages (2002), and Wilbertz (2005). Moreover, Pages et al. (2003) computed efficient functional quantizers for Brownian diffusion and through collaboration with Luschgy applied functional quantizers in option pricing (Luschgy and Pages, 2002; Delattre et al., 2006). Pages and Luschgy (2005) proposed a quadrature expression, based on a Romberg logextrapolation of functional quantization. The Romberg log-extrapolation is covered in detail in Romberg (1955): It applies an iterative superposition of the trapezoidal rule for Freeden and Gerhards (2017), further formulated Romberg different grid sizes. extrapolation for Euler summation-based cubature on arbitrary q-dimensional regular lattices. Luschgy and Pages (2002), then carried out various pricing scenarios within a Heston stochastic volatility framework (Heston, 1997; Heston and Zhou, 2000). Their findings suggest that functional quantization is a very efficient integration method for various path-dependent functions of diffusion processes, because it produces deterministic results which outperform Monte Carlo simulations within the usually expected error tolerance levels (Caflisch 1998; Dupire, 1998; Pages et al., 2003; Dick et al., 2013; Rebentrost et al., 2018).

Levental et al. (2016) argue that all uncertainty is generated by a *d*-dimensional standard Brownian motion B over the finite time horizon [0, T], supported by a probability space (Ω, F, P) , where all processes are assumed to be progressively measurable with respect to the augmented filtration $\{F_t : t \in [0,T]\}$ generated by B. For any subset $V \subseteq \Re$ (respectively $V \subseteq \Re^{n \times m}$), with L(V) denoting the set of V-valued progressively measurable processes, and for any $p \in R$, given in well defined terms⁶¹. They used a subspace, and considered the portfolio maximization problem of a single agent in

 $^{{}^{61}} L_p(V) = \left\{ x \in L(V); E\left[\int_0^T ||x_t||^p \right] < \infty \right\}, \text{ where } ||x_t||^2 = x_t' x_t \text{ (respectively, trace(} x_t' x_t)).$

complete markets with an aggregator that depends on current wealth.

Benaïm and Raimon (2003) studied the convergence in law properties of self-interacting diffusions on a compact Riemannian manifold, where self-interacting diffusions are continuous time stochastic processes living on a Riemannian manifold M, which can be typically described as solutions to a stochastic differential equation $(SDE)^{62}$, with the implication of a special family of Brownian motions, and $(F_{\alpha})_{\alpha}$ a family of smooth vector fields on M such that $\sum_{\alpha} F_{\alpha}(F_{\alpha}f) = \Delta f$, for $f \in C^{\infty}(M)$, where Δ denotes the Laplacian on M, and $V_u(x)$ a "potential" function.

The "potential" function is an important construct around our own concept of replicative function identities in asset pricing, and play an important role in the dynamics of solution-finding. According to Benaïm and Raimon (2003), these processes are characterized by the fact that the drift term depends both on the position of the process X_t , and its empirical occupation measure up to a stopping time t⁶³. Benaim and Raimon (2003) also argue in favour of the asymptotic behaviour as a further development of their earlier work: Benaïm et al. (2002).

$$[\]label{eq:constraint} \begin{split} ^{62}\, dX_t &= \sum_a F_a(X_t) \; \mathrm{o} \; dB_t^\alpha - \frac{1}{t} \Bigl(\int_0^t \nabla V_{X_s}(X_t) ds \Bigr) \, dt \\ ^{63}\, \mu_t &= \frac{1}{t} \int_0^t \delta x_s \, ds \end{split}$$

Benaïm et al. (2002) describe the long-term behaviour of $\{\mu_t\}$ in terms of the long-term behaviour of a certain deterministic semi-flow $\{\Psi_t\}_{t\geq 0}$ defined in the space of a probability measure on M. This includes situations (depending on the shape of V) in which $\{\mu_t\}$ converges almost surely to an equilibrium point μ^* of Ψ . In other situations the limit set for $\{\mu_t\}$ coincides almost surely with a periodic orbit for Ψ . In the simple case when μ_t converges to μ^* , one expects (X_{t+s} , $s \geq 0$) to behave like a homogeneous diffusion of a generator⁶⁴.

Benaïm et al. (2002) explored self-interacting diffusions on a smooth compact manifold using e Girsanov transfer technique to distinguish interactions that reflect symmetry⁶⁵ with a positive or negative self-adjoin operator, making use of Borel probability measure and gradient of interactions (Berberian, 1988; Srivastava, 1998). These interactions are self-repelling when the gradient is positive, and self-attracting when the gradient is negative. They show that , if V₁ is a constant function, for all repelling cases or weakly attracting cases ($\alpha > -\alpha G$, with $\alpha G > 0$), the empirical occupation measure of the associated self-interacting diffusion converges towards λ , and when $\alpha < -\alpha G$, this is not the case, and μ_t may converge towards $\mu * \neq \lambda$ with the interaction, on the n-dimensional sphere S_n. Benaïm and Raimond (2002) showed that in the cases when $\alpha \ge -(n+1)/4$, μ_t converges towards λ and when $\alpha < -(n+1)/4$, there exists an S_n-valued random variable v

 $^{^{64}}G(x) = \int V_y(x)\mu^*(dy)$ and $\langle \cdot, \cdot \rangle$ denotes the Riemannian inner product.

⁶⁵ V is symmetric and defines a positive or a negative self-adjoin operator acting on L²(λ), that can be written in the form $V = \alpha \int_{ci}^{Cf} G(u, x) G(u, y)v(du)$, where C is compact, written Borel probability measure, G: C × M →R is continuous and $\alpha \in R$, which known as gradient interactions that produce examples for which $P_{x,r,\mu}(\tilde{\Omega}) = 1$ and the limit μ^* may be random.

such that μ_t converges towards $\exp[\beta n(\alpha)\cos(d(x,v))]\lambda(dx)/Z_n,\alpha$, where Zn,α is the normalization constant and $\beta n(\alpha)$ is a constant depending only on n and α . They articulated an example of an interaction on S_n , which is not a gradient interaction, for which $Px,r,\mu(\widetilde{\Omega}) = 0$.

Benaim and Cloez (2015) conducted analysis using a stochastic approximation algorithm in order to simulate quasi-stationary distributions on finite state spaces. They found out that the asymptotic behaviour of an empirical occupation measure⁶⁶ is precisely related to the asymptotic behaviour of some deterministic dynamical system induced by a vector field on the unit simplex. This represents new proof occonvergence of asymptotic rates in a constrained topological space. It led to a generalization of the method introduced previously by Aldous et al. (1988). In their 1988 work, Aldous et al., focused on a process $(Y_n)_{n\geq 0}$, that maps out to a Markov chain on a finite state space F with a transition matrix $P = (P_{i,j})_{i,i\in F}^{67}$, which is not assumed to be aperiodic⁶⁸.

Benaim and Cloez (2015) concluded that a process behaves like $(Y_n)_{n\geq 0}$ until it diminishes, which occurs when it hits 0. After it diminishes, it then comes back to life in a state randomly chosen according to its empirical occupation measure; this process is

⁶⁶ Empirical occupation measure of the process, $\mathcal{F}_n = \sigma\{X_k, k \le n\}$.

⁶⁷ Under the assumptions that, (i) the system admits an (attainable) absorbing state, say 0, such that $F^* = F \setminus \{0\}$ is an irreducible class for P, which means Pi, 0 > 0 for $i \in F^*$, $P_{0,i} = 0$ for all $i \in F^*$, and $\sum_{k>0} P_{i,j}^k > 0$ for all $i, j \in F^*$; (ii) P could be either period or aperiodic67; (iii) process $(X_n)_{n\geq 0}$ is limited on F^* , for every $n \ge 0$, and stands for an empirical occupation measure of the process, $\mathcal{F}_n = \sigma\{X_k, k \le n\}$.

⁶⁸ For all $i \in F$, and any probability measure μ on F (or F^{*}), $\mathbb{P}_i(\cdot) = \mathbb{P}_i(\cdot | Y_0 = i), \mathbb{P}_i(\cdot) = \mathbb{P}_\mu = \sum_{i \in F} \mu(i) \mathbb{P}_i$ and $\mathbb{E}_i, \mathbb{E}_\mu$ denote the corresponding expectations.

not Markovian and can be understood as a reinforced or extended random walk. They used a natural embedment of such process onto continuous-time multi-type branching processes. Aldous et al. (1988) provisioned the central limit theorem and further on proved the convergence of $(X_n)_{n\geq 0}$. Under the condition $\gamma_n=1/n$, Benaim and Cloez (2015) also provisioned the central limit theorem, using similar techniques, that allow for convergence rates to compare with a discrete-time version of the algorithm.

Mantegna and Stanley (1995:2000) provide a complete statistical characterization of different markets ⁶⁹, including important aspects of PDF evaluation. There is a considerable amount of on-going work to develop the most satisfactory stochastic model, that could describe the main features encountered in empirical analysis, where the non-Gaussian shape of price returns' PDF is one of common research venues (Milevsky and Posner, 1998; Madan et al., 1988; Kuchler and Tappe, 2008). Nastasiuk (2015) argues that a major area of research concerns the development of a theoretical model that is able to encompass the essential features of a real financial system characterized by such PDF. Financial economics borrows results in statistics (Huang and Litzenberger, 1988; Cuthbertson and Nitzsche, 2004; Bodie et al., 2009; Jacus, 2008; Franke and Heardle, 2011; Berenson et al., 2012), and in addition to the statistical-mechanical description, a quantum mechanical representation has also emerged (Baaquie, 2004; Haven, 2005; Nastasiuk, 2015).

⁶⁹ Stock, commodities, foreign exchange, etc.

Researchers such as Matia et al, (2003), attribute distribution particularities to asset classes and industry sector categorisations. They studied price behaviour of stock and commodities and concluded that commodities have a significantly broader multifractal price spectrum compared to stocks. They proposed that such multifractal properties for both stocks and commodities are mainly due to the broad probability distribution of price fluctuations and secondarily to their temporal organisation. Furthermore, they concluded that for commodities it is the stronger higher-order correlations in price fluctuations that cause the multifractal spectra to be broader.

Market prices change randomly, while remaining near the same on average. By taking a sequence of price changes x (logarithmic returns commonly used in financial analysis), one can obtain the dispersion measure $\sigma^2 = E[(x - E(x))]^2$. Its square root is used to compute the volatility (Markowitz, 1952;Geske and Roll, 1984; Hull and White, 1987; Bodie et. al.,2009; Elton and Gruber, 2011; Hull, 2014). Nastasiuk (2015) elaborates on the concept of volatility with a quantum mechanics framework. He argues that the variance is the expectation of the deviation squared or $\sigma^2 = E[(x - E(x))^2]$, which itself matches in meaning and expression the classical form of the same concept. He explores this further and provides a well-defined probability distribution⁷⁰ with $\psi(x) \equiv \sqrt{p(x)}$ forming the lower bound for the dispersion and in accordance to the Cramer and Rao inequality. This

⁷⁰ I =
$$\int dxp(x) \left(\frac{d\ln p(x)}{dx}\right)^2 = 4 \int dx \left(\frac{d\psi(x)}{dx}\right)^2$$

is expressed as $\sigma^2 \cdot 1 \ge 1$, and serve as a quality metric of price change predictability (Cramér, 1946; Rao, 1945; Fabian and Hannan, 1977; Rao, 1994).

Many of contemporary research papers, in the area of quantum asset pricing, make use of models that include Physical tangibles (Haven, 2002:2005; Baaquie, 2004; Nastasiuk, 2015; Moreira and Wichert, 2016; Haven and Khrennikov, 2016). Interactions between tangibles cannot be fully delegated to finance, at least in the reported form. Although they implicate the financial system's states, one cannot overcome the limitation of their application in a proper and purely financial context. Literature does not provide a clear-cut explanation on how such interactions could be conceptualised in a financial system with no interactive objects. This work focuses at possible connections between probability, space-time geometry, quantum mechanics, and intergration of $\hat{\mathbb{O}}_{\downarrow\uparrow}$ and $\hat{\mathbb{O}}_{\downarrow\uparrow}$ worlds in a way and manner that makes pricing of financial assets and financial options possible without the involvement of physical tangibles.

Canessa (2007) stipulates an association of an analogous probabilistic description with a space-time geometry in the Schwarzschild metric from a macro domain to to a micro one. In the context of my own research work here, it implies the $\mathbb{O}_{\downarrow\uparrow}$ and $\mathbb{O}_{\downarrow\uparrow}$ worlds. He argues that that there is a possible connection among normalised probabilities within a space-time geometry in the form of Schwarzschild radii r_{s} , and quantum mechanics in the form of complex wave functions, along nested probability surfaces (Schwarzschild, 1996). This is in-line with the works of Klauck (2003), Plotnitsky (2009), and Cabello

(2018). It has first order relevance to this research, however concepts such as inhomogeneous density of matter have no equivalence in finance.

In this study, I attempt to fit a Sturm-Liouville system to nested probability surfaces. Bailey (1966) describes well the Sturm-Liouville system in terms of a well-defined expression, where p(r) can be a function of the stochastic process random variable (i.e. rate of return) or can be simply a constant and $q(r, \kappa)$ is a function of the random variable r and the Eigen-value κ . This has since been explored by various researchers (Pruess, 1973; Pruess et al., 1995; Kong and Zettl, 1996; Tharwat, et al, 2013; Hira and Altinisik, 2014; Yang et al., 2015). I expand this concept considerably in the theoretical chapters.

2.4 LITERATURE REVIEW CLOSING REMARKS

Part of this research work is revisionary, intended to (i) to enhance what is already known, (ii) to abstract and unify known theories through a complete equilibrium PDE. I have particularly made choices on publications from 1900s to date. The first 40 years of the 20th century include the introduction of quantum mechanics, in the context of more traditional sciences, with the initial contribution in quantum theory by Planck and Einstein in early 1900 and mid-20s, followed by mathematical formalism and modern style reporting on its use across disciplines (Neumann, 1955; Jammer, 1966; DeWitt, 1970; DeWitt and Graham, 1973).

The development of quantum models has seen financial theorists and practitioners attempt to include quantum option pricing. They are always looking for ways to overcome the resulting performance issues that arise when pricing options with classical and quantum models. This has led to research that employs alternative computing techniques in finance. Focused has been placed on the quantization of the classical Black–Scholes–Merton PDE with emphasis on the instrumentation of the known Schrödinger PDE (Ilinski, 2001; Baaquie, 1997:2002:2004:2005:2007:2013:2014; Haven, 2002:2004:2005; Accardi and Boukas, 2007; Nastasiuk, 2015; Benaim and Cloez, 2015; Haven and Khrennikov, 2016; Khrennikov, 2007a:2007b:2018; Callegaro, et al., 2018a).

Haven (2002) builds on the work of Chen (2001) and others, but considers the market from the perspective of the Schrödinger equation.-The key message in Haven's work is that the Black–Scholes–Merton PDE is a special case of the Schrödinger PDE with the assumption that markets are efficient. The Schrödinger-based equation that Haven derives has a parameter h that represents the amount of arbitrage present in a market. It is driven by non-infinitely fast price changes, non-infinitely fast information dissemination, and unequal wealth among market participants. Haven argues that by setting the parameter h appropriately, a more accurate option price could potentially be derived. However in reality markets are not truly efficient.

In existing physics-finance research, I have noted the use of prevailing classical physics. This is exhibited mostly with researchers attempting to find Financial meaning to truly Physics or Physical quantities and parameters. I believe this is a "hard" limitation. There
have been a limited number of research papers on the application of elements of quantum physics in Finance. Young (1999) used the Gauge theory to a simplified foreign exchange problem. Whereas Ilinski (2001) used Gauge theory to explain non-equilibrium pricing and further proposed the Schrodinger equation when dealing with dynamics of money flows for single investors and horizons. Baaquie et al. (2002) and Haven (2002) proposed a Black–Scholes–Schrodinger equation, with the "hard" limitation prevailing.

The overarching technique in this study is the removal of the "hard" limitation, by formulating an augmented generalization of Schrödinger PDE, without the use of parameters such as Energy, ħ, etc. It suffice to mention that Black and Scholes (1973) classical option pricing model did in a similar manner abstract early derivatives pricing prototypes through a process of removal of parameters that had dual meaning i.e. Physics and Finance (Black, 1989).

It is possible that a quantum option pricing model could be more accurate than a classical one. Baaquie has published many papers on quantum finance that brings relevant concepts together (Baaquie, 1997:2004:2005:2007:2013:2014). Core to Baaquie's research and others like Matacz (2000) were Feynman's path integrals (Feynman and Hibbs, 1965). Baaquie applied path integrals to a number of exotic options and compared analytical results to those acquired through the application of the Black–Scholes–Merton equation (Black and Scholes, 1973; Merton, 1974), showing that they are very similar (Rubinstein et al., 1995; Falloon and Turner, 1999; Bormetti et al., 2018).

Piotrowski et al.(2001) took a different approach by changing the Black–Scholes–Merton assumption regarding the behaviour of the stock underlying the option. Instead of assuming a Wiener-Bachelier process, they consider that stock price follows an Ornstein-Uhlenbeck process (Doob, 1953; Karlin and Taylor, 1981; Shafer, 2002; Hull, 2014; Gillespie, 1996; Schöbel and Zhu, 1999). Subsequently, they derived a quantum finance valuation model as well as a European call option expression.

Other models such as Hull and White (1990) and Cox et al. (1985) have successfully used the same approach in the classical setting with interest rate derivatives. Khrennikov (2007a:2007b) builds on the work of Haven (2002:2004:2005) and others and further bolsters the idea that the market efficiency, assumed in the Black–Scholes–Merton equation, may not be appropriate. To support this idea, Khrennikov proposed a framework of contextual probabilities using agents as a way of overcoming criticism of applying quantum theory to finance. Accardi and Boukas (2007) proposed a quantized Black–Scholes–Merton equation and considered the underlying stock to follow both Brownian and Poisson processes.

It is only in the last decade or so that publications on the use of quantum mechanics and its underlying theory could be applied in financial domains, but to no particular useful effect in pricing despite the general claims to the contrary. I attempt to rectify this here by providing a conceptual and measurable framework that considers the $\mathbb{O}-$ and $\mathbb{O}-$

worlds combined in a financial pricing gauge with tangible effects. Additional effects of inference on literature review are also included in the theoretical chapters.

3.0 RESEARCH METHODOLOGY

I provide justification for the selection of research tools, strategies, approach, and philosophy demployed throughout this study. The selection is based on an adaptive - layered research framework.

From the outset, a good portion of this research is driven by deductive reasoning with model building, finalised at the last stage with numerical simulations, and illustrations using the pricing analytics put forward in this work. I include a detailed treatment of all aspects of the research methodology relevant to this type of research and most suitable for this domain, with all of the research elements that have made the research in this domain possible, by making extensive use of secondary sources (Saunders et al., 2003; Pelissier, 2008; Snieder and Larner, 2009; Wilson, 2010).

Where books are used, they serve the purpose of setting the stage for more detailed and contemporary research powered by dated and recent articles. In summary, this chapter provides the research 'frame' which is made up of my choice of the research philosophy of interpretivism, with a deductive approach. I make use of quantitative techniques for data and information acquisition and carry out research analysis (Saunders et al., 2003;

Pelissier, 2008; Snieder and Larner, 2009; Wilson, 2010). These are embedded in subsequent chapters.

3.1 RESEARCH PHILOSOPHY

I start with the outer-most layer of the research framework, that of the research philosophy, and proceed to the application of a broad and balanced spectrum of views on research philosophy driven by my understanding of what constitutes the best philosophy construct for this body of research work, and that leads to acceptable new knowledge in the domain.

The interpretivism epistemology view is valid due to the fact that this research is based on inferred knowledge, where research elements and outcomes follow interpretations according to the views formed in the process; typically, I consider the views to come from own knowledge acquisition and inference over time and through experience. Expectations are formed within a rather 'fluid' process where what is perceived as acceptable knowledge is revised over time within own knowledge base, which allows for revision of expectations accordingly.

I seek to match my internal expectations with those implied by other collaborated work in a published form and which are broadly acceptable by the research community i.e. external expectations. I believe that with proper management of all elements of this research and the knowledge inferred, I seek not only to align those external expectations, but also to exceed them. The process of the external research expectations' alignment makes it possible to revise the logic over a time horizon with new developments and experiences that drive varied interpretations.

Such research epistemology view is only taken in partial consideration, when adapted to a rigorous domain such Finance allows for the augmentation of a strong view, particularly on the establishment of the interface and 'interactivity' between two 'worlds' under consideration; the D-world⁷¹ and the C-world⁷². Although these two worlds were not put together in the past, they are explored independently in prior literature with substantial theoretical foundations to support one another. To that end I utilise deductive reasoning in a reasonable part and in line with Karl Popper's view (Popper, 1974). There is some theory construction which again is based in part in previously existing axioms.

I discard the effects of a possible objectivist view that separates the two "worlds". It is essential to determine and understand the factors that impact, govern, and affect my very own interpretation of the two worlds, their separation, and betweenness. Existing research

⁷¹ Refers to the quantised worlds or the zero-dimension discretised 'universe', which represents each price-point along the market line.

⁷² Refers to 'our' world, the continuous time-space world, or the 'real' world, in which information or filtrations (outputs of events) develop that have a direct impact in the quantised world or the p-point 'universe'.

either focuses on the ©-world, or the D-world, or when the two are connected, the justification is no well-formed or fully justified for financial valuation purposes. The relevant concepts of betweenness, the nonlinearity of probability, and the likely connection between probability and space-time geometry in a quantum system are explored in the works of Camerer and Ho (1994), and Canessa (2007).

In this work and in minor capacity, I make use of a dual-track research philosophical view, where conventional logical positivism is applied as a major effect and in a forward manner, with the minor effect of inductive phenomenology applied within a reversed track approach to concrete outcomes. The former is based on the fact that my starting point in this research is the creation of a new theoretical framework with merging of alternative theories, and subsequent numerical testing. This leads to the compilation of a finance model through deductive reasoning. However, I also use an extensive amount of trial-by-error numerical scenarios as a means to generate new elements of theory.

It provides a good basis for exploration of properties and internal price-related dynamics within the D-world. It also helps provide clarity on the actual interface between the two worlds within Finance and for the purpose of asset pricing. This is evident in the numerical analysis section, further allowing for a minor use of an inductive approach within a phenomenology philosophical view to a partial effect.

Data and events are utilised within such 'worlds' and at the interface between the two. I

consider a very minimal regard for any other 'world' beyond the two considered. Moreover, in my consideration of the D-w and the C-w, I place greater emphasis on specific elements with minimum regard to other factors, thus counting in only those variables and factors that we expect to have an impact or that are relevant to our purpose set beforehand, rather than effects and outcomes in hindsight. This allows for new elements of theory, also through the means of numerical simulations and testing, as a way to compliment the data emulations and the different financial pricing scenarios.

There is some fragmental symbolic interactionism in the research philosophy deployed here, such as the consideration of the ©-world and D-world interactions and the eventual effect of pricing. It is first developed within the D-world and observed in the market line in the ©-world. This is in full alignment with our interpretivist belief in multiple realities (i.e. multiple universes or worlds) as articulated in the worlds of Schwandt (998), and Denzin and Lincoln (2003).

Furthermore, I not only consider different 'worlds', but focus on each in line with phenomenology, then draw and create meanings and apply different points of view in order to analyse different valuation scenarios and validate the existing theories and practices as reported in published academic literature (Hatch and Cuncliffe, 2006).

I believe that by the greater extend, the research philosophical view with the conventional deductive phenomenology is best suited for this research study; it provides the

framework that factors-in the effects of change in price or asset price behaviours at each quantized point, along the market line. It also allows for adjustments to a new layer of knowledge acquired and inferred, and contributes to the development of new elements of theory.

It does permit for utilisation of data simulated and analysis as a form of proof and validation of the new concepts and the model, which I hope will add to the acceptance of the new knowledge. Its general outlook is more 'realistic' and as such it allows for consideration of some elements of "positivism". It provide reliance on scientific evidence, intermitted with a variant of Karl Popper style philosophical falsifiability in weighting validity through carefully fitted numerical experimentations. It facilitates the acquisition of necessary and relevant results for the purpose of proving or disproving new elements of our theory. Moreover to gain understanding of the factors that determine the validity of the new theory introduced in this research (Popper, 1974; Easterby-Smith et al., 1991; Saunders et al., 2003; Gulati, 2009; Creswell, 2013).

It is evident from the research philosophy, the research problem, and the deducted financial models that the correct research instruments are deployed to seek suitable solutions. In good part, it is due to the complex nature of this research; it crosses over to subdomains of finance and in cases to theories that to-date are minimally or only partially adapted in financial asset pricing. It provides a greater degree of flexibility and agility in pursuit of the answers to the research problem.

Thus, where necessary I switch to a scorched ground-up approach with elements of empirical testability. To have such flexibility I also take a pragmatist position, which enables me to deduce and induce new layers of knowledge and adjust freely around any bias in order to mitigate it and related effects. Furthermore I place greater focus on qualitative strategies. I switch the research philosophical position as needed withing the deductive research framework (Saunders et al., 2003; Pelissier, 2008; Snieder and Larner, 2009; Gulati, 2009; Wilson, 2010).

Where and when needed, I address "bias" in the application of any constituent philosophies within this study's framework. Creswell (2013) explores the likelihood of bias. Less so in the deductive approach, but expected in follow-up interpretations that directly come from the closeness of the researcher and the researched domain. It also prescribes self-reflection as a solution to such bias.

The trial-by-error approach in this study's numerical analysis is an instance of "bracketing the truth" through self-reflection. Moreover I try to ascertain a relationship between two very different "worlds" in which the asset price is developed. Furthermore it allows me to enquire deeper in the discretised universe as a way to unravel the mechanics of price setting, exhibitory of information dissipation and reflection by the market. Thus I attempt to get "close" to the acumen of the 'reality' of such worlds (^(D)-world), although in a partial capacity due to the lack of predictability of

future states of nature (Gleason,1957; DeWitt, 1970; DeWitt and Graham, 1973; Stenger, 1995; Barrett, 1999; Polley, 2001; Saunders, 2004:2006; Jaeger, 2009; Barrett and Zalta, 2010; Bokulich and Jaeger, 2010; Rummens, and Cuypers, 2010).

3.2 RESEARCH APPROACH

In this section I consider the next layer of our research methodology framework. Further I justify the adapted research approach. This research involves developing a theory, testing, and revising it. New theory elementsare added, following numerical trials, through mathematical means and empirical cases.

I have also set the research problem with an extensive use of a deductive approach. However in the last stage, although the data are collected and analysed to reinforce the deduced theory or aspects of it, there are some small elements of theory adjustments that are driven by the empirical testing, which allows for an added inductive approach attribution to this research, although in much smaller consideration; mostly a set of theory testing scenarios, but where necessary, some upgrade of the developed theory or aspects of it takes place. It adds an element of inductive diversion, although by and large the deductive approach is retained.

I am aware of the shortcomings of both approaches, hence I attempt to combine them to for optimum research effect. Previously, researchers have criticised the deductive approach due to it generating cause-effect links between specific variables without consideration of human interpretation (Saunders et al., 2003; Creswell, 2013), arguing that the "world" observed is independent of the researcher, in line with previous work such as Dummett (1978).

In this study it means that the cause-effect links between asset price change behaviour in two interrelated worlds (the \bigcirc and \bigcirc worlds) should be independent of my own interpretations, which is not possible, due to several 'realities' or patterns of price development within each such world, however my main research probe leads to a generalisation, one at the high level of abstraction (the first two theory chapters), and I try to validate the theory in the context of specific scenarios, under clear boundary conditions, in the last theory chapter on empirical testing of the new model for various price cut-off geometries.

This is in line with views from Robson (1993) and his recommendation that a deductive approach is a theory testing practice, arising from an established theory or generalisation. It allows for validation of theory in the context of specific instances, scenarios, and cases. Although this research is not just a test of an established theory, it further develops new elements as extensions and generalisations of existing theories.

I attempt to amend and enhance exiting theories, with additional features that require proof of validity, to develop new aspects of them and to further generalise them. It subsequently leads to a new theoretical model and framework in pricing, which are exhibitions of deductive theories and formal theories as two subsequent outcomes of that very process (Frege, 1884; Dedekind, 1901; Kennedy, 1974; Gillies, 1982; Edwards, 1983; Hausman, 1990; Kamp and Reyle, 1993; Zaitsev, 1994; Segre, 1994; Nicolle, 2003).

The deductive approach represents a suitable methodical paradigm for this research study due to breadth and depth of knowledge as well as the challenge of knowledge management in an efficient and representative manner at a time when published literature has increased exponentially, especially in the recent decades in the financial domain and more specifically in pricing related subdomains (Jashapara, 2004, Creswell, 2013).

According to Jashapara (2004), knowledge management is the central challenge of research, and has been around for millennia. Creswell (2013) suggests that a deductive approach would be a better approach when there is a large amount of literature in a specific research domain. Following a deductive approach, ensures a highly structured methodology (Hausman, 1990; Politzer and Macchi, 2000; Dummett, 2002; Nicolle, 2003; Rosen, 2009; Brenner, 2010). This would serve well the theoretical developments and applications in extension to the third theoretical chapter of this study.

Probably the most well-known is the ontological-metaphysical problem of the nature of mathematical objects where schools differ. Quine (1948) consistently argued against

Logical-Empiricism, specifically on the analytic-synthetic distinction. His argument is seen as part of a holistic view of the world with no clear distinction between empirical science and philosophy, Quine (1948) proposed that the mathematical schools were disputing essentially on an update-to-numbers version of the medieval debate on the nature of universals, developed further in the work of Benacerraf (1964), and the more recent debate on the same by Ebert (2007).

The three schools represent three doctrines: logicism presents the realist position holding that numbers exist in a sort of platonic world which is more real than the phenomenical flow of appearances of the empirical world; intuitionism holds a conceptualist view in which numbers are considered the product of human creativity; formalists use the nominalist version for the belief that numbers are just names, flatus vocis. Suppes (1986:1999) noted that the metaphysical-ontological dispute concludes with three failures: Russell's paradox – Frege (1903) argues that the logicist entities are contradictory. The fact that many important mathematical objects are not intuitionistically plausible excludes the conceptualist view from the scenario. Gödel's results reduce significantly the ontological-metaphysical position in order to take particular stances in other domains such as epistemology, semantics, and methodology for a more effective philosophical discourse (Gödel, 1941:1951; Benacerraf, 1967; Auerbach, 1985; Smullyan, 1991; Franzén, 2005).

The postmodernist attention for a sharp ontological-metaphysical position on the nature of mathematical entities aims at the production of a consequence of rigorous definitions,

which for millennia have been assumed and work out without any trouble. In shifting the attention from the metaphysical/ontological problem on the nature of numbers to the definitions, where definitions play an important role in the deductive reasoning by them and the general objectivity, the challenging problem in my own research study is trying to identify a notion of private grasp of the concepts which will bear the weight of being the source of meaning and truth concerning it (Politzer and Macchi, 2000).

Pylyshyn (2002) recollects the prima facie reasonable theories for such a notion and very briefly explains why they seem to be inadequate to the task, focusing on the understanding as consisting of a definition or mental picture, or understanding as consisting of some mental state. This is fully in line with earlier publications, such as the work of Benacerraf and Putnam (1964), Gillies (1982), Frege (1984), Suppes (1988), etc.

In conclusion to this section, the technique of generating theory through a deductive method is consistent with Pylyshyn (2002) Specifically this form of practice cannot change, although I may change my technique, or part of it as is the case with the trial-byerror approach in the theoretical injection of a new concept in the empirical testing, such that to validate Frege's view: There is no contradiction in an inference being valid that everybody takes to be invalid. For there is no contradiction in the supposition that we are dealing with both the © and D worlds with the former our own, whereas the latter is very different form of "life" and system from the one we in fact have (Frege, 1902:1903; Bokulich and Jaeger, 2010; Rummens, and Cuypers, 2010; Quilty-Dunn and Mandelbaum, 2018).

3.3 METHODOLOGICAL FOUNDATIONS

In this section I articulate the methodological foundations of this research. This itself represents the inner-core layers of my research framework, such as own methodological choice, research strategy(ies), time horizon, techniques and procedures, including also limitations.

Saunders et al. (2003) explains that research methods are aligned with the methods used for data and information acquisition and the actual analysis. Furthermore quantitative research is associated with numeric data collection and analysis, while qualitative methods are used with non-numeric data or data that are gained through inference.

This study allows in part for a methodological choice that could reflect well from a theoretical and methodical perspective, such as in the form of methodological triangulation. Considering them in a mutually exclusive manner would not be good for this research work, thus I combine the them in a suitable pattern or manner. Such combinations are valid and also suggested by literature, such as in the work of Creswell (2013). The main advantage is to get a different perspective while attempting to address the research problem, the deduced theoretical "volume", and also make reliable interpretations (Saunders et al., 2009).

While understanding what these methods offer individually for this research, I define my methodical position by virtue of utilisation of quantitative and qualitative methods, although the large part is based in the quantitative paradigm. I find the methodology configuration optimal to drive me to the essential layers or depths of knowledge. Other postmodernist work or work written in the modern style apply different research methods separately with almost no implied linkage, articulated in the work of Creswell and Clark (2007), who also provides a good distinctive description of the methods.

I rely on a generalised master expression with relevant testing scenarios carried out with data generated through algorithmic applications. This is emphasised across theoretical chapters, where I demonstrate numerical analyses and graphical models with empirical testing. Information is also acquired from theories and relevant literature, which I analyse qualitatively.

However, within the adapted and standard deductive approach explained in preceding sections, my starting position is in the general master formulation with replicative mathematical function identities (price cut-off potentials). Thus I carry-out my research analysis, in subsequent theory and empirical chapters, in a structured manner as well as with articulated inferences from theories and other quantitative analysis (Saunders et al., 2009).

The research strategy provides a rough picture about how the research problem and

research hypothesis will be addressed. It also specifies the sources for the main mathematical expressions, the data and information acquisition and hindrances faced throughout this research, such as data access limitations, time constraints, economical, and ethical issues. Saunders et al. (2003) explain that research strategy is concerned with the overall approach one can adopt, while the tactics involve details such as data and information acquisition methods. This is the under-theme in my own consideration of the processes I deploy in this research. This includes the cluster of financial models that follow the master formulation with consistency in logic, algorithmic emulation of data, random data, and other relevant published data as well as the data analysis methodology.

There are several strategies that I can employ - deductive, inductive, or adopted. I have articulated in previous sections that I have adopted a deductive approach in this study. Strategies following deductive reasoning, emulatation of data, and other secondary data methodological processing, are well suited (and commonly used) in this type of research (Saunders et al., 2009; Creswell, 2013).

I acquire and use a considerable and varied amount of data to determine the relationships between the price-change variables and the interlink between \bigcirc and \bigcirc worlds in which price-related variables exist. Much of the discussion in my methodology review in centred on the deductive theory. My main goal is to develop a new theory through mathematical inferences using a postulated master-expression as a starting point, followed by financial model validation and asset pricing. I choose a language with identity and elemental attributes: variables; connectives, a highlevel mathematical symbolism, and expressions, in line with modern style writing and representation; the quantifiers; some primitive terms, etc., and in Tarski's style. Incidentally, the argument of using the correct language as discussed in Suppes (1988), and Givant (1991) bears also on Fodor's idea of a private language that he calls a language of thought. From this view one tries to mentally implement such language within some mental "machine"; the human storage and processing unit (the brain), which itself does not readily and in its natural form contain a criterion for the correct use of language, symbols, and expressions.

However, it can be normalised through a natural source of norms and symbols when addressing an object, which is expected to be implemented. As such it would no longer be a private language, but a language I use to address an object, which, in Fodor's hypothesis, I accept as the normalised language for use when addressing the object, in this case the mathematical and financial object (Burgess, and Rosen, 1997; Pylyshyn, 2002).

The choice of the language normalisation when treating a mathematical and/or financial "object" depends on the fact that must agree on what counts as correct in both application and proof. Nevertheless, such agreement does exist where the criterion for the use of language is the same when addressing objects of the same classification and is not

subjective. However, what is known to-date is subject to change either individually or collectively. The simplicity but also the expressive limit of this language consists in the fact that it allows quantification on individual variables only. Meanwhile, scholars will continue to generate new mathematical and financial theories. The introduction of new financial models contributes to new knowledge (Hersh, 1997; Benacerraf and Putnam, 1983).

According to Givant (1991), Smullyan (1961), Hausman (1990), and Burgess and Rosen (1997), the deductive apparatus of the formal theory is the calculus, serving the purpose of formalising and making explicit the deductive rules. They are implicit in the deductive theory I wish to formalise. The particular choice of the calculus does not affect the development of the theory. I simply assume that one of them is at my disposal. Thanks to the deductive apparatus of the formalised theory, every time a proof is produced, it can be inspected and checked for correctness. In this way proofs themselves become very precise objects, logical objects (Finey et al., 2000).

I try to include calculus as much as possible, to further establish the main theoretical model at a high abstraction level, where the various sub-theories link to. However, I certainly do not aim to include everything in it. One significant challenge is on how to interconnect the theories in a proper and justifiable manner. For instance, I consider in a form of probability description of an implied probability space, which I interconnect to other implied surfaces, such as the volatility surface (Finey et al., 2000).

The probability space deployed here is structured as a probability triple $\Pi = (\Omega, F, P)$, consisting of a non-empty set Ω , the sample space, a δ -field F of subsets of Ω and a probability measure P defined on F, in a non-empty set of subsets (events) of Ω with closed under taking complements: $A \in F$ implies that $Ac = \Omega \setminus A \in F$, with countable unions: $A_i \in F$, i = 1, 2, ..., implying that $\bigcup_{i=1}^{\infty} A_i \in F$ (Stein and Stein, 1991; Shephard, 1991; Madan et al., 1998).

Furthermore, I formulate around a general function $\Psi = (\Pi, R, \Gamma, E)$, whose dependency is on function structures Π , R, Γ , E; where the existence of a replicative function identity set $\Gamma = (\Pi, R, \Psi, E, Q)$ is possible and where $\Psi \in \Gamma$. Evidently both structures contain a substructure R, which is the structure of the stochastic value field within a spherical transformation. This helps to see that the interconnection between different theories is the intersection between the sets of numbers which represent the quantities assigned to the objects of the theories. In this study, one Ψ and Γ intersection occurs on R, however there is more than one intersection present, evidently on E and Π , where the latter is itself a triplet substructure (Kuchler and Tappe, 2008)

While more challenging, it allows for consideration on how the "world" must configure if all possible intersections of the theories are to be established. I also take into account relevant theory modifications. I recognise that much depends on the historicalphilosophical orientation. In fact, there are two opposite positions disputing over this point. One considers the science of mathematics behind finance to be a discontinuous enterprise and the other a cumulative one (Ladyman et al., 2007).

A central layer of this study's research framework is the time horizon. Saunders et al. (2003) provides good argumentation on the time-paradigm; cross-sectional vs longitudinal time frames. In the cross-sectional approach prominence is given to occurrences, with data and information outputs, around specific time-points (snap shot-alike). The longitudinal approach adds "movement" due to the ©-world's time-dimension of events and information, I find it reasonably fitting to adapt both time-frame positions in this work.

The "snap shot – like" or cross-sectional time frame, is best suited in the investigation of price change dynamics within the \bigcirc -world (price-point universe), whereas the longitudinal is best used when investigating price changes, financial instruments' liquidity, and forecasting over time, thus adding the time dimension in an appropriate measure and with relevance to the \bigcirc -world, which is the environment where market prices are fully observed.

The O-world is the quantized topological space, where price behaviour is developed prior to any price exhibition in the O-world. The consideration of both "worlds" combined, allows for a well-formed and holistic time-horizon perspective (crosssectional and longitudinal). Adam and Schvaneveldt (1991) explain in good detail the advantages of a longitudinal research especially when focusing on the development of a variable of an entire "world".

Longitudinal studies do come with limitations; time itself is a constraint. In crosssectional research, a certain phenomenon in a specific "world", such as asset price behaviour or rate of return can be re-defined in a reduced topological space (D-world). It is also observed longitudinally in the C-world. It allows one to further explore the D -world as a 3-dimension system; "real" time is diminished or put at a complete phase to the other dimensions under consideration (Saunders, 2004:2006; Bokulich and Jaeger, 2010).

This study explores the financial asset price-development within the two worlds under consideration - the ©-world and the D-world in a holistic manner (Saunders, 2004:2006; Bokulich and Jaeger, 2010); it further computes Eigen-prices and asset prices at a market-observed point by factoring-in dissipated longitudinal information filtrations at time nodes, along the market line. Such process requires leveraging of quantitative methods in good measure (Easterby-Smith et al., 1991; Saunders et al., 2003; Pelissier, 2008; Gulati, 2009).

Data collection and analysis are an important element in this study's research framework. Secondary data (documentary and test-complied) are acquired through various channels, including Bloomberg. Relevant trial data sets are generated through computer coding of existing and new theories (presented here), either as stand-alone programming solutions and/or integrated with external capabilities, such as NAG-routines. Price data cannot truly be observed in the D-world due to the very "nature" of such world; its reduced dimensionality and greater disparity with our own world (Bokulich and Jaeger, 2010).

However, I implement partial differential expressions through computer programs using adapt numerical methods to inference the D-world and generate asset pricing data, which is then emulated in the C-world with probable price path scenarios (Keller, 1992; Kloeden et. al., 1994; Saunders, 2004). It is absolutely necessary to test the new pricing model or test essential aspects of it, which is best done with secondary data. Moreover, I develop and test various cases on asset and financial derivative pricing where secondary data are utilised to a great extent. Use of secondary data is in-line with Saunders et al. (2003), and is referred in this study at times as documentary data and test-complied data.

The data collection process does not come without its limitations, ranging from the sample size, secondary data errors (including statistical), to research "bias" such as our choice of the "reflective" function identities in the GSE (General-Schrödinger-Equation), and the algorithms when emulating stochastically progressing price development paths. This is so, despite the inclusion of additional features such as shuffling and safeguards in our random –generating algorithm, which still falls short of emulating an absolute random measure or an absolute random congruent sequence, because simply such is impossible, but I do settle for improved algorithmic routines from those in existing literature such as

those in Kerningham and Richie (1988), and Flannery et al. (2002a:2002b) that I have adapted and extended here. Generally, the reliability and validity of data depend on methods used to collect the data, but also on the source of such data (Saunders et al., 2003; Gulati, 2009; Creswell, 2013).

4.0 AN ABSTRACT STOCHASTIC ASSET PRICING AND CONTIGENT CLAIM VALUATION FRAMEWORK WITH SHRÖDINGER PDE AUGMENTATION

I introduce the research problem as an abstract and probabilistic stochastic asset pricing and contingent claim formulation. I then solve it for various common cases that are in line with contemporary pricing models. The problem is a financial instrument valuation challenge, springing from our master expression, with subsequent Sturm-Liouville adapt solutions.

It provides the basis to incorporate price quantization effects at each point along the market line. Through the model, I add new valuation dynamics to existing asset pricing. The approach and results here are related to classical and contemporary work in quantitative asset pricing. However, I develop the theory within the literature gap, with consideration of the price function within time periods, while implicating orthogonality in the fitted probability distribution system at each price point along the market line.

4.1 INTUITION AND RATIONALE

The master expression related rationale is to address the main hypotheses; (1) that the continuously compounded price evolution effects are due in good part to underlying discrete time-space effects, described by dynamics within a quantized medium, and (2) that discrete price evolution can be explained within a continuous time-space medium. This is in line with information market hypotheses, however it develops the condition that price development reflects filtrations, dissipated at each point in the market line. Subsequently "tunneling" information through quantum "walls" from one zero-object onto the next along the market line. Thus, I consider a general equilibrium relationship, where the space domain is in phase with price effects in the time domain.

I am interested to broaden the treatment of the subject from existing approaches of hedging, market tracking, and self-financing strategies to a more generalised function, whose identity is both replicated and reflected within the equilibrium.

The starting point is to re-consider the highly irregular, continuous, paths of a standard Brownian motion $\{W_t, t \ge 0\}$, with the limitation that they are not differentiable; in addition, they are of unbounded variation on every finite time interval with probability 1. Recall that a function f: $[0, \infty) \rightarrow R$ is of unbounded variation on the interval [0, t] when

$$\sup_{P_n} \sum_{i=1}^n |f(t_i) - f(t_{i-1})| = \infty$$
 (1)

Where the supremum is taken over all finite partitions.

Evidently, construction of the stochastic integral is a non-trivial exercise. Although basic properties of stochastic integral can be established to simplify the process, I treat the problem at a partial differential level, without the need to pursue integration. Subsequently the problem has a solution in a PDE form - the Sturm-Liouville. The Sturm-Liouville PDE is itself a class of a problem, however for the postulate-implied problem, it serves well as a solution due to known numerical PDE methods available for such a problem-class (Bailey, 1966; Pruess, 1973; Pruess and Fulton, 1993; Pruess et al., 1995; Bailey et al, 1996; Kong and Zettl, 1996; Zettl, 1997; Kong et al., 2000:2001:2004; Agarwal and Wong, 1995; Tharwat et al., 2013; Zhang, et al., 2014; Yang et al., 2015).

Thus, within the scope of this research, integrals would not provide an extra helpful layer in the problem resolution. Where necessary numerical methods are fitted to solve the problem. This underlines the approach in this research work, although with properties established, working with ordinary integrals of stochastic processes would no longer be a challenge. These would be integrals of the form $\int_0^t Y_s ds$. For an adapt stochastic processs $\{Y_s, s \ge 0\}$ for which the integral is defined, this may be arranged in a differential notation; a stochastic differential equation of the form $dX_t = Y_t dt + Z_t dW_t$, which is shorthand for the statement that $\{X_t, t \ge 0\}$ is the process defined by

$$X_{t} = X_{0} + \int_{0}^{t} Y_{s} ds + \int_{0}^{t} Z_{s} dW_{s}, \text{ for } t \ge 0$$
 (2)

where X_0 is the initial position of the process. When working with stochastic integrals the principal difference from ordinary calculus is in the treatment of quantities of $(dW_t)^2$ order (Giller, 1982; Harrison and Pliska, 1981; Kotelenez and Curtain, 1982; Weidmann, 1987; Karatzas and Shreve, 1998a:1998b; Revus and Yor, 2004).

A differentiable function f(t) is an infinitesimal quantity of the same order of magnitude as dt. When W_t is a standard Brownian motion, the differential dW_t should be regarded as a stochastic infinitesimal quantity of order \sqrt{dt} , but $(dW_t)^2$ may be worked with as if it is the *deterministic* quantity db (Giller, 1982; Weidmann, 1987; Karatzas and Shreve, 1998a:1998b; Øksendal, 2000). To get a heuristic idea of why this is the case, we recall that an increment of Brownian motion $\Delta W_t = W_{t+\Delta t} - W_t$ is a random variable having a normal distribution with mean 0 and variance Δt ; then the increment may be represented as $\Delta W_t = \sigma \sqrt{\Delta t}$, where σ has the standard normal distribution with mean 0 and variance 1. By Chebychev's inequality we write

$$P(|(\Delta W_t)^2 - \Delta t| \ge) = P(\Delta t |\sigma^2 - 1| \ge) \le (\Delta t)^2 E(\sigma^2 - 1)^2 / \varepsilon^2$$
(3)

so that for any \in which is of larger order than Δt the right-hand side tends to 0 as $\Delta t \rightarrow 0$. The most important result in stochastic calculus for this study's purposes is *Itô's Lemma* (Ito, 1951; Doobs, 1953; Spitzer, 1970; Snyder and Miller, 1991; Seneta, 1996; Parzen, 2015).

THEOREM 1.0: Suppose that $\{X_t, t \ge 0\}$ is a stochastic process that may be represented as $dX_t = Y_t dt + Z_t dW_t$ and that f(x,t) is a function with continuous second partial derivatives. The stochastic process f(X_t,t) may be represented as

$$df(X_t, t) = \left(Y_t \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}Z_t^2 \frac{\partial^2 f}{\partial x^2}\right)dt + Z_t \frac{\partial f}{\partial x}dW_t$$
(4)

where the partial derivatives are evaluated at (X_t, t) .

To appreciate the difference from the deterministic case, when x_i , y_t , z_t and W_t are deterministic and linked by $dx_t = y_t dt + z_t dw_t$, then

$$df(x_t, t) = \frac{\partial f}{\partial x} dx_t + \frac{\partial f}{\partial t} dt = \left(y_t \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \right) dt + z_t \frac{\partial f}{\partial x} dw_t \qquad (5)$$

In the stochastic case, the extra term $\frac{1}{2}Z_t^2 \frac{\partial^2 f}{\partial x^2}$ on the right-hand side is picked up. While I will not provide proof of Itô's Lemma here, an understanding can be acquired from the use of Taylor's Theorem, such that

$$\Delta f(X_t, t) = f(X_{t+\Delta t}, t+\Delta t) - f(X_t, t) = \frac{\partial f}{\partial x} \Delta X_t + \frac{\partial f}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (\Delta X_t)^2 + \frac{\partial^2 f}{\partial x \partial t} \Delta X_t \Delta t + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (\Delta t)^2 + \cdots \quad (6)$$

But, $\Delta X_t = Y_t \Delta t + Z_t \Delta W_t$ and by the explanation above $(\Delta X_t)^2 = Z_t^2 \Delta t + o(\Delta t)$, with terms like (ΔX_t) (Δt) being $o(\Delta t)$, so that they are of smaller order than Δt . If only the terms of order no smaller than Δt are retained, this becomes

$$\Delta f(X_t, t) = \frac{\partial f}{\partial x} \Delta X_t + \left(\frac{\partial f}{\partial t} + \frac{1}{2} Z_t^2 \frac{\partial^2 f}{\partial x^2}\right) \Delta t \quad (7)$$

which gives the correct expression when one substitutes for ΔX_t . A formal proof of Itô's Lemma requires use of Taylor's Theorem along the lines of the above and the definition

of the stochastic integral as a limit over approximating $Z_t(P_n)$ (Karlin and Taylor, 1975:1981; Malliaris, 1982; Øksendal, 2000; Bru et al., 2009; Bru and Bru, 2018).

COROLLARY 1.0: Suppose that $\{X_t, t \ge 0\}$ is a stochastic process that may be represented as $dX_t = Y_t dt + Z_t dW_t$. For (suitably nice) functions f(x, t) and g(x, t), the stochastic differential d(fg) is given by

$$d(fg) = fdg + gdf + Z_t^2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} dt \quad (8)$$

where f, g and the partial derivatives are evaluated at (X_t, t) .

PROOF:

I first set F = fg, then apply Itô's Lemma to F to obtain

$$dF(X_t, t) = \left(Y_t \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + \frac{1}{2}Z_t^2 \frac{\partial^2 F}{\partial x^2}\right)dt + Z_t \frac{\partial F}{\partial x}dW_t \quad (9)$$

I then substitute in

$$\frac{\partial F}{\partial x} = f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x}; \quad \frac{\partial F}{\partial t} = f \frac{\partial g}{\partial t} + g \frac{\partial f}{\partial t}; \quad \frac{\partial^2 F}{\partial x^2} = f \frac{\partial^2 g}{\partial x^2} + g \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x}$$
(10)

then

$$dF(X_t, t) = \left(Y_t \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + \left(f\frac{\partial g}{\partial t} + g\frac{\partial f}{\partial t}\right) + \frac{1}{2}Z_t^2 \left(f\frac{\partial^2 g}{\partial x^2} + g\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right)\right)dt + Z_t \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)dW_t \quad (11)$$

and re-arrange and group terms to produce the result.

COROLLARY 2.0: Suppose that $\{W_t, t \ge 0\}$ is a stochastic process both of whose differential and integral representations exist. Then the stochastic integral $\int_0^t W_s dW_s$ is given by

$$\int_{0}^{t} W_{s} dW_{s} = \frac{1}{2} (W_{t}^{2} - t)$$
 (12)

PROOF:

Suppose that the integral equals $f(W_t, t)$ by Itô's Lemma

$$W_{t}dW_{t} = df(W_{t}, t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^{2} f}{\partial x^{2}}\right)dt + \frac{\partial f}{\partial x}dW_{t} \quad (13)$$

which gives

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0 \text{ and } \frac{\partial f}{\partial x} = x \quad (14)$$

from which I see that

$$\frac{\partial^2 f}{\partial x^2} = 1; \frac{\partial f}{\partial t} = -\frac{1}{2} \qquad (15)$$

Integrate to get f = -t/2 + g(x), with g'' = 1 so that $g(x) = x^2/2 + c$; because f(x, 0) = 0 we see that c = 0. It follows that

$$\int_{0}^{t} W_{s} dW_{s} = \frac{1}{2} (W_{t}^{2} - t) \qquad (16)$$

I consider here a form of probability description of either an observed or implied probability space, which I interconnect to other implied surfaces such as the volatility surface (Gnedenko and Kolmogorov, 1954; Neveu, 1965:1975, Milnor, 1968; Loeve, 1977; Heson, 1993; Billingsley, 1995; Schobel and Zhu, 1999; Saonov, 2001; Gulisashvili and Stein, 2010; Keller-Ressel, 2011; Grasselli, 2017).

The probability space considered here is structured as a probability triple, represented as (Ω, F, P) , consisting of a non-empty set Ω , the sample space, a δ -field F of subsets of Ω and a probability (measure) P defined on F, in a non-empty set of subsets (events) of Ω with closed under taking complements: $A \in F$ implies that $Ac = \Omega \setminus A \in F$, with countable unions: $Ai \in F$, i = 1, 2, ..., implying that $\bigcup_{i=1}^{\infty} A_i \in F$ (Doobs, (1953; Spitzer, 1970; Snyder and Miller, 1991).

I seek to recreate the concept in the new context to include the fact that, in an expanding degrees of freedom representation, I seek some form of pattern in the under-domain curvature in the probability-implied surface; this is more obviously represented over a martingale and in implied volatility surfaces driven by market prices (Hull and White, 1987; Stein and Stein, 1991).

In considering uncertainty and the ways it impacts on asset prices, emphasis are placed on the measure of the outcome of such uncertainty, through random variables, the properties of which are of great interest and are used in this work. The incorporation of a random variable in a pricing function is fully justified, where the sample space is probability-implied by a non-empty set of events or filtrations, observable in real terms and whose expectation is a well-defined and measurable random variable x in a function $x : \Omega \rightarrow R$ measurable with respect to F (Malliaris, 1982; Øksendal, 2000, Bru et al., 2012; Bru and Bru, 2018; Khrennikov, 2018).

The event $(x \le c) = (\Omega : x(\Omega) \le c) \in F$ for all real numbers $c \in R$, and the smallest δ -field with respect to which a random variable χ is measurable, is the δ -field generated by x, which is denoted $\delta(x)$ and the expectation for a non-negative random variable x, is defined as $E(x) = \int x(\Omega) dP$, which may be $+\infty$ (Levy 1925; Billingsley, 1954, Gnedenko and Kolmogorov, 1954; Hazewinkel, 2001; Bernard, 2007), Shiryaev et al., 2006).

I have adopted the usual convention for the relations between two random variables, x, y, under the condition $x \le y$, or x =y which without any further qualification may be taken to hold with probability 1, or represent an "almost certain" event. The uncertainty, random variable representation and the underlying stochastic process representations need further clarity in terms of how one would obtain augmented stochastic variables. Doobs (1953), Malliaris (1982), Øksendal (2000), and Bru et al. (2012) provide a good background on arbitrary random variables. An arbitrary random variable x can be obtained as the difference of two non-negative random variables, x = x + -x - where $x + = \max(x, 0)$ and $x - = \max(-x, 0)$ under the provision that at least one of E(x +) and E(x -) is finite and defines E(x) = E(x +) - E(x -) to be finite. Moreover random variable

x is integrable when $E |x| < \infty$, that is when both E(x+) and E(x-) are finite.

Stochastic financial variables represent randomizer effects that are attributed filtration diffusion processes, although one would wish to identify the deterministic component of a process of this type, the non-deterministic part will be a source of contribution to the random measure. Benaim and Raimon (2003) investigated convergence properties of self-interacting diffusion on a compact Riemannian manifold, and considered self-interacting diffusions to be continuous time stochastic processes living in a Riemannian manifold M, and defined the process mathematically through the use of a "family" of Brownian motions, smooth vector fields, and a potential-like function (Doobs, 1953; Fox, 1962; Karlin and Taylor, 1981; Van-Buskirk, 1985; Shafer, 2002; Benaim et al., 2002; Sheldon, 2003; Hunt and Kennedy, 2004; Gillespie, 1996; Schöbel and Zhu, 1999; Baaquie et al., 2003; Piotrowski and Sladkowski, 2005; Nastasiuk, 2015; Fiorin et al., 2018).

This is supported by other authors (Malliaris, 1982; Karatzas and Shreve, 1998b; Øksendal, 2000; Bru et al., 2012; Doobs, 1953; Lamberton and Bernard, 2007; Shiryaev et al., 2006), who more specifically consider the conditional expectation E(x | g) to only be defined up to sets of probability 0 and for any random walk variable x for which the unconditioned expectation E(x) is defined. This is also extended in validity in cases of chained conditioning of such expectation, or the conditional form of Jenson's inequality that $f(E(x | g)) \leq E(f(x) | g)$ for a convex function $f : R \rightarrow R$ when f(x) is integrable; the inequality is reversed when f is a concave function. For a convex function operational in a real and measurable system, such as pricing and the effects of a bond's term structure. Moreover for two random variables where one is conditioned and measured in relation to that very condition, the conditioned expectation of the two random variables' product, would relate the condition dependency of the originally conditioned random variable and pass on the condition dependency to the first random variable. More specifically, we extend that argument for any x for which E(x) is defined and by taking $A = \Omega$ that E(E(x | g)) = E(x) (Merton 1974:1976; Jarrow and Turnbull, 1998; Parzen, 2015).

For a random variable x and g-measurable random variable y for which both E(x) and E(xy) are defined, then E(xy | g) = E(x | g)y, which shows that when y is g-measurable it may be treated effectively as a constant when conditioning on g and taken outside the conditional expectation. In other words, given g, y is known (Lamberton and Bernard, 2007; Parzen, 2015;).

In the case of two δ -fields g and h, with $g \subseteq h \subseteq F$, E(E(x | h) | g) = E(x | g), according to what is often known as the tower property of conditional expectations. The conditional form of the random variable x is independent of the δ -field g when $\delta(x)$, the δ -generated by x, and h are independent δ -fields; in this case E(x | g) = E(x). The δ -fields g and h are independent when $P(G \cap H) = P(G)P(H)$ for all events $G \in g$ and $H \in h$ (Karatzas and Shreve, 1998b; Gillespie, 1996; Schöbel and Zhu, 1999; Shiryaev et al., 2006; Parzen, 2015).

These effects are applicable in pricing structured products (custom investment products of broad use), both growth based and yield enhancement products, particularly in cases where a bond is combined with vanilla options, or barrier options. Moreover in cases when spread option strategies are combined with a fixed income instrument making use of two or more random variables, where at least one variable is conditioned, while considering all filtrations and probable event indicators, subsequently effecting price development (Jackwerth and Rubinstein, 1996; Jarrow and Turnbull, 1998; Falloon and Turner, 1999; Batten et al., 2004; Shiryaev et al., 2006; Blumke, 2009; Palmer, 2010; Parzen, 2015; Ho and Lee (2015).

It is in this context that I define the asset price function, contingent claims and rate of returns, as random variables defined on the underlying probability space. In the case of a contingent claim or financial derivatives, it may be regarded as the payoff at time 1 of some contract; the value of c is not observed until time 1. The ubiquitous example to keep in mind is a call option at some strike price c on one of the risky assets, asset 1 say; this would pay (S1, 1 - c) + at time 1 (Karatzas and Shreve, 1998b; Shiryaev et al., 2006; Kanniainen and Piché, 2013).

I also consider an extension to sets of random variables as uncountable collections with one random variable representing the essential supremum of the collection, $\{x\gamma : \gamma \in \Gamma\}$, the quantity sup $\gamma \in \Gamma x\gamma$ may not necessarily be a random variable; however, there exists a random variable y, known as the essential supremum of the collection and written as $y = \text{ess} [\text{sup}\gamma \in \Gamma (x\gamma)]$ which has the properties (i) $y \ge x\gamma$ for all $\gamma \in \Gamma$; and (ii) if z is a random variable with $z \ge x\gamma$ for all $\gamma \in \Gamma$ then $z \ge y$ (Karatzas and Shreve, 1998b; Gillespie, 1996; Shiryaev et al., 2006).

Sondermann (2007) defines a filtration { F_t , $0 \le t \le n$ } to be an expanding sequence of δ fields $F \ 0 \subseteq F \ 1 \subseteq ... \subseteq F \ n \subseteq F$, under the supposition that (Ω , F, P) is the underlying probability space, and explains that when one makes observations of a process over times t = 0, 1, ..., n, F_t can be thought of as the cumulative information available at time t with the δ -fields F_t corresponding to the increasingly finer partitions of the sample space Ω . A sequence of random variables { x_t , $0 \le t \le n$ } is adapted to the filtration { F_t } when x_t is F_t - measurable for each t = 0, 1, ..., n; intuitively, this means that when the information in F_t has been observed, the value of x_t is known.

Jacob and Shiryaev (2003) provide a detailed definition of integrable random variables $\{\chi_t, 0 \le t \le n\}$, which according to them represent a martingale (relative to a given filtration $\{F_t, 0 \le t \le n\}$ and a probability P) if the sequence is adapted and for $E(x_t + 1 | F_t) = x_t$ for all $0 \le t < n$. Similar definitions may be provided for sub-, and super - martingales (Doobs, 1953; Feller, 1971; Fuk and Nagaev, 1971; Hull, and White, 1987; Stein and Stein, 1991; Siminelakis and Paris 2010; Parzen, 2015).

A stopping time (relative to the filtration $\{F_t, 0 \le t \le n\}$) is a random variable taking values in the set $\{0, 1, ..., n\}$ such that the event $(T \le t) \in F_t$ for each t = 0, 1, ..., n. This relation
is equivalent to requiring that the event $(T = t) \in F_t$ for each r. Intuitively, a stopping time is a rule which tells us when to stop, based only on knowing the history up to the instant of stopping; that is, it does not look into the future. We note that if τ and T are stopping times then $\tau \wedge T = \min(\tau, T)$ and $\tau \vee T = \min(\tau, T)$ are also stopping times; in particular, $T \wedge t$ is a stopping time when t is a constant time. For a stopping time T the σ -field \mathcal{F}_T is defined to be the set of those events $A \in F$ such that $A \cap (T \leq t) \in Ft$ for each t = 0, 1, ..., n. It is straightforward to check that F_T is a δ -field and it should be noted that it represents the information available through observing the history up to the stopping time T (Doobs, 1953;; Stein and Stein, 1991; Parzen, 2015).

Furthermore, when a sequence of random variables $\{x_t, 0 \le t \le n\}$ is adopted then the random variable x_T is F_T -measurable. When τ and T are stopping times with $\tau \le T \le n$ and $\{x_t, 0 \le t \le n\}$ is martingale, then we have $E(x_T | F_{\tau}) = x_{\tau}$, which shows that the martingale property is preserved at stopping times; this is known as the Optional Sampling Theorem (Doobs, 1953; Parzen, 2015; Hull, and White, 1987; Stein and Stein, 1991; Lamberton and Bernard, 2007).

It may be deduced that for any stopping time T, the sequence $\{x_T \land t, 0 \le t \le n\}$ is a martingale. When $\{x_t, 0 \le t \le n\}$ is a sub-martingale, one may use the inequality $E(x_T | F_{\tau}) \ge x_{\tau}$, and in the case of a super-martingale by $E(x_T | F_{\tau}) \le x_{\tau}$, note that a sequence of random vectors $\{xt, 0 \le t \le n\}$ taking values in R_{τ} with $x_t = (x_1, t, ..., x_{\tau}, t)T$, is a martingale relative to the fixed filtration and probability if each coordinate sequence $\{xi, t, 0 \le t \le n\}$ is a martingale, i = 1, ..., s (Fuk and Nagaev, 1971; Stein and Stein, 1991; Hira and

Altinisik, 2014; Parzen, 2015).

A collection { F_t , $0 \le t < \infty$ } of sub δ -field of F is a filtration when $F_\tau \subseteq F_t$ whenever $0 \le \tau \le t$. Furthermore, a stochastic process { x_t , $0 \le t < \infty$ }; that is, a collection of random variables indexed by $t \ge 0$) adapted to the filtration { F_t } is a martingale when x_t is integrable for each t and $E(x_t | F_s) = x_\tau$ when $0 \le \tau \le t$. I assign a general meaning to the stopping time in this study - the zero-time of pricing (or trading) interest in which a quantifiable $\hat{\mathbb{O}}_{\downarrow\uparrow}$ - system can be fitted (Doobs, 1953; Hull, and White, 1987; Stein and Stein, 1991; Lamberton and Bernard, 2007; Hira and Altinisik, 2014; Parzen, 2015).

In addition to the scaling property, a Brownian motion has independent increments with future price value dependent not only on the present price value, but also on the entire history up to the stopping time which itself implies a stronger than Markov property, and moreover, the price displacement between two time points is independent of the position at the first point (the first of any two time points). It allows one to obtain a well-defined expression for the transition probabilities, the probability density function of the normal distribution, and a joint probability density function (Doobs, 1953; Spitzer, 1970; Snyder and Miller, 1991; Seneta, 1996; Parzen, 2015).

Brownian motion pathsare continuous. With probability of one, these paths are not differentiable. Boobs (1953), Spitzer (1970), Snyder and Miller (1991), Jacob and Shiryaev (2003), and Sondermann (2007), consider some properties of the Brownian

motion process using elementary arguments; many of these results may be obtained more easily using the machinery of martingale theory, but they argue that it is instructive to get a feel for working with Brownian motion from first principles. By the symmetry of the normal distribution it is immediate that when $\{W_t, t \ge 0\}$ is a standard Brownian motion then $\{-W_t, t \ge 0\}$ is again a standard Brownian motion. Also, when $s \ge 0$ is any fixed time $\{W_t + s - W_s, t \ge 0\}$ is a standard Brownian motion. According to them, what is also true is that for certain random times T, called stopping times of the process, $\{W_t + T - W_T, t \ge 0\}$ is again a standard Brownian motion and is independent of the process $\{W_s, 0 \le s \le T\}$.

The information available at time t is the history of the price process, $\mathcal{F}_1 = \sigma$ (S_u, $0 \le u \le t$), that is the information obtained by observing the movements of the stock price process up to time t; equivalently, it is $\sigma(W_u, 0 \le u \le t)$, the information obtained by observing the driving Brownian motion in the stochastic differential equation (Giller, 1982; Weidmann, 1987; Øksendal, 2000).

4.2. THE MODEL

In this section, I present the new abstract and stochastic valuation model. It is presented as the master formulation. The model is applied to special cases, leading to probability density function variations, and asset pricing. Where necessary the classic concepts and frameworks that have contributed much to existing pricing theories are blended in here for comparison purposes to the new model; in part this will give weight to the validity of the new abstract model. From the centre-piece formulation to each asset-pricing model, case, and scenario, I make use of at least one transformable stochastic process and differentiable function.

4.3 PROPOSAL PROBABILITY DENSITY AND ASSET PRICING PDEs

I seek to formulate the problem in partial differential form, present it as a master expression, and subsequently fit a solution without involving integration. This intuition is based on the concepts in the above sections, however I contemplate the inclusion of the phase factor which is not evident in the Taylor's expression. I stipulate a compact PDE formulation that takes on relevant and significant effects; a framework from where known asset pricing models can be derived with varying assumptions. After many trials and tribulations, I have finalised it as a postulate of the form

$$\forall q \in \{1, \dots, Q\}, \forall m \in \{0, \dots, M\}, \forall n \in \{0, \dots, N\} \left\{ \left[\left(\frac{1}{2} \frac{\partial^2}{\partial \chi^2} - \gamma(\chi, t) + (i^n)^m \frac{\partial}{\partial t} \right) f(\chi, t) \right]^q \right\} = 0$$
 (17)

This master expression represents a general equilibrium. It incorporates two sets of effects - the continuous and discretised price related space - time effects, which balance each other out. It is also relatively easy to see that the two effects can be treated as additive with small adjustments to the expression and lowering of the power q term. "Space" here will refer to the zero-dimensional topological space or quantum price depth. In contingent claims, this will include also the depth of "moneyness".

At q=1, I generate a sufficient formulation to explain most of the current asset pricing formulations. This simplifies the expression to

$$\forall m \in \{0, \dots, M\}, \forall n \in \{0, \dots, N\} \left[\left(\frac{1}{2} \frac{\partial^2}{\partial \chi^2} - \gamma(\chi, t) + (i^n)^m \frac{\partial}{\partial t} \right) f(\chi, t) \right] = 0 \quad (18)$$

This is significantly relevant, because it does include the second order effects to match the equivalent Taylor expansion around f(x, t). Furthermore it also includes the phase factor. Of the two effects, one has to be the imaginary term. If one concentrates on the market price timeline, then the market depth is in phase with it and subsequently with the imaginary effect, and vice versa.

Equation (18) appears to be a generalised version of Schrodinger's equation (GSE). I refer to it in this work as GSE or the master expression, interchangeably. In its most simple representation, it has been previously used to study the behaviour of microscopic systems within a branch of quantum mechanics known as wave mechanics (Bailey, 1966; Barrett, 1999). My intuition has led to the confirmation of Schrodinger's equation to deal adequately with the research problem in asset pricing in this study. However, I make no use of quantities with physical attributes in its original form.

This is contrary to the works of Chen (2001:2003), Haven,

(2002:2004:2005:2008a:2008b), and other contemporary research work in the domain. Such relaxation of variables is in line with own intuition and supported by previous cases such as the option pricing models of Black and Scholes (1973), and Sprenckle's (1961). Sprenckle had derived an option pricing model prior to Black and Scholes. However he included parameters of partial relevance, whose measurability proved problematic in a purely financial context. These parameters were set to one on the Black and Scholes model and were assigned no meaning. Black discusses this in detail in his 1989 paper. I subsequently refer to new asset pricing model as the generalised Schrödinger equation (GSE) or more generically as the master equation. I proceed to treat the use of GSE in asset pricing through a sequence of models, cases, and specific scenarios.

Within the theoretical framework, I make use a system of assumptions, updated throughout this work and in various sections to support the model augmentations for pricing purposes. The initial assumptions are (i) $\{\chi_t, t \ge 0\}$ is a stochastic process, (ii) $f(\chi,t)$ is a composite function with an identity characterised by "memory-less-ness", and (ii) $\{\chi_t, t \ge 0\}$, and $f(\chi,t)$ have continuous second partial derivatives of a generic form (eq. 17).

The q power takes any integer values from 1 onwards, although this is built in the GSE to include higher order terms, in actuality q=1 is sufficient to draw a simpler formulation that allows one to compile the PDE versions of the contemporary asset pricing models, under sufficient terms' inclusion and significance. Under q=1, equation (17) takes a simpler form:

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \gamma(\chi, t)\right)f(\chi, t) = (i^n)^m \frac{\partial}{\partial t}f(\chi, t) \qquad (19)$$

The term $(i^n)^m$ on the right hand side of Equation (19) can be treated in two probable ways; the first approach is subjectively intuitive, and is explained below, whereas the second approach blends power integers $(i^n)^m \equiv i^s$, where *s* is linked to the quantized rate-of-return system i.e. the price change behaviour at quantized levels. The second approach is evident at the later stages, but either-way their effects are not conflictual with each other in the theory put forward here.

In the first approach, I consider a power coefficient *n* with discrete numerical values of 0, 1, 2, etc., to be dependent on the identity of function $f(\chi, t)$ and whether $f(\chi, t)$ measures the probability density function, or more precisely the square root of it; 0 if it does not and 1 if it does. If the probability density function has a dual outlook and is treated jointly for a financial security and its protection⁷³, then *n*=2, etc...

The *m* coefficient takes integer values, 0, 1, 2, etc., is associated with the identity of function f (χ , t), more precisely the number of constituent pricing functions.; (a) If the composite function f (χ , t) is a pricing function and its relations or constituent functions are financial instruments such as financial securities, financial derivatives, etc., then *m*=1,

⁷³ In the context of investments, terms such as protection, insurance, contingent claim and financial derivative are used interchangeably.

if there is one and the corresponding number of 2, 3, etc., if there are 2, 3, etc. constituent pricing functions. If $f(\chi, t)$ measures the square root of the probability density function and contains only one pricing constituent function, then $(i^1)^1 = i = \sqrt{-1}$, and if the function to be established is the square root of the probability density, then n = 1, thus $(-1)^{\frac{1}{2}} = \sqrt{-1}$, subsequently the right hand side of the equation becomes a phase factor.

In simple terms, if the function f (χ , t) contains one asset pricing constituent function in its composition, m=1; if there are two different constituent functions of a pricing nature, then m=2, and the number is increased linearly with the number of constituent functions. The γ (χ , t) function is central and reflective of the identity of composite function f (χ , t) and its relations to the constituent function(s). The precise nature of the γ (χ , t) function depends on the identity of function f (χ , t) and will be established for each case considered here. This is different from the interpretations given in existing literature, including works of Segal and Segal (1998), Haven, (2002:2004:2005:2008a:2008b), Haven and Khrennikov (2016), etc., that have been reviewed in preceding chapters.

4.3.1 CASE 1.0 : TIME - INDEPENDANT GSE

From the initial assumptions, I also consider that it is possible to consider an additional assumption - (iv) function $f(\chi,t)$ can be de-composed by means of variable separation, such that $f(\chi,t)$ is transformed and represented in a time-independent differential form, as shown here below:

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \kappa\eta(\chi)\right)\psi(\chi) = \alpha\psi(\chi) \qquad (20)$$

Where $\alpha = (i^n)^m \left[\frac{1}{\xi(t)} \frac{\partial \xi(t)}{\partial t}\right]$ is a constant and η is a time independent function with an observable identity. Solution provided in Appendix II – Case 1.0.

4.3.2 CASE 2.0 - CAPITAL ASSET PRICING PDE

Inline with assumption (i) with respect to $\{\chi, t \ge 0\}$, I consider $\{X, t \ge 0\}$ also to be a stochastic processes with known transformation relationships between them. By the current system of assumptions, $\psi(\chi)$ function is a time-independent function with (v) continuous second partial derivatives, (vi) representing an important part of the square root of the probability density function, (vii) $\xi(t)$ is a time dependent function with continuous derivatives, representing a risk-free asset price function. The system of seven assumptions is sufficient to establish a generalised capital asset pricing PDE expression in equilibrium.

I consider the augmented equation (20) and assume (viii) an economy that contains at least one risk free investment with a rate of return r_f , and a risky investment (i.e. equity) with a rate of return of μ and risk measured through its standard deviation, σ . These are risk and return concepts described in the works of Markowitz (1952), Sharp (1964), Treynor. and Black (1973), Bodie et. al. (2009), and Elton and Gruber (2011), typified in a classical two-fund separation problem. Starting with the expression (20), where α is defined previously.

Although the constant can take any value, I consider the special case that leads to the known stochastic risk-free asset price function, and so I define

$$\left[\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right] = r_f \quad \Longrightarrow \quad \xi(t) = \xi(0) e^{(r_f t)} \qquad (21)$$

In this case, function $\psi(\chi)$ is a constituent function of $f(\chi, t)$ in the GSE and retains the square root of the probability density function, thus n=1. I also define $\chi = (X - \mu)/\sigma$, in line with Gurajati and Porter (2010), where both $\{\chi, t \ge 0\}$ and $\{X, t \ge 0\}$, are themselves both random variables, the identities of which are well explained in Daughterly (2011). Furthermore, Berenson et al. (2012) provide empirical cases of the random behaviour of such variables. These are observables, where the X represents a stock's rate of return variable, and χ a standardized statistical measure. Using the expression for χ , I obtain its partial derivative $\partial \chi/\partial X = 1/\sigma$ and $(\partial \chi)^2 = \sigma^{-2}(\partial X)^2$, which I substitute back in equation (20) to obtain

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial X^2} + \kappa\eta(X)\right)\psi(X) = \alpha\psi(X) \qquad (22)$$

The existence of price change behaviour identities (i.e. the explicit risk-free and implicit risky asset pricing functions), makes it possible to establish the value m=2, subsequently

$$\alpha = (i^1)^2 \left[\frac{1}{\xi(t)} \frac{\partial \xi(t)}{\partial t} \right] = i^2 r_f = \left(\sqrt{-1} \right)^2 r_f = -r_f \qquad (23)$$

It follows that

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial X^2} + \kappa\eta(X)\right)\psi(X) = -r_f\psi(X) \qquad (24)$$

which can be expressed in the following compact form:

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial X^2} + r_f + \kappa\eta(X)\right)\psi(X) = 0 \qquad (25)$$

4.3.2.1 SCENARIO 1.0

With the existing system of assumptions, I consider $\{r, t \ge 0\}$ to be a stochastic process, where $\psi(r)$ function is a time-independent function with continuous second partial derivatives representing an important part of the square root of the probability density function, and a two-fund separation problem of a portfolio constructed with a risk-free asset (i.e. a bank account, or a treasury bill) with return r_{f} , and an equity account with a rate of return of r . I further assume that (viii) market tracking and portfolio factor-loading are possible.

Such concepts are discussed in good detail in various sources. Hillier et al. (2011) articulate the concepts of factor reduction and portfolio factor loading, in that while there might be more than one economic and financial factor that can determine the return on an equity account, it is also possible to factor-load the portfolio around one significant factor at a time. This is based in part on the work of Christoffersen et al. (2009). The number of factors matches the degrees of freedom in the price behaviour system and can only partially serve as inference to market price dynamics and can be used to measure

investor's expectations.

Consider equation (25) where $X \equiv r$ to obtain

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + r_f + \kappa\eta(r)\right)\psi(r) = 0 \quad (26)$$

and taken k and $\eta(r)$ to be two matrices with *n* x *l* and *l* x *n* dimensions, respectively. Then equation (26) can be expressed as

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + r_f + \kappa_{[nx1]}\eta_{[1xn]}(r)\right)\psi(r) = 0 \qquad (27)$$

Such that

$$\kappa\eta(\mathbf{r}) = \kappa_{11}\eta_{11} + \kappa_{21}\eta_{12} + \kappa_{31}\eta_{13} + \dots + \kappa_{n1}\eta_{1n} = \beta_{11}F_{11} + \beta_{21}F_{12} + \beta_{31}F_{13} + \dots + \beta_{n1}F_{1n}$$
(28)

Where the variables are simply re-labled to facilitate an easier recognition of the terms as those on the known multi-index capital asset pricing model. This can be set to represent the rate of return, r, of the equity account. The equity account can be conceptualised as an equity portfolio following a multifactor model, where pure-factor portfolios and subsequently momentary arbitrage are observables (Ross, 1976; Ross and Roll, 1980; Conner and Korajczyk, 1995; Delbaen and Schachermayer, 2006; Hillier et al., 2011). This allows us to simplify expression (27) to

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + r\right)\psi(r) = 0 \qquad (29)$$

Consider that a pure factor portfolio similar to those articulated by Hillier et al. (2011) can be engineered and assume that only factor η_{11} can be tracked, then equation (27), can be simplified to

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + r_f + \kappa_{11}\eta_{11}(r)\right)\psi(r) = 0 \quad (30)$$

One can intuitively pick up the term $\kappa_{11}\eta_{11}$, set it to be βr_m , where r_m is the market rate of return, and subsequently obtain

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + r_f + \beta r_m\right)\psi(r) = 0 \qquad (31)$$

It is easy to see that the expression under the normal brackets in equation (31) represents the PDE form of the capital asset pricing model for the rate of return, where r_m is the rate of return of a market-tracking portfolio. This is a differential expression that can be solved relatively easy. Further compare the expression within brackets on the left hand side of the equation (31) with the classical form of CAPM

$$-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + r_f + \beta r_m = r_f + \beta r_m + \epsilon \qquad (32)$$

to obtain an expression for the stochastic residual effect

$$\varepsilon = -\frac{1}{2}\sigma^2 \frac{\partial^2}{\partial r^2} \qquad (33)$$

Expression (33) is an important finding because the residual effect in the classical capital asset pricing model is associated with real time effects from unanticipated future events. Whereas the right-hand side represents the quantization effect within the quantum box. In equilibrium the two are equal. One can draw from this that the cumulative effect of

filtrations or present and past events when dissipated within the zero-object are fragmented and exist within new quantized patterns. This leads to conclusion that future prices are decided by the present state of the price. In-line with the market efficiency hypothesis the spot price reflects all relevant past and present events (Asquith, 1983); Bachelier, 1990; Bernard and Thomas, 1990; Bodie et al., 2009; Khrennikov, 2018). This is an exhibition of the Markov property (Markov, 1971; Mura and Swiatczak, 2007; Busemayer et al., 2009; Fiorin et al., 2018) within this study's conceptual framework.

4.3.2.2 SCENARIO 2.0

One can similarly treat the same elements in the alternative scenario, where the equity account is replaced by a fixed-income account. Elton and Gruber (2011) explore the single and multi-index models for bonds, under the assumptions of expectation theory and consider the total return of the portfolio to be the sum of (i) the expected return (i.e. expectation theory), (ii) return due to an unanticipated shift in the yield curve, and (iii) the stochastic residual return term, expressed as

$$r_i = E(r_i) + \beta_i [r_m - E(r_m)] + e_i$$
 (34)

where e_i is independent of the bond index, and $\beta_i = D_i/D_m$; D_i and D_m are durations of the bond and market index, respectively. Derivation of equation (34) is evident in great detail in Elton and Gruber (2011). I can now re-write the equation (29) as

$$\left\{-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + \left[E(r) + \frac{D}{D_m}[r_m - E(r_m)] + e_i\right]\right\}\psi(r) = 0 \quad (35)$$

where subscript i is dropped for simplicity. Expression (35) is simplified when

considering the equality of equation (33). More specifically (35) becomes

$$E(r) + \frac{D}{D_m}[r_m - E(r_m)] = 0$$
 (36)

Expression (36) appears to be free of quantization effects. However this is related to several issues in equation (34). There are two specific issues with the expression (34); (a) it ignores the convexity effect due to the curvature on the bond yield, and (b) it ignores the multifactor effects and sensitivities often regressed out of data in common practice. The first issue can be relaxed under the assumption that (ix) a preference-driven investor expects bond price changes to be small, an argument articulated in greater depth by Elton and Gruber (2011), although the expression could be generalised to include the convexity effect, whereas the second issue can be resolved by considering expression (35), with $\alpha = E(r)$.

When I consider 3x degrees of freedom in the development dynamics of the rate of return, then the operator can be generalised as

$$\nabla^2 = \left(\frac{\partial^2}{\partial r_{\rm dim1}^2} + \frac{\partial^2}{\partial r_{\rm dim2}^2} + \frac{\partial^2}{\partial r_{\rm dim3}^2}\right) \qquad (37)$$

Using the operator expressed in (37), equation (29) is generalised with extended degrees of freedom. The expression takes the form.

$$\left(-\frac{1}{2}\sigma^2\nabla^2 + r\right)\psi(r) = 0 \qquad (38)$$

$$\nabla^2 \psi(\mathbf{r}) = 2\left(\frac{\mathbf{r}}{\sigma^2}\right)\psi(\mathbf{r})$$
 (39)

Subsequently it allows me to obtain the observable expression for the operator:

$$\nabla \equiv \pm \frac{1}{\sigma} \sqrt{2r} \qquad (40)$$

Equation (39) provides the general equilibrium between two systems with their own degrees of freedom, the first contributed by the known multifactor model, whereas the second is attributed to a quasi-zero environment. The latter contributes 3x additional degrees of freedom to the model, adding price behaviour quantization effects at each time-node along the market line. It is best represented in the expression by the

$$\frac{1}{2}\sigma^2\nabla^2\psi(\mathbf{r})\qquad(41)$$

4.3.2.3 SCENARIO 3.0

Suppose that $\{\chi, t \ge 0\}$, $\{x, t \ge 0\}$, and $\{s, t \ge 0\}$ comply to assumption (i), thus are stochastic processes with known transformation relationships between them. By assumptions (v) and (vi), $\psi(\chi)$ is a time-independent function with continuous second partial derivatives, representing an important part of the square-root of the probability density function. By assumption (vii) $\xi(t)$ is a time dependent function with continuous derivatives, representing a risk-free asset price function, and further consider *s* is the price of a risky asset at time *t* (assumption x).

We consider equations (20), (21), and assumption (viii). We consider the same risk and rate of return concepts following the observation and measurement mechanics as those

typified in the studies of Bodie et. al. (2009), Elton and Gruber (2011), and based on the original Markowitz framework. In this case the function $\psi(\chi)$ is a constituent function of the f(χ , t) (i.e. GSE) and retains the square root of the probability density function, thus n=1. We define

$$\chi = \frac{x-\mu}{\sigma}$$
, where $\{x, t \ge 0\}$ (42)

which are stochastic processes and

$$d\chi = \frac{1}{\sigma} dx = \frac{1}{\sigma} d\left(\int \frac{ds}{s}\right) = \frac{1}{\sigma s} ds$$
 (43)

where *s* is the risky asset price (i.e. stock price), and based on known relations found across statistics and econometrics literature, among others, in Daughterly (2011), Gurajati and Porter (2010), Franke and Heardle (2011), Berenson et al., (2012). The existence of pricing concepts allows us to establish the value m=2. Subsequently we obtain the same result as that in equation (23). It follows that

$$\left(-\frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + \kappa \eta(s)\right)\psi(s) = -r_f\psi(s) \qquad (44)$$

4.3.2.4. SCENARIO 4.0 - PART I

Suppose that by (i) $\{\chi, t \ge 0\}$ is a stochastic process, and assume $\psi(\chi)$ has a probability density function identity with continuous second order partial derivatives and is at a phase factor to any additional asset pricing function present (assumption xi). The probability density function identity of $\psi(\chi)$ is inline with the main assumption that the postulated

expression (18), and subsequently the master expression (19) are polymorphic in identity.

We consider the variable separation solution (Appendix II - Case 1.0), where

$$f(\chi, t) = \psi(\chi)\xi(t) \qquad (45)$$

To determine the meaning of the constant α , for this specific case we consider equations (19), and (20) with the given expression for α and a probability density function present in the formulation (n=1); $\psi(\chi)$ function retains the nature of the function in its square root form and there is one asset pricing function (m=1) in the formulation

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \gamma(\chi)\right)\psi(\chi) = (i^n)^m \left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right)\psi(\chi)$$
(46)

where the right-hand side of (46) is determined through the following expression

$$(i^{n})^{m}\left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right) = (i^{1})^{1}\left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right) = i\left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right) = i\alpha \qquad (47)$$

Subsequently, the solution is found as

$$\xi(t) = \xi(0)e^{i\alpha t}$$
 (48)

This result assures that when the probability density function $u(\chi)$ is sought, the phase factor in expression (45) and the effect of ξ is diminished entirely, and that

$$u(\chi, t) = f(\chi, t) \cdot f^*(\chi, t) = |\psi(\chi)|^2 \xi(t) \cdot \xi^*(t) = |\psi(\chi)|^2 \xi(0) e^{i\alpha t} \cdot \xi(0) e^{-i\alpha t} = \xi(0)^2 |\psi(\chi)|^2 = |\psi(\chi)|^2 \quad (49)$$

Where $f^*(\chi,t)$ is the conjugate of $f(\chi,t)$. From (49) we can see that $u(\chi, t)$ is independent of time, or $u(\chi, t) \equiv u(\chi, 0) \equiv u(\chi)$. Under such condition we set $\xi(0) = 1$ to obtain

$$u(\chi) = |\psi(\chi)|^2$$
 (50)

In such case, the true nature of α can be established, but it plays no role in the establishment of the probability density function. Its true nature is important in identifying the dynamics of the asset pricing when treated separately. This is in congruence with the fact that the primary identity of the differential expression is the square root of the probability density function.

4.3.2.5. SCENARIO 4.0 - PART II

We consider $\xi(t)$ to represent the stock price at time t. We further simplify its notation to ξ_t , where ξ_t is determined by the stochastic differential equation

$$d\xi_t = \xi_t(\mu dt + \sigma dW_t) \quad (51)$$

with {W_t, t ≥ 0} being a standard Brownian motion, $\sigma > 0$, and μ are constants, in-line with Doobs (1953), Spitzer (1970), Snyder and Miller (1991), Revus and Yor (2004), Sondermann (2007), Parzen (2015). In the financial trading context and in line with literature (Elton and Gruber, 2011), parameter σ is known to represent the volatility of stock. We consider next a stochastic differential equation for the exponential Brownian motion equivalent to the Euler (Karatzas and Shreve, 1998a:1998b; Gobet, 2000) discretised solution of equation (66). Solving (66) through integration, we obtain the exponential Brownian motion $\xi_t = \xi_0 \exp(\sigma W_t + \mu t)$, where W_t is the standard Brownian motion and ξ_0 is a constant. Using the argument in Kennedy (2010), we further replace μ by $\mu - \sigma^2/2$ and obtain the discretised solution of the form

$$\xi_{t} = \xi_{0} \exp\left[\left(\mu - \frac{\sigma^{2}}{2}\right)t + \sigma W_{t}\right] \qquad (52)$$

which evolves the state variable using an Euler discretisation scheme. It follows that the stock-price process $\{\xi_t, t \ge 0\}$ is an exponential Brownian motion similar to the ones reported across literature, specifically in the works of Kennedy (2010), Shreve (2004), Kijima (2013), Lamberton and Bernard (2007), Karatzas and Shreve (1998a:1998b), to name a few. Further on and in accordance with Jackson and Staunton (2004), we set

$$W_t = Z_t \sqrt{t} \qquad (53)$$

where Z_t is ~N(0, 1), typified by a shuffling randomiser procedure in practical application, suggested by Flannery et al., (2002). Combining expressions (52) and (53), we obtain

$$\xi_{t} = \xi_{0} \exp\left[\left(\mu - \frac{\sigma^{2}}{2}\right)t + \sigma W_{t}\right] = \xi_{0} \exp\left[\left(\mu - \frac{\sigma^{2}}{2}\right)t + \sigma Z_{t}\sqrt{t}\right]$$
(54)

Under the effect of the master expression, the right-hand-side phase factor affects the function where ξ_t is the asset price proxy and the α is an expectation term

$$\alpha = E\left[\left(\mu - \frac{\sigma^2}{2} + \frac{\sigma Z_t}{\sqrt{t}}\right)t\right] = \mu - \frac{\sigma^2}{2} \qquad (55)$$

Subsequently

$$\xi_{t} = \xi_{0} \exp\left[\left(\mu - \frac{\sigma^{2}}{2}\right)t\right]$$
 (56)

and

$$S_{t} = S_{0} \exp(\sigma Z_{t} \sqrt{t}) \xi_{t} = S_{0} \exp(\alpha t + \sigma Z_{t} \sqrt{t})$$
 (57)

After substituting the new expression for α we obtain

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \kappa\eta(\chi)\right)\psi(\chi) = \mathbf{i} \times \left(\mu - \frac{1}{2}\sigma^2\right)\psi(\chi) \qquad (58)$$

Where ξ is an asset price equivalent function and ψ_{χ} is the square root of probability density function. Both functions are postulated in a phase factor relationship as described in the GSE.

4.3.3 CASE 3.0 - GENERALISED OPERATOR

Suppose that $\{\chi, t \ge 0\}$ is a stochastic process, $S(\chi)$ and $\Pi(\chi)$ have quasi-zero timedimension asset and portfolio pricing function identities, respectively. We establish that they have continuous second order partial derivatives and can be expressed in differential forms:

$$O^{s} \cdot S(\chi) = 0$$
 where operator $O^{s} = \left[-\frac{1}{2} \frac{\partial^{2}}{\partial \chi^{2}} + \left(\mu - \frac{1}{2} \sigma^{2} \right) \right]$ (59)

$$O^{\Pi} \cdot \Pi(\chi) = 0 \text{ where operator } O^{\Pi} = \left[-\frac{1}{2} \frac{\partial^2}{\partial \chi^2} + \left(\mu - \frac{1}{2} \sigma^2 \right) - r_f \right]$$
(60)

To determine the meaning of the $k\eta$ and α we can consider GSE with no probability density function identity present (n=0) and the two-fund separation problem; the portfolio of a risk free investment and a risky asset, such as stock.

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \kappa\eta(x)(\chi)\right)\psi(x) = (i^n)^m \left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right)\psi(x)$$
(61)
$$(i^n)^m \left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right) = (i^0)^m \left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right) = \left(\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right) = \alpha$$
(62)

We set α =0, justified by the fact that at a quasi-zero time point, the sensitivity of ξ on an infinitesimal time increment should diminish (Eugene and O'Donnell, 1997). Basically ξ is constant and the asset price identity beyond the price-point and over the time horizon is assigned to the function, ξ , that also leads to the asset pricing identity assignment to composite function $f(\chi, t)$ i.e. refer to the master formulation (GSE). It is exactly this that sets (i^0)^m=1, due to the lack of a probability density function identity in this consideration. Such considerations lead to the purpose of augmentation of the GSE to entirely represent an asset pricing differential equation, endowed by properties of ordinary differential equations (Birkoff and Rota, 1962; Nagle et al., 2004). This allows us to express (76) as

$$-\frac{1}{2}\frac{\partial^2 S(\chi)}{\partial \chi^2} + \kappa \eta(\chi)S(\chi) = 0$$
 (63)

where $s_{(\chi)} \equiv_{\Psi(\chi)}$ representing the asset p-point⁷⁴ function, which in this case is an equity account function. The $s_{(\chi)}$ value is dictated by the filtration effect at time t. The more holistic form of it would be denoted $S_{\mathcal{F}@t}(\chi)$, where \mathcal{F} @t represents the filtration effect at time t, however, we will carry on using the short hand notation for simplicity. The concept of filtrations, both in formulation and reflection of events' processing at the market interface is discussed in greater length in Malliaris (1982), Øksendal (2000), Bru

⁷⁴ Price-Point; a specific time-node reading of price in the market line and within a time horizon.

et al. (2012). Furthermore the events' reflection on the asset price are in line with efficient market hypothesis (Asquith, 1983); Bachelier, 1990; Bernard and Thomas, 1990; Bodie et al., 2009, Khrennikov, 2018).

It is possible to consider a non-zero solution by setting $\alpha = -r_f$, which can be possible under the consideration of a model that consists of an economy in which there are just two assets, a bank account paying a fixed continuously compounded interest rate r_f per unit time and the second risky asset with a price given by a stochastic process; we will refer to the latter asset as a stock but it may be any other tradable asset, such as a foreign currency (Doobs, 1953; Spitzer, 1970; Parzen, 2015).

One unit in the bank at time 0 grows to $\exp(r_f t)$ by time t and we will assume that $r_f \ge 0$, although from the mathematical viewpoint this is not a requirement for all that follows. The bank account ensures that there is positive riskless borrowing so that for example, at time τ a bond paying off one unit at time t may be bought at a positive price $\exp(-(t-\tau)$. This implies that the discount factor is $\exp(-r_f t)$ at time t. Under the two-fund portfolio basis, $\psi(\chi)$ would represent the portfolio value function, $\Pi_{\mathcal{F}@t}(\chi)$ under the filtration effect at time, t. The subscript is dropped in the following consideration for simplicity (Doobs, 1953; Spitzer, 1970; Parzen, 2015).

The price in both cases can progressively evolve along the time dimension, however we

additionally consider a zero-dimension reset effect, which allows us to dwell on possible price quantization effects at any given time point; the quasi zero quantized price pattern will be independent of the time; this does not invalidate our understanding of the spatial price evolution nor its spatial distribution, but instead sheds light on additional price behaviour pattern dynamics, complimentary to the spatial one (Möller and Zettl, 1996; Yan et al., 2017; Fiorin et al., 2018). This can be assigned to the strength of filtrations, the dissipative effect of information by the market, and the subsequent reflections on the market price in line with an efficient market hypothesis (Jordan, 1983; Fama, 1991; Malkiel, 2003; Bodie et al., 2009). This implies a composite price pattern evolution within the probability space, attributed to the probability triple (Ω , \mathcal{F} , P), a non-empty set Ω , the sample space, a σ -field \mathcal{F} of subsets of Ω , and a probability (measure) P defined on filtrations \mathcal{F} .

From our mathematical workings (see Appendix II - CASE 3.0), equations

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \left(\mu - \frac{1}{2}\sigma^2\right)\right]S(\chi) = 0$$
(64)

$$\left[\left(r_f - \frac{1}{2}\frac{\partial^2}{\partial\chi^2}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\right]\Pi(\chi) = 0 \quad (65)$$

are easily recognized harmonics and in a form ready to be used for price valuation work at a quasi-zero dimension point, along the price evolution path. Furthermore on close inspection, we evidently see that a solution of Sturm-Liouville type can be sought out (Pruess, 1973; Pavel, 1975; Combescure and Ginibre, 1976; Parthasarathy, 1992; Pruess and Fulton, 1993; Kong and Zettl, 1996; Kong et al., 2000:2001:2004; Savchuk and Shkalikov, 1999; Albeverio and Kurasov, 2000; Pages et al., 2003). We recall that stock price has an initial price assumed to be observed at time 0. The information available at time t is the history of the price process, $\mathcal{F}_1 = \sigma(S_u, 0 \le u \le t)$. That is the information obtained by observing movements of the stock price process up to time t; equivalently, it is $\sigma(W_u, 0 \le u \le t)$; information obtained by observing the driving Brownian motion in the stochastic differential equation (Ito, 1951; Karatzas and Shreve, 1998b; Parthasarathy, 1992; Øksendal, 2000, Jacus, 2008).

4.3.4 CASE 4.0 - TIME INDEPENDENT EIGEN-STATE FORMULATION

Suppose that $\{\chi_t, t \ge 0\}$ is a stochastic process, $f(\chi,t)$ is a time-independent function with continuous second order partial derivatives, $\gamma(\chi)$ is a simple harmonic function. The probability density function $u_n(\chi_t,t)$ may be represented in the time-independent discretised and Eigen-state form:

$$u_n(\chi) = |\psi_n(\chi)|^2 = \left(\frac{\alpha}{\sqrt{\pi}} 2^n n!\right) |H_n(\alpha \chi)|^2 e^{-(\alpha \chi)^2}$$
 (66)

where $\chi = r - \mu$, $a = 1/(\sigma\sqrt{2})$, and n = 0 represents the classical expression of the probability density function at the lowest Eigen-state level:

$$\mathbf{u}_{0}(\mathbf{r}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mathbf{r}-\mu}{\sigma^{2}}\right)} \tag{67}$$

For n = 0, we obtain the classical expression for the probability density functions $u_0(\chi)$ and it has a maximum at $\chi = 0$. From our mathematical workings (Appendix II - CASE 4.0), we conclude that as n increases, the result moves away from the classical result implied. Further more equation

$$\psi_{n}(\chi) = \left(\frac{\alpha}{\sqrt{\pi}} 2^{n} n!\right)^{\frac{1}{2}} H_{n}(\alpha \chi) e^{-\frac{1}{2}(\alpha \chi)^{2}} \qquad (68)$$

is a solution of the differential equation

$$\frac{\partial^2 F}{\partial \chi^2} - 2\chi \frac{\partial F}{\partial \chi} + (2\alpha - 1)F = 0$$
 (69)

See Appendix II – CASE 4.0 for the transformation ψ to F).It implies harmonic behaviour and its solutions are Hermite functions. Equation (68) includes the square root of the weight function , so that the functions $\psi(\chi)$ are orthogonal when integrated from - ∞ to + ∞ , which is required by theory (Szego, 1939; Gupta et al., 1974; Walter, 1980; Greblicki, 1981; Walter, 1980; Puig, 2003).

The orthogonality of the Hermite polynomials is expressed by

$$\int_{-\infty}^{+\infty} e^{-\chi^2} H_n H_m d\chi = \delta_{nm} 2^n n! \sqrt{\pi}$$
(70)

where δ_{nm} is the Kronecker delta which is a function of two variables, usually just positive integers (Szego, 1939; Bailey, 1966; Gupta et al., 1974; Greblicki, 1981). The function is 1 if the variables are equal and 0 otherwise

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$
(71)

is zero if m is not equal to n, and unity if m is equal to n.

Schwartz (1967) provides the full mathematical work-out, and we do not reproduce it here. In essence to prove it, one would need express the exponential times the Hermite polynomial of larger order as an nth derivative using the Rodrigues formula, and then use integration by parts until the polynomial of smaller order is differentiated to zero (Schwartz, 1967). If the orders are equal, the final integral, and subsequently the result is the integral of $\exp(-\chi^2)$ times a constant, and the normalisation constant becomes $(\sqrt{2\pi})n!$ (Walter,1977). The orthogonality attribution can be used to expand an arbitrary function in a series of Hermite polynomials, in exactly the same way as a Fourier series (Sansone, 1939; Walter, 1980; Brigham, 1988; Puig, 2003; Fang and Oosterlee, 2009; Callegaro, et al., 2018a).

5.0 ASSET PRICE RAPPROCHEMENT: SPLIT PDF IDENTITIES AND STURM-LIOUVILLE QUANTUM FITTING

The main aims of this chapter are to (1) fit a quantum price system to our probabilistic stochastic finance asset price and contingent claim valuation model, and (2) establish the dynamics of Eigen-prices of an information-conservation and dissipation environment such as an efficient market affected by stochastic randomness with price moves skewed by market price impact of filtrations. The idea put forward is that in order to understand the dynamics of time-horizon based market-motive price moves, we need to make a strong assumption - that in the absence of a 3-dimensional physical confinement of the price (that would be illogical to consider), it is reasonable to consider a factor-based

model and a zero-dimensional quantum price-system or view. This represents an actual point in the market-motive price line, modelled through a zero-dimensional quantum system with a measurable density of price states. It implies that the continuous marketmotive price line is a continuous exhibition of the discrete nature of prices within a quantum-price system.

This will lead eventually to the identification, modelling, and measuring of a highly discretised quantum price distribution function to replace (or combine with) the probability density functions used by the existing models, such as the on suggested by Black and Scholes (1973). The essence of the chapter itself is in the creation of a new adaptive theory, as well as in the identification, measuring, and modelling of a parameterized zero-dimensional quantum price framework that would provide the theoretical basis for future research with the intention to extend it to alternative financial derivatives' pricing models.

The fundamental hypothesis is that the continuous price line in a conservative and dissipative information system such as an efficient market is a result of the discrete time-space underplaying, where price jumps in the market surface are the result of continuous time-space effects, i.e. ex-dividend dates, cash flow projections, new technological innovations, etc. The focus here is on the discrete time-space effect which itself is the driving force behind the random or the stochastic nature of prices in a market-motive surface, and that once quantified can be combined or factored in derivative pricing models.

5.1 INTUITION AND RATIONALE

In this study the asset price and contingent claim valuation is conceptualized across a sequence of time nodes and measured in the quasi-zero-dimension level in a model with an extended number of degrees of freedom. Modern pricing theories consider wealth growth along a parametrized "medium" (Cox et al. 1979; Ho and Lee, 1986; Black, 1989; Jarrow and Turnbull, 1998; Elton and Gruber, 2011). Moreover in cases where the price valuation is carried out in a dual manner, at an underlying asset base, and at the asset-protection level through contingent claims, the time implication is of paramount importance in the expectation term. It can be seen more clearly through the Black-Scholes formula for a call expressed in a slightly different way, such as

$$c = e^{-rT} \left[Se^{rT} N(d_1) - XN(d_2) \right]_{or} c = e^{-rT} \left[S_T N(d_1) - XN(d_2) \right]$$
(72)

The expression inside the square brackets represents the expected payoff of the option, $E^{Q}[...]$. $N(d_{2})$ is the probability that the call will be exercised in a risk-neutral world, where a share that pays no dividends has an expected return or the risk-free rate.

In the contingent claim valuation formulations (Ho and Lee, 1986; Jarrow and Turnbull, 1998; Hull, 2014), and the original work of Black and Scholes (1972:1973:1989) using a continuous stochastic process, the expectation term has a probabilistic nature with a time parameter too. Kennedy (2010) articulates that stock price changes may also be represented by a one-parameter stochastic process, whereas interest rates are naturally

represented by a two-parameter process, where the first parameter is the time that the loan matures, and the second parameter is the 'real' time, so that the process modelled is a random surface economical factor tracked usually to the market return and is claimed to explain to a good degree the price movements of other stock in the market. Bodie et al. (2009) also explains to a greater extent the price behaviour at each time-node and measures in terms of either logarithmic price change or a price percentage change. Additional factors (i.e. indices) can be added to the index model, leading to an increase in the degrees of freedom.

Karatzas and Shreve (1998b), Revus and Yor (2004), Shiryaev et al. (2006), and Kennedy (2010) articulate in depth the concept of filtrations and consider the real-time shock effect at the asset price, observable and measurable in the market volatility surface. However we consider an additional and unrelated "orthogonal" filtration effect to the asset price, which we attribute to the strength of the filtrations and the impact the filtrations' degrees of strength have on each p-point⁷⁵, but in symmetry with the "dissipation" ability of the market of each event or stream of events We consider the "orthogonal" filtration attribution to be responsible for the quantized price change behaviour at each p-point.

The stochastic shock effect due to the relevant filtrations, absorbed by the markets and

 $^{^{75}}$ The term p-point refers to a price point or a zero-time price point/system, alternatively the term p-tip is used in equivalence and implies "orthogonally" in the price, or price/value identification at a zero-time point where it is difficult to establish with the current models; normally a supplementary effect in pricing or an additional term unrelated or independent, which is ignored by the classical asset pricing models. The terms are used interchangeably in this paper.

reflected in the stock price, is often modelled through the Weiner process and has a Gaussian distribution (Boobs 1953; Fujihara and Park, 1990). However, this study suggests that the "real" probability distribution system contains various probability distributions with various degrees of distribution mixing, where the system itself exists in various eigen-state levels of mixing or separation; a quantization effect that follows directly from the 3x expansion of the price-system's degrees of freedom. The Gaussian probability distribution function corresponds to the lowest and most stable eigen-state, however at higher quantized levels (i.e. higher volatilities and irregularities in filtrations patterns), there are distribution splits, evident both mathematically and through numerical illustrations (Luschgy and Pages, 2002; Haven, 2002; Bally and Pages, 2003; Zhang et al., 2014; Khrennikov, 2018).

Further on the new model considers the additional price change effect due to the quantization effects at each price point, and subsequently models the cumulative pricechange (i.e. more precisely the return rate) effects through the classical pricing models with the additional quantization effects on the price-change at a quasi-zero time dimension of each point along the market line. This has profound implications in financial instrument pricing, especially in financial derivatives because the classical models such as Black and Scholes option pricing (Black and Scholes, 1972:1973; Ho and Lee, 1986; Jarrow and Turnbull, 1998) among others, use a probability distribution with normal or lognormal distribution considerations, which under the new model needs to be upgraded to include the probability distribution system mixing and the additional Fermi-Dirac probability distribution for the quasi-zero price point system along the market line. Hence, the objective of this chapter is threefold: First, reconsider pricing problems starting from our previously augmented generalisation of the Schrödinger PDE, leading to asset price and financial derivative valuation formulations in partial differential forms; secondly, we derive valuation expressions that are in line with a rational investor's expectations, although derived from an abstract and unified formulation; thirdly, we incorporate quantization effects in the probability distribution and the price-change systems, and subsequently provide a Sturm-Liouville solutions (Bailey, 1966; Zettl, 1997; Zhang et al., 2014).

In this chapter we use a postulated problem with a very abstract formulation and various common cases that are in line with contemporary pricing models (Bodie et al., 2009); the problem is a financial instrument valuation challenge springing from an axiom, with Strum-Liouville adapt-solutions, in order to observe quantization effects at a price's point in the time line.

We seek to pick on patterned curvatures as an effect of price quantization at each time point in a market price line, assigned to internal mechanics of the \mathbb{D} -world. Attempts have been made by other researchers to link this concept to asset pricing and portfolio valuation, although Levental et al. (2016) claim that all uncertainty is generated by a multiple-dimensional standard geometric Browning motion over finite time horizon [0, T], supported in probability space (Ω , F, P), given tangible observations in real time of augmented filtration {F_t:t \in [0,T]}. However, there is no factorization of any quantization effect or linkage to a possible regularly patterned probability surface curvature. We seek such D-world implied curvature pattern in exhibitions of implied volatility surfaces.

Stochastic financial variables represent randomizer effects that are attributed to filtration diffusion processes, although we would wish to identify the deterministic component of a process of this type, the non-deterministic part will be a source of contribution to the random measure.

Although the properties of random variable and attributes of financial products are reported in modern times, there is a status-quo in terms of the analytics, most are centred around the Black-Scholes option pricing (and indexing for bonds and stock). We attempt to connect the two here and reapply the stochastic concepts in order to leverage deeper knowledge and revisit the known models with intent to upgrade them.

The concept of $\uparrow \downarrow$ direction of a random variable is argued well in various existing literature, such as Lamberton and Bernard (2007), and Shiryaev et al. (2006), but there is a lack of literature in terms of attributing such as properties to other than our own 'real' universe, or the continues time-space world (the ©-world), where randomized residual effects are exhibited and financial instruments' prices are developed and observed.

We consider both the application of such attribution to our own world, but also on each information dissipation point along the market line, or within the \mathbb{D} -world. We thus consider uncountable collections of random values (i.e. price behaviours in a typical case) within a supremum representation in both the $\mathbb{O}_{\downarrow\uparrow}$ and $\mathbb{D}_{\uparrow\downarrow}$ -world. This is a new concept and there is little of significance in existing literature. With the new construct, we also provide the internal mechanics and the necessary vocabulary to be used along.

The $\mathbb{O}_{\uparrow\downarrow}$ system focus is on effects of dissipation and the appropriation of that on price density of states. It represents the essential supremum within the system and under the price-related information dissipation effects. No existing literature makes such connection or treats such constructs in this way, least so on the internal dynamics of the price development within the $\mathbb{O}_{\uparrow\downarrow}$ -system, and the quantification of information dissipation within such system.

A stochastic process on its own and in its application in asset pricing or contingent claim valuation is typically composed of a deterministic part and an additional process known as the Weiner process. The Wiener process describes behaviour of a variable that is subject to random shocks that are completely uncorrelated over time. It satisfies the Markov property, which entails that the current observation of a variable summarizes all the relevant information we need to predict its future values; past history of the process is irrelevant. Such process is said to be a process without memory. It is on the basis of Markov property that we consider the stopping time and the O_{14} system; a zero-time 3-

dimension expansion where information is dissipated by the market. We also consider that market 'tunnelling' effect is present in well-regulated markets (Doobs, 1953; Spitzer, 1970; Ho and Lee, 1986; Snyder and Miller, 1991; Seneta, 1996; Bally and Pages, 2003; Parzen, 2015; Rustemovich and Mukmino, 2018).

This is something that is not included in existing literature in finance or related domains. The information "tunneling" effect is quantified well in literature related to applied mathematics, but no feasible adaptation of it in finance has been reported. The stochastic process aligns to price development dynamics with the O_{++} -world, and typically does not consider O_{++} system measures, which represents a missing link (matched by lack of literature, hence a research gap partition), that we integrate in our abstract analytical framework.

Information available at time t is the history of the price process, $\mathcal{F}_1 = \sigma(S_u, 0 \le u \le t)$, that is the information obtained by observing the movements of the stock price process up to time t; equivalently, it is $\sigma(W_u, 0 \le u \le t)$, the information obtained by observing the driving Brownian motion in the stochastic differential equation. Most authors, (among whom Karatzas and Shreve, 1998b; Øskedal, 2000; Shiryaev et al., 2006; Sondermann, 2007; Lamberton and Bernard, 2007), consider the contingent claim valuation from basic assumptions of a one time period horizon and a discrete-time underlying process (i.e. not to be confused with the "orthogonal" price change or displacement within the discretised \mathbb{O} -system, although they interface at each discretised stop time or time node), and seek

an equivalent martingale measure (or probability), which is an equivalent probability Q under which the discounted stock-price process $\{e^{-\rho t}S_t, 0 \le t \le t_0\}$ is a martingale.

5.2. PROPOSED FINANCIAL DERIVATIVES' PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

We start with the GSE, defined in the previous chapter and restated here for q = 1:

$$\forall m \in \{0, \dots, M\}, \forall n \in \{0, \dots, N\} \left[\left(\frac{1}{2} \frac{\partial^2}{\partial x^2} + \gamma(\chi, t) + (i^n)^m \frac{\partial}{\partial t} \right) f(\chi, t) \right] = 0 \quad (73)$$

5.2.1 CASE 5.0 – TIME-DEPENDENT ASSET PRICING PDE

Suppose that $\{\chi_t, t \ge 0\}$ is a stochastic process and that $f(\chi,t)$ is a time-dependent function with continuous second order partial derivatives. The composite function $f(\chi_t,t)$ may be represented in the differential forms:

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + (\gamma(\chi, t) - \alpha)\right)\psi(\chi, t) = \left((i^n)^m\frac{\partial}{\partial t}\right)\psi(\chi, t) \quad (74)$$

We do not make a claim on the identity of the function $f(\chi,t)$ at this point or for this theorem's general effect, nor do we intend to do so in this formulation; it is a broadly generic expression. However, we seek a solution that considers it to be a composite function (Eugene and O'Donnell, 1997). This is suitable in cases where focus shifts to the identities of the constituent functions $\psi(\chi, t)$, and $\xi(t)$; typically they could be considered to be asset and contingent claim valuation functions. Refer to Appendix II – Case 5.0 for
mathematical workings.

5.2.1.1. SCENARIO 1.0 - PART I

Suppose the usual assumptions such that $\{\chi_t, t \ge 0\}$, $\{x_t, t \ge 0\}$, $\{s_t, t \ge 0\}$, are stochastic processes with transformational inter-dependent relations and that $f(\chi,t)$ and $\psi(\chi,t)$ are time-dependent functions with continuous second order partial derivatives. We then consider a scenario where one single contingent claim, with a function $\psi(s, t)$, carries the legality of operating on (purchase or sell) a fraction of a stock, s (t). We also assume a financial environment where borrowing and lending are possible at the risk free rate r_f . Alternatively, we assume there is one risk free investment in the economy, and it is possible to action a claim on an asset (such as stock) on the basis of an underwritten contract between parties (Black and Scholes, 1973; Merton, 1973:1974:1976; Hull and White, 1987; Black, 1989; Hull, 2014). We also consider random variable χ to possess all the properties outlined earlier and articulated in good detail by Kennedy (2010).

Next we define

$$\chi = \frac{x - \mu}{\sigma}$$
 and $x = \int_{s} \frac{ds(t)}{s}$ (75)

Where $x = \sigma \chi + \mu$, is also a random variable incorporated as a function $x : \Omega \to R$, measurable with respect to \mathcal{F} . That is all events $(x \le \sigma c + \mu) = (\Omega : x(\Omega) \le \sigma c + \mu) \in \mathcal{F}$ for all real numbers $\sigma \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $c \in \mathbb{R}$ (Kennedy, 2010). It follows that

$$d\chi = \frac{dx}{\sigma} = \frac{ds}{\sigma s}$$
 and $d\chi^2 = \left(\frac{1}{\sigma s}\right)^2 ds^2$ (76)

We consider a pricing identity for $\psi(s, t)$. More specifically a contingent claim, such as a financial option (Ross, 2003). We then substitute (76) into equation (74) to obtain

$$-\frac{1}{2}\sigma s \frac{\partial^2 \psi(s,t)}{\partial s^2} + [\alpha - \gamma(s,t)]\psi(s,t) = \frac{\partial \psi(s,t)}{\partial t}$$
(77)

It follows that

$$\alpha = (i^0)^2 \left[\frac{1}{\xi(t)} \left(\frac{\partial \xi(t)}{\partial t} \right) \right] = r_f$$
(78)

Further, we substitute (78) into (77) and move all terms onto the same side of the expression to obtain

$$\frac{\partial \psi(s,t)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \psi(s,t)}{\partial s^2} - [r_f - \gamma(s,t)]\psi(s,t) = 0 \qquad (79)$$

Next we establish the identity $\gamma(s, t)$ by considering a portfolio, Π , under a risk-neutral environment and within one wealth-generating time horizon (Hull, 2014). Under this case $\psi(s, t)$ represents the value of a one tradable contingent claim, where the portfolio contains one option and -h (i.e. $\partial \psi/\partial s$) units of the underlying stock (Treynor and Black, 1973; Black, 1989; Bustamante and Contreras, 2016). The term r Π is the growth of riskless portfolio Π in the time horizon. This allows us to set

$$r_{f}\psi(s,t) - \gamma(s,t)\psi(s,t) = r_{f}(\psi - hs) = r_{f}(\psi - \frac{\partial\psi}{\partial s}s) = r\Pi e^{r_{f}t}$$
(80)

Expression (80) is consistent with the case of a self-financed portfolio, where $h = \partial \psi / \partial s$, represents the hedge. Substitute (80) into our initial PDE we obtain

$$\frac{\partial \psi(s,t)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \psi(s,t)}{\partial s^2} - r_f \left[1 - s \frac{\partial}{\partial s}\right] \psi(s,t) = 0 \qquad (81)$$

where the identity of γ is evidently

$$\gamma = rs \frac{\partial}{\partial s} = rs\Omega$$
 (82)

The non-zero operator, Ω , underlines the fact that the price of the option is quantized at a quasi-zero dimensional point in the price evolution path, under the full effect of filtrations.

We can write

$$\frac{\partial \psi(s,t)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \psi(s,t)}{\partial s^2} - , (s,t) + rs\Omega \psi(s,t) = 0 \quad (83)$$

Replacing the expression for Ω , we obtain the Black-Scholes PDE.

$$\frac{\partial \psi(s,t)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \psi(s,t)}{\partial s^2} + rs \frac{\partial \psi(s,t)}{\partial s} - r\psi(s,t) = 0 \quad (84)$$

Equation (84) is a significant result, due to the fact that it validates the GSE and gives us flexibility in terms of the identity of the function $\gamma(s, t)$, which allows one to investigate different scenarios.

In the classical approach and in alignment with Black and Scholes (1973), Black (1989), and Shiryaev et al. (2006), the resulting expression (84) is achieved by using Ito's Lemma for a function with multiple variables (typically, variables s and t), relating a small change in a function of a random variable to a small change in the variable itself with a deterministic component dt and a random component d χ . Further our finding that the Black-Scholes option pricing PDE is a special case of Schrödinger's equation is inline with the research work of Haven (2002).

5.2.1.2. SCENARIO 1.0 - PART II

Consider the usual assumptions that $\{\chi_t, t \ge 0\}$, $\{x_t, t \ge 0\}$, $\{s_t, t \ge 0\}$, are stochastic processes with transformational inter-dependent relations and that $f(\chi,t)$ and $\psi(\chi,t)$ are time-dependent functions with continuous second order partial derivatives. Given suitable relations between the stochastic processes, we may write $\psi(s, t)$ in the form:

$$-\frac{1}{2}\frac{\partial^{2}\psi(s,t)}{\partial s^{2}} + \frac{r}{\sigma^{2}s^{2}}\psi(s,t) - (\mu - \frac{1}{2}\sigma^{2} + r_{f})\frac{h}{\sigma^{2}s} = 0$$
(85)

where r_f is the risk free rate of return and μ the rate of return on a risky asset such as stock.

Following from SCENARIO 1.0 – PART I, we seek a solution that can easily be presented in the Sturm-Liouville form for easy valuation at a quasi-zero dimension price point (Bailey, 1966). We start with expression (84) and with minor arrangements obtain

$$-\frac{1}{2}\sigma^{2}s^{2}\frac{\partial^{2}\psi(s,t)}{\partial s^{2}}-rs\frac{\partial\psi(s,t)}{\partial s}+r\psi(s,t)=\frac{\partial\psi(s,t)}{\partial t}$$
(86)

then expand the right-hand side of (86)

$$-\frac{1}{2}\sigma^{2}s^{2}\frac{\partial^{2}\psi(s,t)}{\partial s^{2}} + r_{f}\psi(s,t) - r_{f}s\frac{\partial\psi(s,t)}{\partial s} = s\frac{\partial\psi(s,t)}{\partial s}\left(\frac{1}{s}\frac{\partial s}{\partial t}\right)$$
(87)

and rearrange

$$-\frac{1}{2}\sigma^{2}s^{2}\frac{\partial^{2}\psi(s,t)}{\partial s^{2}} + \left[r_{f}\psi(s,t) - s\frac{\partial\psi(s,t)}{\partial s}\left(\frac{1}{s}\frac{\partial s}{\partial t}\right) - r_{f}s\frac{\partial\psi(s,t)}{\partial s}\right] = 0 \quad (88)$$

The price of the stock at time t is S_t , where St is determined by the stochastic differential equation

$$dS_{t} = S_{t} \left(\mu dt + \sigma dW_{t} \right)$$
(89)

with {Wt, $t \ge 0$ } being a standard Brownian motion and $\sigma > 0$, μ are constants (Shiryaev et al., 2006; Parzen, 2015). It follows from equation (89) that stock-price process {St, $t \ge 0$ } is an exponential Brownian motion and may be represented as

$$\mathbf{S}_{t} = \mathbf{S}_{0} \exp(\mu - \frac{1}{2}\sigma^{2} + \sigma \mathbf{W}_{t})$$
 (90)

where S_0 is the initial price of the stock, assumed to be observed at time 0. The information available at time t is the history of the price process, $\mathcal{F}_1 = \sigma(S_u, 0 \le u \le t)$. That is, the information obtained by observing the movements of the stock price process up to time t; equivalently, it is $\sigma(W_u, 0 \le u \le t)$, the information obtained by observing the driving Brownian motion in the stochastic differential equation (Ito, 1951; Giller, 1982; Weidmann, 1987; Øskedal, 2000; Parzen, 2015). It is evident that

$$\left(\frac{1}{s}\frac{\partial s}{\partial t}\right) = \mu - \frac{1}{2}\sigma^2 \qquad (91)$$

We then substitute back into (88) and obtain

$$-\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \psi(s,t)}{\partial s^2} + r\psi(s,t) - (\mu - \frac{1}{2}\sigma^2 + r)sh = 0$$
(92)

After re-arrangement, we get

$$-\frac{1}{2}\frac{\partial^2\psi(s,t)}{\partial s^2} + \frac{r}{\sigma^2 s^2}\psi(s,t) - (\mu - \frac{1}{2}\sigma^2 + r)\frac{h}{\sigma^2 s} = 0$$
(93)

A special case arises when the right-hand side of equation (86) is set to zero. This is possible as we seek a solution of a quantized value pattern at zero-dimension price point, resulting in simplified mathematical expression

$$-\frac{1}{2}\sigma^{2}s^{2}\frac{\partial^{2}\psi(s,t)}{\partial s^{2}} + r_{f}\psi(s,t) - r \bullet s \bullet h = 0 \qquad (94)$$

5.3. PRELIMINARY PROPOSAL FOR THE POTENTIAL 'IDENTITY' FUNCTION

In our previous sections and chapter, the function $\gamma(\chi)$ in the GSE and subsequent cases and scenarios, was identified. It reflects the main underlying idea behind the GSE in terms of what it does and how the functions $\psi(\chi)$ and $f(\chi, t)$ were identified, thus we refer to it at times as the "identity" function. That is because in order to get to identities of $\psi(\chi)$ and $f(\chi, t)$, we need to establish the identity of $\gamma(\chi)$ first. We introduce preliminarily such function here and provide a much more detailed explanation in the next chapter. We consider it to stem from the following general expression.

where $\chi = \frac{1}{L} \Big(\frac{r-\mu}{\sigma} \Big) \,$, or

$$\gamma(\mathbf{r}) = \sum_{n=-\infty}^{n=+\infty} \kappa_n \left(\frac{1}{L} \left(\frac{\mathbf{r}-\mu}{\sigma}\right)\right)^n = \dots + \kappa_{-i} \left(\frac{1}{L} \left(\frac{\mathbf{r}-\mu}{\sigma}\right)\right)^{-i} + \dots + \kappa_{-1} \left(\frac{1}{L} \left(\frac{\mathbf{r}-\mu}{\sigma}\right)\right)^{-1} + \kappa_0 + \kappa_1 \left(\frac{1}{L} \left(\frac{\mathbf{r}-\mu}{\sigma}\right)\right)^1 + \dots + \kappa_i \left(\frac{1}{L} \left(\frac{\mathbf{r}-\mu}{\sigma}\right)\right)^i + \dots$$
(96)

The generalized potential "identity" function can be broken down to specific functions and solutions, dependent on the values of the coefficients κ_n , and the value of n (Eugene and O'Donnell, 1997). The choice of the "identity" function is quite important as it is linked with the density of states and reflects the effect of the filtrations and patterns in a volatility surface of a financial market (Möller and Zettl, 1996). Furthermore it can drive price dynamics towards a stable state and lead to a reasonable cut-off price. All of the "identity" functions used in Table 2.0 can be generated by careful selection of values of n and κ_n . For example when n=0, $\kappa_{+1} \neq 0$ and all the other coefficients are set to zero, it yields the harmonic case $\gamma(\chi) = \kappa_{+1}\chi^2$. Using the logic above we can generate many more testable functions (Möller and Zettl, 1996), some of which are tabulated below:

TABLE 1.

	Name	Axiom Idenity Function
1.	GAUSSIAN	$\gamma(\chi) = \gamma(0)[1 - \exp(-\chi^2)]$
2.	HARMONIC	$\gamma(\chi) = \frac{1}{2} \kappa \chi^2$
3.	DECAYING EXPONENTIAL	$\gamma(\chi) = \gamma(0)[1 - \exp(-\chi)]$
4.	LINEAR	$\gamma(\chi) = \gamma(0)\chi$
5.	CONSTANT SQUARE WELL	$\gamma(\chi) = \gamma(0)$
6.	ARCTAN	$\gamma(\chi) = \frac{1}{2}\gamma(0) \left[\frac{2}{\pi}\arctan\left(\frac{1}{b}\chi\right) + 1\right]$
7.	COSH-2	$\gamma(\chi) = \gamma(0)[1 - \cosh^{-2}(-\chi^2)]$
8.	INVERSE	$\gamma(\chi) = -\gamma(0)\frac{1}{\chi}$

All the functions listed in Table 1 represent alternative identities of functions, $\psi(\chi)$, and subsequently $f(\chi, t)$, (Eugene and O'Donnell, 1997). We previously proved that when $\gamma(\chi)$ function has a harmonic identity (equations 64 and 65), the solution obtained is the classical probability density function (Black and Scholes, 1972:1973). However, the harmonic function can be seen as the first order approximation of the Gaussian. Similarly, the linear function found in CASE 2.0 (equations 29 and 38), can also be seen as the first order approximation (Protter and Weinberger, 1984).

5.3.1 CASE 6.0 - ASSET PRICING PDE WITH A 'HARMONIC' POTENTIAL

Suppose that $\{\chi_t, t \ge 0\}$ is a stochastic process, $\psi(\chi,t)$ is a time-independent function with continuous second order partial derivatives, tied to an identity of a probability density function, $\gamma(\chi)$ is assumed or observed to be an approximated harmonic function. We

consider expression under CASE 4.0, the general effects of the master equation, and polynomial expression (95) with coefficients set to the creation of a Gaussian identity. Dependent on whether the formulae are expressed in term of χ or morphed to that of r, the Gaussian can be approximated by the harmonic function, which is useful because the quantization effects of a harmonic function are known (Abramowitz and Stegun, 1972).

This can be seen by using the Taylor expansion

$$\gamma(\chi) = \gamma(\chi)|_{\chi=0} + \left(\frac{d\gamma(\chi)}{d\chi}\Big|_{\chi=0}\right) \frac{\chi}{1!} + \left(\frac{d^2\gamma(\chi)}{d\chi^2}\Big|_{\chi=0}\right) \frac{\chi^2}{2!} + \dots + \left(\frac{d^n\gamma(\chi)}{d\chi^n}\Big|_{\chi=0}\right) \frac{\chi^n}{n!} + \dots$$
$$1 - \exp(-\chi^2) = 0 + 0 + 2\frac{\chi^2}{2!} - 0 - 12\frac{\chi^4}{4!} + \dots \text{ h. o.t} \quad (97)$$

For $\chi \ll 1$, terms χ^4 , χ^6 ... approach zero much faster than χ^2 . Thus Gaussian is approximated by the harmonic function

$$1 - \exp(-\chi^2) \approx \chi^2 \qquad (98)$$

where $\chi = \frac{1}{L} \left(\frac{r-\mu}{\sigma} \right)$.

Subsequently expression (98) is valid when $(r - \mu) \ll \sigma$ or $r \ll \mu + \sigma$. On the other hand, larger values of χ imply smaller volatility σ (i.e. χ is inversely proportional to σ). Subsequently for $\chi \ll 1$, $r - \mu$) $\ll \sigma$ becomes less "true", thus more terms on the right hand side of the expansion would be needed. Hence, a Gaussian identity function would be better suited (Abramowitz and Stegun, 1972).

This shows that for small values of χ or high volatilities, the harmonic is a good identity

representation of γ , but a poor approximation for large χ , (small volatilities), in which case the Gaussian is more realistic (Abramowitz and Stegun, 1972). The approximations would be sensitive not only to the value of σ , but also that of μ , and a hypothesised rate of return, r, thus the Gaussian identity function would be all around a more generalized and inclusive representation than the harmonic (Eugene and O'Donnell, 1997). We replace expression (92) in equation (98) to obtain

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \frac{1}{2}\kappa(1 - e^{-\chi^2})\right)\psi(\chi) = \alpha\psi(\chi) \qquad (99)$$

The market specifics and volatility influence the choice of $\gamma(\chi)$ (Joshi, 2008). In addition, a harmonic function's suitability depends on many-asset correlation factors.

We may apply the same logic for other expressions, however a similar transformation of significant impact is that of expression (84) (i.e. CASE 5.0), which is a complete match to the Black-Scholes PDE. Thus any update on the function would also be reflected in an update on the identity of the option pricing function $\psi(s, t)$ (Malliaris, 1982; Øksendal, 2000, Bru et al., 2012).

5.3.1.1. SCENARIO 1.0

Suppose $\{\chi_t, t \ge 0\}$ is a stochastic process, $\psi(\chi,t)$ is a time-independent function with continuous second order partial derivatives, tied to an identity of an asset's indemnity or a contingent claim, $\gamma(\chi)$ is assumed or observed to be an approximated linear identity.

We consider the same logic and mathematical work-out as that of CASE 5.0 to reach expression (84), recreated below:

$$\frac{\partial \psi(s,t)}{\partial t} + \frac{1}{2}\sigma s \frac{\partial^2 \psi(s,t)}{\partial s^2} + r \left[s \frac{\partial}{\partial s} - 1 \right] \psi(s,t) = 0 \quad (100)$$

We also consider the general effects of master expression, and the polynomial expression (85) with coefficients set in such a way to facilitate the creation of a decreasing exponential identity for $\gamma(\mathbf{r})$. It is evident that a decreasing exponential can be approximated by a linear potential "identity" function, which is useful because it leads to classical Black-Scholes PDE (Karatzas and Shreve, 1998a:1998b).

The linear $\gamma(\mathbf{r})$ is only a good choice for small \mathbf{r} or large σ , and not a good choice for larger \mathbf{r} (and small volatility). The linear identity function has presumed observable or measurable quantization effects and is subsequently easier to treat mathematically or apply as the classical result shows. However, we seek to generalise with better inclusion of parameter sensitivities. We can see from the Taylor expansion of a decreasing exponential

$$(1 - \exp(-r)) = +\left(\frac{1}{1!}\right)r - \left(\frac{1}{2!}\right)r^2 + \left(\frac{1}{3!}\right)r^3 - \dots + \text{h.o.t}$$
 (101)

For very small r (large σ), it can be approximated to a linear function

$$1 - \exp(-r) \approx r \tag{102}$$

Which is then substituted back to equation (100) to obtain

$$\frac{\partial \psi(s,t)}{\partial t} + \frac{1}{2} \sigma s \frac{\partial^2 \psi(s,t)}{\partial s^2} + (1 - e^{-r}) \bullet \left[s \frac{\partial}{\partial s} - 1 \right] \psi(s,t) = 0$$
(103)

Equation (103) provides a modification at a slightly more generalised level of the well reported and applied Black-Scholes PDE (Black and Scholes, 1973; Black, 1989). Another level of generalisation can be achieved by revisiting the problem treated previously (i.e. CASE 5.0).

5.3.2 CASE 7.0 – SUPERSET DERIVATIVES' PRICING PDE

Suppose $\{\chi_t, t \ge 0\}$, $\{s_t, t \ge 0\}$ are stochastic processes, and $\psi(\chi,t)$ is a time-dependent function with continuous second order partial derivatives. The composite function $\psi(\chi_t,t)$ may be represented in the following differential form:

$$-\left(\frac{\partial}{\partial t} + (1+r)\right)^2 \psi + \frac{1}{2}s\left(\frac{\partial}{\partial s}\right)\psi + \psi = 0 \qquad (104)$$

Expression (104) is a more generalised and abstract solution than the Black-Scholes PDE. Interesting in this case, the potential has a constant identity. See Appendix II – CASE 7.0 for the full derivation of equation (104).

5.3.2.1. SCENARIO 1.0

Suppose that $\{s_t, t \ge 0\}$ is a stochastic process, $\psi(\chi, t)$ is a time-independent function with continuous second order partial derivatives, tied to an identity of an asset's indemnity or a contingent claim, γ is assumed or observed to have a constant identity. We Consider equation (104) and apply the operator $\partial/\partial s$ to obtain

$$\frac{1}{2} \left[s \frac{\partial^2}{\partial s^2} + 3 \frac{\partial}{\partial s} \right] \psi = \frac{\partial}{\partial s} \left[\frac{\partial}{\partial t} + (1+r) \right]^2 \psi \qquad (105)$$

or

$$\frac{1}{2} \left[s \frac{\partial^2}{\partial s^2} + 3h \right] \psi = \frac{\partial}{\partial s} \left[\frac{\partial}{\partial t} + (1+r) \right]^2 \psi \quad (106)$$

Previously we have considered cases that led to PDEs that could allow us to establish the quantization effects (Bailey, 1966; Giller, 1982; Weidmann, 1987), often those of price or log price. However in this case it would be the operator on the right hand side, which under the variable separation effect it is a constant. We denote it by $[\Xi]$, as shown below

$$[\Xi] = \frac{\partial}{\partial s} \left[\frac{\partial}{\partial t} + (1+r) \right]^2 \qquad (107)$$

and after re-arrangement, we obtain

$$s\frac{1}{2}\frac{\partial^2 \psi}{\partial s^2} + \frac{3}{2}h = [\Xi]\psi$$
 (108)

Which can be generalized with various different Eigen-states dependent on l and k as

$$\forall l \in [0, ..., L], \forall k \in [0, ..., K] \ [\Xi]_{l,k} = \left(\frac{\partial}{\partial s}\right)^l \left[\left(\frac{\partial}{\partial t}\right)^k + (1+r)\right]^2$$
(109)

and

$$\forall l \in [0, ..., L], \forall k \in [0, ..., K] \ s \frac{1}{2} \frac{\partial^2 \Psi}{\partial s^2} + \frac{3}{2} h = [\Xi]_{l,k} \Psi$$
 (110)

A special case rises for l = 0, and k = 0,

$$[\Xi]_{0,0} = \left(\frac{\partial}{\partial s}\right)^0 \left[\left(\frac{\partial}{\partial t}\right)^0 + (1+r) \right]^2 = [1+(1+r)]^2 \quad (111)$$

Subsequently, we obtain

$$s\frac{1}{2}\frac{\partial^2 \Psi}{\partial s^2} + \frac{3}{2}h = [1 + (1 + r)]^2 \Psi \quad (112)$$

These appear to be variants of the Sturm-Liouville system (Bailey, 1966; Pruess and Fulton, 1993; Bailey et al., 1996; Kong and Zettl, 1996; Zettl, 1997; Agarwal and Wong, 1995; Kong al et., 2001:2004).

5.4. DENSITY OF STATES AND EIGEN-PRICE SYSTEM

The prices 'confined' in an infinitesimally small interval in an efficient market with normal information dissipation are strongly quantized. That is, the price spectrum, spread or pattern is discrete. As shown in Figure1.0, the quantization of price, or alternatively, the reduction of dimensionality of the system, is directly reflected in the density of states function. The density of states for a three-dimensional Eigen-price system (i.e. eigen-price, α) has the form

$$\frac{\mathrm{dN}}{\mathrm{d\alpha}} \propto \frac{\mathrm{d}}{\mathrm{d\alpha}} \left(\alpha^{\frac{1}{2}} \right)^3$$
 (113)

Where we have established previously that

$$\forall n \in [0, ..., N], \forall m \in [0, ..., M] \ \alpha = (i^n)^m \left[\frac{1}{\xi(t)} \frac{\partial \xi}{\partial t}\right] (114)$$

Similarly, for a two-dimensional Eigen-price system is a step function of the form

$$\frac{\mathrm{dN}}{\mathrm{d\alpha}} \propto \frac{\mathrm{d}}{\mathrm{d\alpha}} \sum_{\epsilon_j < \alpha} (\alpha - \epsilon_i) = \sum_{\epsilon_j < \alpha} 1 \quad (115)$$

Both of the previously established $(i^0)^2 \left[\frac{1}{\xi(t)} \frac{\partial \xi}{\partial t}\right] = r_f$ and $(i^1)^2 \left[\frac{1}{\xi(t)} \frac{\partial \xi}{\partial t}\right] = -r_f$ satisfy condition $\forall j \in N \ \epsilon_j < \alpha$. On the other hand, a one-dimensional system has a singularity,

$$\frac{\mathrm{dN}}{\mathrm{d\alpha}} \propto \frac{\mathrm{d}}{\mathrm{d\alpha}} \sum_{\epsilon_i < \alpha} (\alpha - \epsilon_i)^{\frac{1}{2}} = \sum_{\epsilon_i < \alpha} (\alpha - \epsilon_i)^{-\frac{1}{2}} \quad (116)$$
193

In this research we are particularly interested in a zero-dimensional system, which turns out to have the shape of δ -peaks,

$$\frac{\mathrm{d}N}{\mathrm{d}\chi} \propto \frac{\mathrm{d}}{\mathrm{d}\chi} \sum_{\epsilon_i < \alpha} \Theta(\alpha - \epsilon_i) = \sum_{\epsilon_i < \alpha} \delta(\alpha - \epsilon_i) \quad (117)$$

where, ε_j are the discrete eigen-price levels, Θ is the Heaviside step function, and δ is the Dirac function. This is actually the definition of the density of states! The sum of delta functions turns into the other forms, once it is integrated over the phase space in 1, 2 or 3 dimensions. i.e. in limit the summations turn into integrals.



Although the 0D are considered to be semi-zero dimensional systems, in mathematical terms, they are dealt with as 3-D coordinates. This is so, because we are dealing with 3-D market price confinement. The price system is said to be confined, when it confines Eigen-price to regions comparable in size to 1/100 of 1% of 1bp in finance terms. When such systems have confined Eigen-prices on the spaces mentioned above, then the quantum price effects arise. On 0D price systems, the space is quantized and the Fermi-Dirac statistic have full validity. Also the prices we are treating in this research are prices

with a spin; both positive and negative price changes occur. The number of 0D Eigen prices per unit "volume" is given by

$$n(\alpha) = g(\alpha)f_{FD}(\alpha)$$
(118)

Where $g(\alpha)$, is the density of state of a 0D market price system and f_{FD} is the Fermi-Dirac distribution function

$$f_{FD}(\alpha) = \frac{1}{\left\{ \exp\left[\frac{1}{2} \left(\frac{\alpha - \mu}{\sigma}\right)\right] + 1 \right\}}$$
(119)

Referring to Eigen-price density instead of density of states is of relevance for the market information absorption and continuous-time price spectrum to achieve price change predictability along an efficient market price line (Merton, 1990). The treatment above is in a regulated and conditioned market context.

The density of states of price in an unconditioned market can also be established for 3D, 2D, 1D, and 0D systems. The density of states of a 3-dimentional unconditioned market price is proportional to the square root of the rate of return.

$$g(\alpha) = \sqrt{2\alpha} \quad (120)$$

The density of states, for an ideal 2-dimentional system, is constant.

$$g(\alpha) = 1 \quad (121)$$

Whereas the density of states of a one-dimensional system has a square-root singularity at the origin, as shown below

$$g(\alpha) = \sqrt{\frac{2}{\alpha}}$$
(122)

The case which is of special interest to us in this research is that of the density of states

of 0-D unconditioned market price systems. An ideal zero-dimensional price is one that exists in a single state of fixed price change or range, α_0 . The density of states is then given simply by

$$g(\alpha) = \delta(\alpha - \alpha_0) \qquad (123)$$

where $\delta(\alpha)$ is the Dirac delta-function (Dirac, 1926:1958). We can refer to our previous finding for α_0 , subsequently setting it to r_f , thus ideally we would expect the price change to be in the α - r_f range.

5.5. INFORMATION DISSIPATION, REFLECTION AND RELAY FROM 0D PRICE SYSTEM

On understanding the market information dissipation and information reflection on price concepts, one needs to recall the fact that we are dealing here with infinitesimally small intervals. On most of 0D price systems and efficient markets, their properties will be dependent on filtrations, the market conditioning, and the degree of uncertainty. Such dependencies can be deficient and hence influence the properties in a 0D price system.

The "size" of the 0DS (zero-dimensional system), the shape and depth of the potential 'identity' function influences the density of states and the Eigen-price density. The information-reflection on the price pattern is linked with the quantity and quality of the information dissipated by the market and reflected at the 0DS level. On the other hand, the 0DS relays part of the information linked with the dissipation and reflection of information from the 0DS. The information dissipation and relay process can dictate the

prediction of a financial instrument's market price.

Information relay from the 0D system is a way of assessing the quality of dissipation by the markets and the information conservation system (0DS) properties. The information relay pattern is also relevant to us, as it is one measurement to which the calculations of the quantum price levels could be compared. For such comparison, one would need to carry out measurements of the spatial information relay with observable historical price horizons and information dissipation, then make the right plots of those quantities with respect to the Eigen-price. All the "significant" peaks on such plots correspond to 0DS of different dissipation depths (for instance, on price arrays/points on the historical line), whereas the rest of the "smoother" peaks would correspond to the transitions within each of the 0DS, i.e. such measurements are outside the scope of this research!

The quasi-zero-dimensional quantum price points are a complete quantization of the price's evolution path. The strong constraints imposed by the parameters recreates a "central" eigen-price tendency, it justifies the potential 'identity', and gives rise subsequently to the market motive dynamics and the price line.

5.6. REBRANDING THE MASTER EXPRESSION

We have used our master formulation throughout our research. Furthermore it has led to 0D quantum fittings with focus on information dissipation, which is key to price change predictability. Thus at this point we prefer to refer to it as the effective information equation, with the potential 'identity' function as the cut-off price potential. Most simply, the price value confined by a cut-off price potential can be treated in the effective information approximation. The positive and negative information causes the price in expectation to be oscillatory in nature. This is a phenomenon that is empirically evident. The Bloch theorem is adapted for such approximation and it tells us that the true wave function Ψ can be written as the product

$$\Psi \approx \psi(\mathbf{r})\mathbf{u}(\mathbf{r})$$
 (124)

Where u(r), the Bloch function describes the rapidly-varying price part of the wave function and ψ , sometimes called the envelope function describes the part which is slowly varying on a 0-dimension quantized price point and scale. The envelope function obeys an expression which is a variant of our main axiom - like the equation which in the simplest cases near q=0, takes the form

$$\left[-\frac{1}{2}\sigma^{2}\nabla^{2} + \gamma(\mathbf{r})\right]\psi = \alpha[\mathbf{p}(\mathbf{r})]\psi \quad (125)$$

Where the price $\alpha(\mathbf{r})$ is measured at the cut-off edge, ψ is the envelope function, and γ does not include the spatial price potential. The entire effect of the spatial price potential is to change the information relevant to shift price from α to α^* , where both values are linked to probabilities. The probability of α^* is worked out from perfect information within the Bayesian framework. The price potential on the equation above contains the effect of all the external price behaviour potentials, and in particular, that due to changes in the market-line cut-off.

We turn our attention now to the boundary conditions. The ideal "classical" boundary conditions for ψ as treated by most quantum mechanics books and related research publications (Bailey, 1966; Bailey et al., 1996; Canessa, 2007; Yan et al., 2017) are given by

$$\psi_{I}(0) = \psi_{II}(0) \quad \text{and} \quad \frac{d\psi_{I}}{dx}\Big|_{x=0} = \frac{d\psi_{II}}{dx}\Big|_{x=0} \quad (125)$$

where ψ_I and ψ_{II} are respectively the "wavefunctions" of spinned prices (spin up and spin down) inside and outside on any "theoretical" quantum price cut-off region. An illustration of a quantum price well is shown below



In the market surface however, the situation is different. There are various ideas proposed for the matching conditions of a function at market surface. The matching conditions presented here are those which are known as the Bastard conditions (Bailey, 1966). Those matching conditions were suggested and used even earlier by Ben-Daniel and Duke in 1955.

At the market surface/interface, the effective information and the cut-off potential are effectively discontinuous. The "proper" boundary conditions, at the market interface (x=0) between information/events I and II are

$$\Psi_{\rm I}(0) = \Psi_{\rm II}(0) \text{ and } \frac{1}{p_{\rm I}} \frac{d\Psi_{\rm I}}{dx} \Big|_{x=0} = \frac{1}{p_{\rm II}} \frac{d\Psi_{\rm II}}{dx} \Big|_{x=0}$$
 (126)

The boundary conditions given above can be shown to imply conservation of the probability information flow J for the full wave-function Ψ ,

$$J = -Re\left[\Psi^* \frac{a}{p_0} i \nabla \Psi\right] \qquad (127)$$

It is averaged over a small section containing the market surface interface. It also implies the existence of steady-state solutions, for the full wave-function Ψ , or over two adjacent market regions.

Having introduced the effective information equation, we will refer to it as EIE from now on. Due to the oscillatory wave-like nature of the price, we establish the normalization condition.

$$p(r,t) = |\Psi(r,t)|^2$$
 and $\int p(r,t)d\tau = \int |\Psi(r,t)|^2 d\tau$ (128)

Integrated over the market region. One example is the case of the infinitely deep rectangular 0DS.

The main postulated master expression has been adapted to a new quantum mechanics context to facilitate strongly quantized asset price valuations. We subsequently transformed it to the effective information equation for quantum-price systems (ODS). Going from the postulated formulation to the effective information equation is trivial. The effective information equation is the postulated main equation modified to read the effective information of market-impacting events, so that it takes into account the effect of the market potential on the price evolution. One alternative interpretation p^* is to see it as representing the confidence that a marketaligned investor holds on the degree of information dissipation by the market on recent events and under the assumption of market efficiency (Latham, 1986). The p^* can be seen as the confidence we have in the information and its impact in the price i.e. 0.90, can be interpreted as 90% confident that the information will have an impact on the price and as such is a subjective measure. However well aligned market makers and investors should freely use such measures based on their strong "views" on the markets. Its value should vary from 0 to 100%. It can also be written as $p^*=b = 1$ -p where p is the significance level of the information. The confidence level in this case is a measure of the quality of the filtration.

6.0 COMPUTATION OF ASSET PRICES IN FORWARD TIME WITH EIGEN-VALUE CONVERSION

This chapter follows from the theoretical framework, developed in previous chapters, for a strongly parameterized zero-dimension quantum price-system. The theoretical model considers zero-dimension quantum price systems of various "geometries", corresponding to the particular strength of the parameter–based potential functions or price potentials. The underlying theory is based on the GSE and modelling using various programming languages such as VBA, MATLAB, and C/C++. We have assigned a reflective price identity to the potential function . Within the 0D price constraint, it represents a physical geometry, emulated by the strength of filtrations and the market absorption of such filtrations. This leads to a confining and measurable system of Eigen-prices for each point along the market line. In order to measure Eigen-price effects, we define the discrete time dimension constraint. An Eigen price can only exist in a system with no dimensions. The time dimension is lost within an interval of less than 10 nseconds, in other terms, the price system is discrete in that confinement. Such systems can be called quasi zero-dimensional. The number of probable Eigen prices can be between 1 and 1000 within such constraints.

The quasi-zero dimension treatment of the market surfaces is not considered in existing derivative pricing models. This chapter follows from our initial investigation on the possible inclusion of the quasi-zero dimension effects in a contingent claim's payoff expectation term, and it intends to draw more attention to quasi-zero Eigen price systems. The small size makes them good at responding to a very narrow price spectrum range, thus making them suitable for more effective price change predictions.

Practically, the 'geometry' of the confining 0D system is not a geometry in the true sense of the word, instead it allows us to establish cut offs in terms of how deep filtrations are absorbed by the market, and to what degree market prices are affected. Hence, we have modelled different geometries. Theoretically the cut-off of a quantum price system can be of any geometry, such as cubic, spherical, cylindrical, pyramidal etc. To build a model that fits such practical applications, one has to solve the GSE derived expressions for different cut-off price potential functions. Further-more we seek to measure price change effects on the basis of fitting a Sturm-Liouville system on our theoretical framework.

Empirically, a price evolution path can be seen to display, however randomized,

oscillatory properties. In order to test the concepts and justify our theoretical framework, we have made the following assumptions:

- (i) A financial asset's price evolution is a randomized oscillation or harmonic in nature.
- (ii) All information is reflected in price and has a two dimensional effect (1) orthogonal dissipation at the price point (time 0) and (2) relay over time and across other assets.
- (iii) The asset's price can be modelled as a randomized outcome with two components(1) an imaginary space, and (2) an observed component in real time.
- (iv) At any price point and under the quasi-zero dimensional constraints, the price and subsequently the rate of return is quantized. The market price is the quantized price at the most stable degenerate level.
- (v) The quasi-zero dimension price point encapsulates a number of possible states with a price spread affected by factors such as uncertainty, and quality of information.
- (vi) Smoothness and continuity of a price evolution path are a result of discrete-time space information dissipation and relay. Discrete price jumps or orthogonal shifts are due to effects of continuous time-space events i.e. ex-dividend, etc.

The stochastic shock effect due to filtrations, absorbed by the markets and reflected in stock price is often modelled through the Weiner process and has a Gaussian distribution (Shreve, 2004). However, this study suggests that the "real" probability distribution system contains various probability distributions with various degrees of distribution mixing. Where the system itself exists in various Eigen-state levels of mixing or

separation, quantization effects follow directly from the 3x expansion of the pricesystem's degrees of freedom and under infinitesimally small time-space constraints.

The Gaussian probability distribution function corresponds to the lowest and most stable Eigen-state, however at higher quantized levels (i.e. higher volatilities and irregularities in filtrations patterns), there are distribution splits, evident both mathematically and through numerical illustrations. Further on the new model considers the additional price change effect due to the quantization effects at each price point. It subsequently models the cumulative price-change (i.e. more precisely the logarithmic price) effects through the classical pricing models with the additional quantization effects on the price behaviour at a quasi-zero time dimension of each point along the market line.

This has profound implications in financial instrument pricing, especially in financial derivatives because the classical models such as Black and Scholes option pricing (Black and Scholes, 1973; Black, 1989), make use of a probability distribution with normal or lognormal distribution considerations. Under the new model this needs to be upgraded to include the probability distribution system mixing and the additional Fermi-Dirac probability distribution for the quasi-zero price point system along the market line. This has interesting pricing implications, where the expectation term need an upgrade to include both the special and orthogonal effects. Therefore each realised market price is the result of a two-way gamble; apart from N(d1) and N(d2) set, one would also need to include the conditioned Fermi-Dirac probabilities.

Hence, the objective of this chapter is threefold: First, we reconsider pricing problems starting from our formulated axiom, leading to asset price and financial derivative valuation formulations in partial differential forms; secondly, we consider valuation expressions that are in line with a rational investor's expectations with various price cut-off identities; thirdly, we incorporate quantization effects in the probability distribution and the price-change systems, and subsequently provide a Sturm-Liouville solution (Bailey, 1966).

In this chapter we set a postulated problem with a very abstract formulation and solve various common cases that are in line with contemporary pricing models (Bodie et al., 2009); the problem is a financial instrument valuation challenge springing from an axiom, with Sturm-Liouville adapt-solutions, in order to compute quantization effects at a price's point along the market line. Our approach and results are related to classical and contemporary work in asset pricing, among others, the work of Black and Scholes (1973), Sharp et al. (1995), Shreve, (2004), Bodie et al. (2009), and Elton and Gruber (2011), but moves in to the gaps in literature and goes beyond market tracking (Hillier et al., 2011), to the dynamics of the price function not only spatially over a time horizon, but also at a zero-dimension point, thus implicating orthogonality in the combined probability distributions.

We use various computational procedures and techniques to illustrate several effects and cases. We set the stage with a VBA generated demonstration of the effect of stock price

simulation using an Euler discretized Brownian motion, generating various price paths (Jackson and Staunton, 2004). Next we use the Metropolis Algorithm described well in Gilks et al. (1996), which allows us to set a proposal probability function and test various scenarios of probability distribution mixing. The MATLAB program implements Monte-Carlo integration and generates various distribution mixings. Further on, various C++ programs invoke NAG routines to compute the price-changes using a Sturm-Liouville solution, which are then plotted in Excel.

6.1. DISCUSSIONS AND NUMERICAL ILLUSTRATIONS

In this section, we illustrate numerically the abstract stochastic asset pricing model, Monte-Carlo integration with Metropolis Algorithm and the quantization effects through Sturm-Liouville fittings with various gamma functions, leading to pricing.

6.1.1. GEOMETRIC BROWNIAN MOTION SIMULATION OF STOCK PRICE

In this section, we simulate stochastic price paths also as a context for further theoretical and practical expansion with quantum system fittings at each point along such simulated paths. The price of the stock at time t is S_t where S_t is determined by the stochastic differential equation

$$dS(t) = S(t) [\mu dt + \sigma dW(t)]$$
 (129)

with {W(t), $t \ge 0$ } being a standard Brownian motion and $\sigma > 0$, μ are constants. In the context of finance, the parameter σ is known as the volatility of the stock (Boobs, 1953;

Karatzas and Shreve, 1998a:1998b; Hull and White, 1987; Stein and Stein, 1991). It follows that the stock-price process $\{S(t), t \ge 0\}$ is an exponential Brownian motion and it may be represented as

$$S(t) = S(0) \exp[(\mu \frac{1}{2}\sigma^2)t + \sigma W(t)]$$
(130)

where S(0) is the initial price of the stock, which it may be assumed is observed at time 0. The information available at time *t* is the history of the price process, $\mathcal{F}_1 = \sigma(S_u, 0 \le u \le t)$, that is the information obtained by observing the movements of the stock price process up to time *t*; equivalently, it is $\sigma(W_u, 0 \le u \le t)$, the information obtained by observing the driving Brownian motion in the stochastic differential equation (Kennedy, 2010). We expand $W(t) = Z(t)\sqrt{t}$, and for the purpose of the numerical illustration, we generate values of Z(t) as a random variate computed as a sum of 12 random numbers, each between 0.0 and 1.0, finally subtracting 6 (Jackson and Staunton, 2004). Further-more the programmable procedure evolves state variables using an Euler discretisation ln(S(t)), where z is ~N(0,1).





Figure 3: Four simulations/paths of a Geometric Brownian Motion for stock price, using 12 random uniforms to generate 'normal' variates⁷⁶.

The orthogonal 0DS in the later sections are fittings along each of the price paths simulated above.

6.1.2 MONTE-CARLO PROBABILITY DENSITY MIXING

Throughout this chapter we have focused on the PDEs (refer to previously explored problem cases and scenarios) of relevance to financial instruments, and as this research does not develop further around expectation expressions, we think it is important to demonstrate simulated valuation at an expectation level as is the case of probability density distribution. We stress the importance of the Gaussian probability density function and its role in the computation of the expectation of the rate of return (Spitzer, 1970; Snyder and Miller, 1991; Seneta, 1996).

⁷⁶ Algorithm can be provided to anyone upon request.

However, the proposed density function can be either a variant of the Gaussian or an entirely different function. We also established in CASE 5.0 that Gaussian is appropriate as a special case and that dependent, we say, on n (and non-regularity in the filtrations impact on the price), the density distribution can be replaced by a mixture of various Gaussians. We construct a Markov Chain fully simulated with the Monte-Carlo method and draw dependent (correlated) states, χ_0 , χ_1 , χ_2 , ..., from the chain, and subsequently perform Monte-Carlo integration with u^T (χ) (Robert and Casella, 2004).

Let χ , $\chi' \in \chi$ be states in the chain and $v^{P}(\chi'|\chi)$ be an arbitrary, easy-to-sample from an arbitrary proposal distribution (Robert and Casella, 2004), which does not satisfy detailed balance, and hence may not be a stationary distribution. However, suppose that

$$\mathbf{v}^{\mathrm{P}}(\boldsymbol{\chi}' \mid \boldsymbol{\chi}) \bullet \mathbf{u}^{\mathrm{T}}(\boldsymbol{\chi}) > \mathbf{v}^{\mathrm{P}}(\boldsymbol{\chi} \mid \boldsymbol{\chi}') \bullet \mathbf{u}^{\mathrm{T}}(\boldsymbol{\chi}') \quad (131)$$

then, there is a factor $r(\chi'|\chi) \le 1$ such that the above inequality is balanced

$$[v^{P}(\chi \mid \chi) \bullet u^{T}(\chi)] \bullet r(\chi \mid \chi) > [v^{P}(\chi \mid \chi) \bullet u^{T}(\chi)] \bullet r(\chi \mid \chi)$$
(132)

and solving for r, yields

$$r(\chi' | \chi) = \min \left[1, \ \frac{u^{T}(\chi') \cdot v^{P}(\chi | \chi')}{u^{T}(\chi) \cdot v^{P}(\chi' | \chi)} \right]$$
(133)

which is then converted into an algorithm. Gilks et al. (1996), describes in good detail the

algorithm itself which is known as the Metropolis Algorithm. In such numerical simulation given a target distribution $\Psi(\chi)$, the proposal q would be valid if

$$\operatorname{supp}(u) \subseteq \bigcup_{\chi} \operatorname{supp}(q(. | \chi))$$
(134)

where $supp(u) = \{\chi : u^T(\chi) > 0\}$ is the support of distribution u i.e. the non-zero probability set (Gilks et al., 1996). The condition itself supports the fact that that our proposal must have non-zero probability of moving to the states that have non-zero probability in the target. The actual distribution of the algorithm is given by

$$p_{M}(\chi' | \chi) = v^{P}(\chi' | \chi) \bullet r(\chi' | \chi) + I(\chi' = \chi) \bullet \left[1 - \int v^{P}(\chi' | \chi) \bullet r(\chi' | \chi) \bullet d\chi'\right]$$
(135)

A common proposal distribution to use is a Gaussian: $v(\chi'|\chi)=N(\chi'|\chi, \sigma^2)$, under an algorithmically simulated procedure known as the random walk Metropolis-Hasting algorithm (Andreieu et al., 2003). In the computational simulation itself, it is important to pick the right volatility σ to ensure that a reasonable number of the proposals are accepted. Under the algorithmic procedure, if the new state χ' is more probable than the current state χ , the proposal is always accepted $r(\chi'|\chi)=1$, otherwise it is accepted with probability $u(\chi')/u(\chi)$, however the proposal could also be independent of the current state: v(x'|x)=v(x') (Gilks et al., 1996). Under the procedure we a mixture of two 1D Gaussians as the target distribution.

$$\mathbf{u}(\boldsymbol{\chi}) = \mathbf{w}_1 \boldsymbol{\mathcal{N}}(\boldsymbol{\chi} \mid \boldsymbol{\mu}_1, \boldsymbol{\sigma}_1) + \mathbf{w}_2 \boldsymbol{\mathcal{N}}(\boldsymbol{\chi} \mid \boldsymbol{\mu}_2, \boldsymbol{\sigma}_2)$$
(136)

where $w_1 + w_2 = 1$ are the mixture weights. The proposal is a 1D Gaussian $v(\chi'|\chi) =$



 $N(\chi' | \chi, \sigma_p)$, where σ_p , is a parameter of the proposal (Gilks et al., 1996).

Figure 4. A Metropolis Hastings algorithm, sampling from a mixture of two 1D Gaussians using a Gaussian proposal with variance $\sigma^2 = 10^2$. Figure produced with Matlab⁷⁷.

The above algorithm can be adjusted to different degrees of probability distribution mixing, but also in cases where there is asymmetry in the proposal distribution and that may differ from the Gaussian (Gilks et al., 1996).

A close look at the expression CASE 5.0 of reveals that the probability distribution

⁷⁷ Algorithm can be provided to anyone upon request.

system possesses different quantized states (Schwartz, 1967), and that such system is absolutely stable at n=0 or the ground state only, however the system can also exist in higher eigen-states, where a distribution split is evident and of profound implications in both asset pricing and contingent claim valuation. We argue that the split in the probability distribution is in part the result of high volatilities and high irregularities in the filtration patterns. The first two eigen states (n=0, and n=1) of probability distribution mixing system are shown in figure 4.



Fig. 5. The probability density functions u_0 and u_1 . A distribution split occurs at n=1 or if there are different distribution contributions, they are only mixed in absolute terms at n=0.,

6.1.3 AUGMENTED MARKET-MOTIVE POTENTIAL FUNCTION AND INFORMATION TUNNELING

Having declared the general potential 'identity' function as a polynomial in the previous chapter, and rebranded it to the price cut-off potential, we further link it to dynamics of market price evolution, by referring to it as the market-motive potential interchangeably.

Furthermore, on our theoretical consideration we are more specific and choose several functions and specify boundary conditions for each of the 'geometries' of the price-cutoff potential. These geometries are important in providing a finance space in market information penetration, but also in consideration of adjacent market regions with region abruptness linked with the degree of information tunnelling to the next market region, subsequently allowing us to contemplate the effect of information tunnelling and impact of fragmented information reflection in a future market price. On all the cases we have applied the same boundary conditions; R(r) is finite for r = 0, since Y = rR = 0 for r = 0, R can be finite and so we define it to be such. R(r) = 0 as $r \rightarrow \infty$ (Bailey, 1966; Pruess and Fulton, 1993; Bailey et al., 1996; Kong al et., 2001).

After converting to spherical coordinates, we establish an absolute finite limit to the depth of information penetration of the market and the quantum space radial consideration. Thus our previous potential functions and our initial potential functions are transformed to

	Name	AxIOM IDENITY FUNCTION	Augmented Axiom Idenity Function
1.	GAUSSIAN	$\gamma(\chi) = \gamma(0)[1 - \exp(-\chi^2)]$	$\gamma(\mathbf{r}) = \gamma(0) \left[1 - \exp\left(-\frac{1}{L^2} \left(\frac{\mathbf{r} - \mu}{\sigma}\right)^2\right) \right]$
2.	HARMONIC	$\gamma(\chi) = \frac{1}{2}\kappa\chi^2$	$\gamma(\mathbf{r}) = \frac{1}{2} \kappa \frac{1}{L^2} \left(\frac{\mathbf{r} - \mu}{\sigma}\right)^2$
3.	DECAYING EXPONENTIAL	$\gamma(\chi) = \gamma(0)[1 - \exp(-\chi)]$	$\gamma(\mathbf{r}) = \gamma(0) \left[1 - \exp\left(-\frac{1}{L}\left(\frac{r-\mu}{\sigma}\right)\right) \right]$
4.	LINEAR	$\gamma(\chi) = \gamma(0)\chi$	$\gamma(\mathbf{r}) = \gamma(0) \frac{1}{L} \left(\frac{\mathbf{r} - \mu}{\sigma} \right)$
5.	CONSTANT SQUARE WELL	$\gamma(\chi) = \gamma(0)$	$\gamma(\mathbf{r}) = \gamma(0)$

TABLE 2.

6.	ARCTAN	$\gamma(\chi) = \frac{1}{2}\gamma(0) \left[\frac{2}{\pi}\arctan(\chi) + 1\right]$	$\gamma(\mathbf{r}) = \frac{1}{2}\gamma(0) \left\{ \frac{2}{\pi} \arctan\left[\frac{1}{bL}\left(\frac{\mathbf{r}-\mu}{\sigma}\right)\right] + 1 \right\}$
7.	COSH ⁻²	$\gamma(\chi) = \gamma(0)[1 - \cosh^{-2}(-\chi^2)]$	$\gamma(\mathbf{r}) = \gamma(0) \left[1 - \cosh^{-2} \left(-\frac{1}{L^2} \left(\frac{\mathbf{r} - \mu}{\sigma} \right)^2 \right) \right]$
8.	INVERSE	$\gamma(\chi) = -\gamma(0)\frac{1}{\chi}$	$\gamma(\mathbf{r}) = -\gamma(0) \left(\frac{\mathbf{L}\sigma}{\mathbf{r}-\mu}\right)$

6.1.4 SUITABLE POTENTIAL FUNCTIONS

We decide on the suitability of potential⁷⁸ by using a criterion that takes account of the market specification combined with a theorised rationale. Different markets have varied information dissipation capabilities and its regulation and governance, which provides for the depth of the quantum dot or its radius. We may apply the same potential for different zero-dimension systems, with market particularities undercutting the impact of information dissipation and the market depth. Thus we would expect the candidate potential functions to change across the interface by depth value. We have used the dispersion measure as a linear measure of that depth and assume it to be homoscedastic.

The higher the market regulatory/self-regulatory nature and external governance, the deeper the effect of information penetration in the market region of investigation. For forecasting purposes of research, we consider two adjacent market regions; the region of reference is a region observed, where all information dissipated is reflected in the observed market price, whereas the second region is a future market region with an abrupt

⁷⁸ It refers to the cut-off price potential or market motive potential function as an alternative short-hand name.

interface between them. Our assumption on the abrupt nature of the "orthogonal" interface between two adjacent market regions, which we may visualise as a form of a quantum well, is in line with the fact that we also expect the tunnelling of the filtration or a fraction of it to the future region with difficulty, which complies with the fact that it would be improbable to forecast market price without prior knowledge of future events.

However, information that is already known and dissipated by the reference region is possible to tunnel through to the next region and may hold the key to the forecasting of the market price. If we take the view that the separation interface is abrupt, then on the same argument we may discard the harmonic and the triangular cut-off market price function (i.e. potential).

The choice of the cut-off price function depends upon the abruptness of the boundary between the two market regions. We may also consider different layers in each market region and assume some separation between each as well as with the spatial market region. Following this argument, we can say that the rectangular potential is a reasonable function to consider. However the spherical is a simple model and we consider it below in comparison with others (Möller and Zettl, 1996).

On a OD quantum price system of a well-regulated or self-regulated market, the market region can be viewed as graded in layers with interfaces between and among the regions. The interfaces are formed by the observed reference /adjacent regions and therefore the interfaces between them are not abrupt. So a continuously varying cut-off potential may be fitted. This filtration dissipation can be very selective, therefore both ARCTAN for
reasonably small values of dispersion, σ and the COSH⁻², potentials would appear to provide a reasonable fit. However for prices evolved by the market under modulated filtrations, one would expect the potential to be smoother at the edges, one could reason that the Gaussian would be ideal for this situation. Dependant on how modulated the information is and internal "texture" of the market region under consideration with the occurrence of information diffusion, the Gaussian cut-off price potential is probably the most suitable one (Möller and Zettl, 1996).

Thus from potential functions considered here, the most suitable are spherical square well (constant potential), Gaussian, Cosh, Harmonic – step function. The Arctan potential, for very small values of dispersion, σ , becomes similar to the square well (constant potential).

We consider the quantized space measurable in terms of $q\sigma$, where $q = 10^{-7}$ of a basis point, where s is taken to be the minimum 10^{-4} , the smallest dispersion in our consideration, subsequently 1qs is equal to 10^{-11} . We call this a space conversion or transition coefficient. In practical terms we can convert market parameters defined in a continuous \mathbb{O}_{11} system to equivalents in the quantized \mathbb{O}_{11} and vice versa. On a spherical representation of a quantized market price, the radius is expressed in $q\sigma$ terms, this way we avoid use of SI units for space that in Finance would not have relevance. The 'space' here is in terms of a price, subsequently it is a price-space. We also avoid use of currency symbols by using % representations, for example μ becomes quantized as $q\mu$ and σ similarly to $q\sigma$. We have put greater emphasis in dispersion and its conversion to a quantized measure as a measure of the 'depth' of the quantized market price point, hence the introduction of the unit $q\sigma$ in our further theoretical and numerical considerations.

The choice of γ also depends on the size of the quantum price point. For small 0D systems defined with a radius of a less than $10^3 q\sigma$, a substantial 'mixing' of market regions and information tunnelling is likely to occur across the region's interface, and the γ within the quantum price point may not be 'flat'. In these cases, a square well would be a poor approximation and a continuously varying γ such as the Gaussian would be a better fit. For larger quantised market points (~ $10^4 q\sigma$ radius) the square well γ , which has $\gamma = 0$ everywhere within the market price point, is better suited.

The Gaussian γ offers a good approximation for small quantum market points. For small values of r the Gaussian can be approximated by the harmonic, which is useful because the eigenvalues of a harmonic γ are known! This can be seen by using the Taylor expansion

$$\gamma(\chi) = \gamma(\chi)|_{\chi=0} + \left(\frac{d\gamma(\chi)}{d\chi}\Big|_{\chi=0}\right)\frac{\chi}{1!} + \left(\frac{d^2\gamma(\chi)}{d\chi^2}\Big|_{\chi=0}\right)\frac{\chi^2}{2!} + \dots + \left(\frac{d^n\gamma(\chi)}{d\chi^n}\Big|_{\chi=0}\right)\frac{\chi^n}{n!} + \dots (137)$$

$$1 - \exp(-\chi^2) = 0 + 0 + 2\frac{\chi^2}{2!} - 0 - 12\frac{\chi^4}{4!} + \dots \text{ h. o.t} \quad (138)$$

For $\chi \ll 1$, terms χ^4 , χ^6 ... approach zero much faster than χ^2 . Thus Gaussian is approximated by the harmonic function

$$1 - \exp(-\chi^2) \approx \chi^2 \tag{139}$$

Where $\chi = \frac{1}{L} \left(\frac{r-\mu}{\sigma} \right)$. The relationship between stochastic variables χ and r is based on the usual convention between two random variables $\chi \leq r$, or $\chi = r$. A relationship that holds with 100% certainty. The uncertainty, random variable and the underlying stochastic process representation χ is obtained through augmentation of the stochastic variable $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. Further-more r can be obtained as the difference of two non-negative random variables, r = r + -r - where r + = max(r, 0) and r - = max(-r, 0) under the provision that at least one of E(r+) and E(r-) is finite and defines E(r) = E(r+) - E(r-) - finite and the random variable r when E | r | $< \infty$, that is when both E(r+) and E(r-) are finite (Doobs 1953; Feller, 1971; Parzen, 2015).

To represent a market price point at a quantum level, the harmonic should be cut off so that $\gamma(\mathbf{r}) = \gamma(0)$, values of r above some critical value, where $\gamma(\mathbf{r}) = \gamma(0)$). We refer to this as the conditioned harmonic γ . On the other hand, when we decrease the value of L (smaller quantum price point), $(\mathbf{r}-\mu)/\sigma \ll L$ becomes less "true", thus more terms on the right hand side of the expansion would be needed. The smaller the radius the more terms should be included on the expansion, hence the Gaussian is better. This shows that the for small quantum price points \mathbb{D}_{11} the harmonic is a poor approximation and the Gaussian is more realistic.

The Cosh⁻² is another smooth cut-off price potential that serves as a good approximation for small quantum price points. Using the Taylor expansion, one can see that this γ function as well as the Gaussian are good approximations for small quantum price points

 $\mathbb{D}_{\uparrow\downarrow}$. The spherical square well γ is a good approximation for large quantum price points $\mathbb{D}_{\uparrow\downarrow}$. On this occasion the smooth price-cut off function such as Gaussian and COSH⁻² lose validity and a rectangular well with 'abrupt' edges may be a better approximation.

The Arctan is a way of simulating "roughness", possibly when diffusion of information across the market line is significant. The Arctan's potential suitability depends on the value of the parameter b and the size of the quantum price point $\mathbb{O}_{\uparrow\downarrow}$. This price cut-off function can be a good approximation for a large price point $\mathbb{O}_{\uparrow\downarrow}$. For such cases the arctan well is very similar to the rectangular well. The Arctan is more favourable than the square well, because it is more "adjustable" and "controllable" with respect to the "smoothness" of the 'edges'. It can be seen as the square well with rounded edges for b very small. For more detailed analysis, this price cut-off γ can be expanded, using the Taylor expansion. The Harmonic γ function becomes a good approximation in the cases when there is no overlap of filtrations, or the sequence of events is orderly with the quantized price point, $\mathbb{O}_{\uparrow 1}$.

The decreasing exponential γ is too smooth for even small quantum price points, $\widehat{\mathbb{D}}_{\uparrow\downarrow}$. Therefore it will be discarded. Moreover, the price gradient in the region near r = 0 could not be possible. This γ approximates to a triangular cut-off price function for small values of $\chi = \frac{1}{L} \left(\frac{r - \mu}{\sigma} \right)$. This can be seen from the Taylor expansion of the decreasing exponential

$$1 - \exp\left[-\frac{1}{L}\left(\frac{r-\mu}{\sigma}\right)\right] = +\frac{1}{L}\left(\frac{r-\mu}{\sigma}\right) - \frac{1}{2L^2}\left(\frac{r-\mu}{\sigma}\right)^2 + \frac{1}{6L^3}\left(\frac{r-\mu}{\sigma}\right)^3 - \dots + h. \text{ o. t} \quad (140)$$

when $\chi = \frac{1}{L} \left(\frac{r-\mu}{\sigma} \right)$ is very small, then

$$1 - \exp\left[-\frac{1}{L}\left(\frac{r-\mu}{\sigma}\right)\right] \approx \frac{1}{L}\left(\frac{r-\mu}{\sigma}\right) \tag{141}$$

The inverse cut-off price function cannot be a good approximation for quantum price points, $\mathbb{D}_{\uparrow\downarrow}$. because it would imply some form of singularity in the quantum price point centre. The internal quantum price point dynamics do not imply such lateral γ effect.

As mentioned above, the suitability of the price cut-off function factors in market specification combined with a theorised rational and our choice of γ depends upon the "abruptness" of the boundary between the two market regions and possibly the different layers in each market region. In addition, the harmonic cut-off price function depends on the many events in filtrations, their sequencing and overlapping effect. Comparing the inverse γ with the lateral equivalents could provide a venue of investigation into the nature of event-sequencing and overlapping and their impact on pricing.

6.1.5 NUMERICAL ANALYSIS

We are interested in the log price distribution of the eigenvalues of the 0D system, or the density of states. We consider logarithmic price, because we consider each of the eigenvalues, in a discretized time-space system, to be part of a continuum of market price points or a network of 0D systems or market price points with a c continuous compounding effect. To look at this we are able to plot the eigenvalues against log price as a histogram. For widely spaced eigenvalues, the range appears to contain a series of delta functions. By varying the size of the logarithmic price "bin range" we are able to

vary the resolution of the density of states plot. With a small enough bin range, the range always consists of discrete states.

We compiled the histograms from the data generated for different programs representing the different symmetric γ functions listed above. In all cases the data were generated by a C++ program containing NAG Routines⁷⁹. Data generated appear to show discreteness of the log price levels for small quantum price points - price conservation dots in the market line. The increase of log price levels is dependent on the cut-off price function depth, quantum price point's radius $(x^2 + y^2 + z^2)^{\frac{1}{2}}$, etc. Running the programs and analysing the data, we observed the following:

The number of price-conservation levels located is controlled by a loop. The number of eigenvalues which exist is controlled by γ_0 , where γ_0 is a certain depth of the well. For the quantum price systems $D_{\uparrow\downarrow}$ and the different price cut-off functions considered in this research, the results appear to indicate price eigenvalues sensitivity to the market 'depth' of the well, radius of the quantum price point and a 'tolerance' parameter. For the Arctan γ , the parameter b appears to additionally influence price eigenvalues.

An increase of the depth of the well increases the values of the price-conservation levels. This is illustrated for the spherical square well potential i.e. data in Table 3.0

⁷⁹ Algorithm can be provided to anyone upon request.

k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р
	γ(0)=15qσ	γ(0)=15qσ	γ(0)=15qσ	γ(0)=15qσ	γ(0)=15qσ	γ(0)=25qσ	γ(0)=25qσ	γ(0)=25qσ	γ(0)=25qσ	γ(0)=25qσ	γ(0)=35qσ	γ(0)=35qσ	γ(0)=35qσ	γ(0)=35qØ	γ(0)=35qσ
0	0.0498250	-0.1497120		-14.9712%	1.051087140	0.0514103	-0.152789		-15.2789%	1.05275475	0.0520918	-0.156647		-15.6647%	1.05347245
1	0.1995370	-0.2475580	0.149712	-4.8923%	1.220837380	0.2041990	-0.256712	0.152789	-5.1962%	1.22654221	0.208739	-0.259046	0.156647	-5.1199%	1.23212337
2	0.4470950	-0.3424060	0.247558	-4.7424%	1.563762850	0.4609110	-0.312318	0.256712	-2.7803%	1.58551773	0.467785	-0.363099	0.259046	-5.2027%	1.59645413
3	0.7895010	-0.4264490	0.342406	-4.2022%	2.202297206	0.7732290	-0.467431	0.312318	-7.7557%	2.16675141	0.830884	-0.464446	0.363099	-5.0674%	2.29534693
4	1.2159500	-0.2849200	0.426449	7.0765%	3.373497365	1.2406600	-0.575770	0.467431	-5.4169%	3.45789492	1.29533	-0.561400	0.464446	-4.8477%	3.652201
5	1.5008700	-0.0069200	0.284920	13.9000%	4.485589836	1.8164300	-0.607480	0.575770	-1.5855%	6.1498642	1.85673	-0.657730	0.561400	-4.8165%	6.40276546
6	1.5077900	-0.0137800	0.006920	-0.3430%	4.516737766	2.4239100	-0.077010	0.607480	26.5235%	11.2899167	2.51446	-0.731660	0.657730	-3.6965%	12.3599326
7	1.5215700	-0.0204900	0.013780	-0.3355%	4.579409227	2.5009200	-0.007340	0.077010	3.4835%	12.193707	3.24612	-0.254770	0.731660	23.8445%	25.6904673
8	1.5420600	-0.0269200	0.020490	-0.3215%	4.674209231	2.5082600	-0.014570	0.007340	-0.3615%	12.2835381	3.50089	-0.007100	0.254770	12.3835%	33.1449378
9	1.5689800		0.026920	2.6920%	4.801747912	2.5228300		0.014570	1.4570%	12.4638194	3.50799		0.007100	0.7100%	33.3811043

Table 3: Data generated from the C++ program for different well depth for the spherical square well γ function.



Figure 6: A plot of the density of states against eigen-price for constant γ (spherical well function) is shown on the graph below.



Figure 7: Density of states plot showing the delta functions for a spherical square well (constant) y.

When the radius of the quantum price point $\widehat{\mathbb{D}}_{\uparrow\downarrow}$ is decreased, the price eigenvalues increase, also the highest and lowest levels have values which increase at different paces. The uppermost value increases only just, whereas the lowest values increases fast. This can be expressed mathematically as $\Delta P_{\text{Eigen}} \approx \frac{(1 \text{ q}\sigma)^2}{\Delta \text{Radius}}$. This also implies that, as the quantum "well" gets narrower near the bottom, the confinement increases. We illustrate this below for the COSH⁻² price cut-off γ function.

k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р
	L= 2000 qσ,	L= 2000 qσ,	L= 2000 qσ,	L= 2000 qσ,	L= 2000 qσ,	L= 1000 qσ,	L= 1000 qσ,	L= 1000 qσ,	L= 1000 qσ,	L= 1000 qσ,
	γ(0)=6 qσ	γ(0)=6 qσ	γ(0)=6 qσ	γ(0)=6 qƠ	γ(0)=6 qƠ	γ(0)=6 qƠ	γ(0)=6 qσ	γ(0)=6 qσ	γ(0)=6 qƠ	γ(0)=6 qσ
0	0.0298122	-0.0601690		-6.0169%	1.030261033	0.098178	-0.185229		-18.5229%	1.10315924
1	0.0899812	-0.0706398	0.060169	-0.5235%	1.094153713	0.283407	-0.185001	0.185229	0.0114%	1.3276454
2	0.1606210	-0.0758020	0.070640	-0.2581%	1.174239848	0.468408	-0.126487	0.185001	2.9257%	1.59744903
3	0.2364230	-0.0774800	0.075802	-0.0839%	1.266710015	0.594895	-0.006823	0.126487	5.9832%	1.81284059
4	0.3139030	-0.0761920	0.077480	0.0644%	1.368756961	0.601718	-0.010873	0.006823	-0.2025%	1.82525189
5	0.3900950	-0.0718850	0.076192	0.2154%	1.477121114	0.612591	-0.018210	0.010873	-0.3668%	1.84520614
6	0.4619800	-0.0639380	0.071885	0.3973%	1.587213559	0.630801	-0.024552	0.018210	-0.3171%	1.87911515
7	0.5259180	-0.0504030	0.063938	0.6768%	1.692011402	0.655353	-0.030330	0.024552	-0.2889%	1.92582221
8	0.5763210	-0.0247910	0.050403	1.2806%	1.779479668	0.685683	-0.035790	0.030330	-0.2730%	1.98512721
9	0.6011120		0.024791	2.4791%	1.824146123	0.721473		0.035790	3.5790%	2.05746162

Table 4: Data generated from the C++ program for $COSH^{-2} \gamma$ for two different sized quantum price points.

The density of states graph for the $COSH^2 \gamma$ well is shown below. The depth value of the well is

taken $\gamma_0=6 q\sigma$ which is a relatively shallow well. The radius is 2000 q σ .



Figure 8: Density of states plot for the COSH⁻² γ , showing delta functions as predicted from the theory.



Figure 9: Eigen Return Rate with various depth.

The quantum price point's depth of $\gamma_0=6 q\sigma$ corresponds to a rather shallow well, but also the eigenvalues at this depth have good resolution for the purpose of demonstrating the density of state –eigen log price plot.

The price eigenvalues generated for Arctan will become similar to the *square* well, when the parameter b is very small. When b is large, the spacing between the eigenvalues becomes very small. This shows how the price quantum levels may be affected by 'roughness' in the quantum price point 'interface'. This is shown by the data in Table 5, where the radius of the quantum price point is kept fixed at 1000 q σ .

Spo	t £100.34																	
k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Quantum Price in qσ	Next Time Period Price in £	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Quantum Price in qσ	Next Time Period Price in £	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Quantum Price in qσ	Next Ti Perio Price i
	Square well	Square well	Square well	Square well	Square well	£	Arctan well of γ(0)=6 qσ,	Arctan well of γ(0)=6 qσ,	£	Arctan well of γ(0)=6 qσ,	, £							
	11-7 - 4-	11-7 - 1-	11-7 - 1-	γ(0)=6 qƠ	11-7 - 4-		b=100 qσ		b=10E12 qσ									
0	0.0464844	-0.1387766		-13.8777%	1.047581737	£87.3382	0.0464844	-0.138525		-13.8525%	1.04758174	£87.3602	0.3005640	-0.004512		-0.4512%	1.35062034	£99.88
1	0.1852610	-0.2231830	0.138777	-4.2203%	1.203532521	£96.1934	0.1850090	-0.223434	0.138525	-4.2455%	1.20322927	£96.1693	0.3050760	-0.009023	0.004512	-0.2255%	1.35672811	£100.1
2	0.4084440	-0.1923920	0.223183	1.5396%	1.504475000	£101.8967	0.4084430	-0.192393	0.223434	1.5521%	1.5044735	£101.9095	0.3140990	-0.013536	0.009023	-0.2256%	1.36902526	£100.1
3	0.6008360	-0.0066470	0.192392	9.2873%	1.823642729	£110.1053	0.6008360	-0.006647	0.192393	9.2873%	1.82364273	£110.1053	0.3276350	-0.018047	0.013536	-0.2255%	1.38768238	£100.1
4	0.6074830		0.006647	0.6647%	1.835804858	£101.0092	0.6074830		0.006647	0.6647%	1.83580486	£101.0092	0.3456820		0.018047	1.8047%	1.41295323	£102.1

Table 5 The Eigenvalues generated by the C++ program for the spherical square well and arctan wells at the depth of $\gamma(0) = 6 q\sigma$.



Figure 10: An explicit density of states graph for the Arctan



Figure 11: Showing the delta functions for the Arctan cut-off price potential, for $b = 0.1 q\sigma$.

A similar plot, but for *b* very large, would show the delta peaks "poorly" resolved; it. can be clearly seen from the data in Table 5.

Based on the data generated for the harmonic potential, we analysed the data and plotted the histograms. In this case we expect the separation between the successive price levels to be equal. The data generated from the program show this too. In functions of this type (harmonic) in quantum mechanics, we often refer to the concept of angular momentum. This is not an applicable concept here, thus our exclusion of it from our definition of the gamma function is equivalent to setting it to zero. We do so for the sake of our C++ program generating the data consistently (Strousrup, 2000). This is done to avoid using the term as in previous cases, for the harmonic γ our main quantum number n = k + 1. Subsequently, and in general, logarithmic prices will be given by α_n and the differences between successive price levels would be constant $d\alpha = \alpha_n - \alpha_{(n-1)}$. The data generated from the program for the Harmonic γ show exactly this.

Spot	£100.34					
k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Quantum Price in qO	Next Time Period Price in £
	Harmonic γ(0)=10 qσ	£				
0	0.149976	-0.2000160		-20.0016%	1.161806359	£82.1501
1	0.349992	-0.2000970	0.200016	-0.0040%	1.419056196	£100.3359
2	0.550089	-0.1999050	0.200097	0.0096%	1.733407284	£100.3496
3	0.749994	-0.2000060	0.199905	-0.0050%	2.116987315	£100.3349
4	0.950000	-0.2000000	0.200006	20.0006%	2.585709659	£122.5563
5	1.150000		0.200000	20.0000%	3.158192910	£122.5556

Table 6 Data generated for the harmonic γ at well depth of $\gamma_0=10 \ q\sigma$



Figure 12: Figure 10: An explicit density of states graph for the Harmonic gamma.

Also a histogram for this case shows the expected result, illustrating once more that the quantum price level separation remains constant and for the selected small enough bin range, the plot is equivalent to the density of states for 0D, illustrated in the. histogram below (Figure 13).



Figure 13: Density of States-Eigeneprice plot

The comparison of the quantum price eigenvalues generated for the COSH⁻² and Gaussian respectively show similarities. The data show that the values in both cases are very similar at the top corner of the well and less similar at the bottom of the well. For the purpose of illustration, this is shown clearer by the data, rather than the plot.

Spot	£100.34											
k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Quantum Price in qO	Next Time Period Price in £	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Quantum Price in qO	Next Time Period Price in £
	Gaussian γ(0)=10 qσ	£	COSH^-2 γ(0)=10 qσ	£								
0	0.1646480	-0.1883320		-18.8332%	1.178978045	£83.1156	0.0981781	-0.185229		-18.5229%	1.10315924	£83.3739
1	0.3529800	-0.1453790	0.188332	2.1477%	1.423302677	£102.5183	0.2834070	-0.185001	0.185229	0.0114%	1.3276454	£100.3514
2	0.4983590	-0.0900210	0.145379	2.7679%	1.646017938	£103.1561	0.4684080	-0.126487	0.185001	2.9257%	1.59744903	£103.3190
3	0.5883800	-0.0132220	0.090021	3.8400%	1.801068320	£104.2679	0.5948950	-0.006823	0.126487	5.9832%	1.81284059	£106.5268
4	0.6016020		0.013222	1.3222%	1.825040174	£101.6755	0.6017180		0.006823	0.6823%	1.82525189	£101.0270

Table 7: Data for Gaussian and $Cosh^{-2}$ price cut-off γ function.



Figure 14: Stock price per quantum level.

We also included an inverse cut-off price function γ , although we have argued that this may not be a good price reflective identify function because it implies a central price point singularity, such as is normally conceptualized in the context of a real physical confinement, but would also need further consideration of the quantum price point dynamics, which would in specific terms require consideration of price effects from filtration sequencing and overlapping within the dissipative quantized market region with the hypothesis that overlapping of events of filtrations within the quantized region creates a singular point. We illustrate it here only for brief comparative reasons.

Spot Price	£100.34											
k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	P in qs	P(future point)	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р	P(future point)
	Square Well, γ(0)=15qσ		Inverse well, γ(0)=15qσ									
0	0.0498250	-0.1497120		-14.9712%	1.051087140	£86.39	-0.4374560	-0.328085		-32.8085%	0.64567694	£72.28
1	0.1995370	-0.2475580	0.149712	-4.8923%	1.220837380	£95.55	-0.1093710	-0.060761	0.328085	13.3662%	0.89639779	£114.69
2	0.4470950	-0.3424060	0.247558	-4.7424%	1.563762850	£95.69	-0.0486102	-0.021500	0.060761	1.9630%	0.95255236	£102.33
3	0.7895010	-0.4264490	0.342406	-4.2022%	2.202297206	£96.21	-0.0271101	-0.014280	0.021500	0.3610%	0.97325408	£100.70
4	1.2159500	-0.2849200	0.426449	7.0765%	3.373497365	£107.70	-0.0128301	-0.018896	0.014280	-0.2308%	0.98725185	£100.11
5	1.5008700	-0.0069200	0.284920	13.9000%	4.485589836	£115.30	0.0060659	-0.024638	0.018896	-0.2871%	1.00608434	£100.05
6	1.5077900	-0.0137800	0.006920	-0.3430%	4.516737766	£100.00	0.0307042	-0.029966	0.024638	-0.2664%	1.03118044	£100.07
7	1.5215700	-0.0204900	0.013780	-0.3355%	4.579409227	£100.00	0.0606702	-0.035053	0.029966	-0.2543%	1.06254843	£100.09
8	1.5420600	-0.0269200	0.020490	-0.3215%	4.674209231	£100.02	0.0957228	-0.039998	0.035053	-0.2473%	1.10045398	£100.09
9	1.5689800		0.026920	2.6920%	4.801747912	£103.08	0.1357210		0.039998	3.9998%	1.14536229	£104.43

Table 8: Data for inverse and square well price cut-off γ function, $\gamma(0) = 15q\sigma$.



Figure 15: Eigen-return values vs quantum level numbe.



Figure 16: Density of States vs Eigen price plot for an inverse gamma.

We define a parameter $N = \frac{1}{n^2}$, where n is the main quantum number (n = k + 1). A plot of the eigenvalues against parameter N exhibits a straight line. The data generated for this situation have been plotted and shown below. This example shows that the numerical method is capable of locating the price eigenvalues and gives confidence in the method.



Figure 17: Inverse price cut-off function γ .

6.1.6 INFORMATION TUNNELLING AND PRICE TRANSITION

To obtain such graphs we need to pick up one of our programs with a "suitable" potential. Out of choice, we pick up the program for the Gaussian γ . The transition market price is given by

$$\forall \mathbf{k} \in \{0, \dots, \mathbf{K}\}, \forall \mathbf{n} \in \{1, \dots, \mathbf{N}\}, \forall \tau \in \{0, \dots, \mathbf{T}\} \therefore \mathbf{K} \models \mathbf{N} \& \mathbf{T} \models \mathbf{N}, \ \mathbf{r}_{\mathbf{k}} =: \frac{1}{2} \left\{ \left[\text{diff} \left(L\mathbf{N}(\mathbf{P}) \right)_{\mathrm{L}} \right] + \left[\text{diff} \left(L\mathbf{N}(\mathbf{P}) \right)_{\mathrm{H}} \right] + q \mu \left[1 + \left(\tau - C_{\triangleright} \left(\bigcirc_{\tau} | F\tau \right)^{\frac{\tau}{n}} \right) \right] \right\}$$
(142)

We define $C_{\triangleright}(\bigcirc_{\tau} | F\tau)$ as the quality of filtration in expectation as a coefficient, related to the degree of congruence of market regulation and governance. This is a constant coefficient that tends to vary from market to market. Tau represents the stoppage times from 0 to T. We also provide an additional interpretation to $C_{\triangleright}(\bigcirc_{\tau} | F\tau)$ as the % of filtration tunnelling to the market adjacent region. For computations in the current region we use tau = 0 and for a considerable number of Eigen states, considering the information is completely contained in the current region with no tunnelling, we get the same result as shown in the previous calculations. However for the next adjacent region $C_{\triangleright}(\bigcirc_{\tau} | F\tau)$ the effect will be non-zero. The value of n is k+1. We may consider a "graded" composition of the market region with different effects of dissipation and tunnelling at each of the quantum levels or may set this value to the largest n or the highest Eigen level if we assume a rather homogeneous market composition in how it dissipates information. Further on

$$\forall k \in \{0, ..., K\} P_{\text{transition} \to II} = P_{\text{spot} \to I} \exp(r_k)$$
(143)

$$\forall k \in \{K, ..., 0\} \operatorname{diff}(LN(P))_{H} = LN(P)_{\forall i \in \{K, ...1\}} - LN(P)_{\forall j \in \{K-1, ...0\}}$$
(144)
$$\forall n \in \{0, ..., N = K - 1\} \operatorname{diff}(LN(P))_{L} = LN(P)_{\forall i \in \{0, ..., N\}} - LN(P)_{\forall j \in \{1 ..., K\}}$$
(145)

And

In this research for the most part, our focus is on any two adjacent quantized market regions. In the computations above, we used
$$1 + [\tau - C_{r}(\bigcirc_{\tau} | F\tau)^{\frac{\tau}{k+1}}] = 0$$
. It is easy to see from the table below that for quantized region 1 (tau-0) this term becomes zero. For a sequence of quantum points along the market line, this effect increases, which translates to larger asset price moves or higher uncertainty.

Information Tunneling %	Nine qu	antised market	points at Egen state 0	Quan	tum region 1 fo	or 9 Eigen states	Quantum region 2 for transition price for 9 Eigen states				
C (Ô_T Ft)	Tau	n=k+1, for fixed k = 0	$1 + \left[\tau - C_{s}\left(\hat{Q}_{+} Pt\right)^{\frac{\tau}{k+1}}\right]$	Tau=0	n=k+1	$1 + \left[\tau - C_{p} \left(\hat{Q}_{\tau} P \tau\right)^{\frac{\tau}{b+1}}\right]$	Tau=0	n=k+1	1+ [t-C,(Q, R) ³⁺		
40.0%	0	1	0.00	0	1	0.0	1	1	1.6000		
40.0%	1	1	1.60	0	2	0.0	1	2	1.3675		
40.0%	2	1	2.84	0	3	0.0	1	3	1.2632		
40.0%	3	1	3.94	0	4	0.0	1	4	1.2047		
40.0%	4	1	4.97	0	5	0.0	1	5	1.1674		
40.0%	5	1	5.99	0	6	0.0	1	6	1.1416		
40.0%	6	1	7.00	0	7	0.0	1	7	1.1227		
40.0%	7	1	8.00	0	8	0.0	1	8	1.1082		
40.0%	8	1	9.00	0	9	0.0	1	9	1.0968		

Figure 9: Quantized market point data at the lowest Eigen state.

We also plot the transition price against the radius of the price quantum point in $q\sigma$ considering that only 40% of the information tunnels through from the current dissipative region to the next adjacent one.

Spot	£100.34					
Radius	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Quantum Price in qƠ	Next Time Period Price in £
in qơ	Gaussian γ(0)=7 qσ	£				
70	1.773766	0.0000000		0.0000%	5.893004680	£100.3400
100	1.773766	0.0364036	0.000000	1.8202%	5.893004680	£102.1831
130	1.737362	0.0251867	-0.036404	-0.5608%	5.682335910	£99.7788
150	1.712176	0.0119029	-0.025187	-0.6642%	5.541003935	£99.6758
160	1.700273	0.0218970	-0.011903	0.4997%	5.475440888	£100.8427
180	1.678376	0.0193021	-0.021897	-0.1297%	5.356848309	£100.2099
200	1.659074		-0.019302	-1.9302%	5.254441400	£98.4218

Table 10: Quantum prices and conversion to a forecast price set in the observed world



Figure 18: Eigen-price vs radius of the quantized topological space.



Figure 19: The quantum price versus radius in $q\sigma$.

In the case of a Harmonic cut-off price function and within the classical conceptualization of quantum systems, one would to resolve a misalignment with the introduction of angular momentum. This is a concept that would not make sense in our work. Although we have coded our C++ program to deal with this, we subsequently avoid consideration of such concept and only consider the effect of the main quantum number n. This in effect meant that we have scaled down our loop structure in our C++ program, by removing the angular momentum effect given through $L^2 = l(l + 1)$, where l is the angular momentum number.

Instead, we use integer k = n-1, where n is the main quantum number.

Ideally we would sketch the relationship between the main quantum number and the angular momentum number, however as the latter is relaxed from our treatment, we showcase it with the main quantum number dependency. For the harmonic γ the price eigenvalue separation f remains roughly constant. This is shown pictorially below and can be established from the data generated with the NAG routine.



Figure 20: A sketch indicating the price Eigen-levels for (n) with l = 0 for a market quantum well.

With the removal of the angular momentum consideration we are left with the challenge of the price spin, a phenomenon that is linked to the price change being negative (spin down) or positive (spin up). We have resolved this through the introduction of the quantum rate of return

$$\forall \mathbf{k} \in \{0, \dots, \mathbf{K}\}, \forall \mathbf{n} \in \{1, \dots, \mathbf{N}\}, \forall \tau \in \{0, \dots, \mathbf{T}\} \quad \therefore \mathbf{K} \models \mathbf{N} \& \mathbf{T} \models \mathbf{N}, \quad \mathbf{r}_{\mathbf{k}} \\ =: \frac{1}{2} \left\{ \left[\text{diff} \left(L\mathbf{N}(\mathbf{P}) \right)_{\mathbf{L}} \right] + \left[\text{diff} \left(L\mathbf{N}(\mathbf{P}) \right)_{\mathbf{H}} \right] + q \mu \left[1 + \left(\tau - C_{\triangleright} \left(\bigcirc_{\tau} \mid F\tau \right)^{\frac{\tau}{\mathbf{k}+1}} \right) \right] \right\}$$

$$(146)$$

Therefore our relaxation of the angular momentum concept in our research has been properly addressed to give tangible results.

To price at the next quantum region, we keep the radius of the quantum market point constant and vary the fraction of information tunnelled to the adjacent quantized market region. For each value of $C_{\triangleright}(\bigcirc_{\tau} | F_{\tau})$ we obtain the values of the lowest transition (k=0) price. The transition price set is computed in the same fashion as above using the same formula. The market quantum well considered is Gaussian and the radius is kept constant at 100 q σ . The data generated by the program are

Information Tunneling %	Nine qu	antised market	points at Egen state 0	Quan	tum region 1 fo	or 9 Eigen states	Quantum reg	ion 2 for trans 9 Eigen states	ition price for	Quantum regi	on 2 for transi state_n= :	isition price for Eigen = 1	
C (Ô_T Ft)	Tau	n=k+1, for fixed k = 0	$1 + \left[t - C_{p}(\mathbf{Q}, \mathbf{R})^{\frac{1}{ \mathbf{N} }}\right]$	Tau	n=k+1	$1 + \left[t - C_{p} \left(\hat{Q}_{1} \mathbb{R} t \right)^{\frac{1}{ w }} \right]$	Tau	n=k+1	$1 + \left[t - C_{0}(\hat{Q}_{1} R)^{\frac{1}{ N }}\right]$	Tau	n=k+1	$1 + \left[t - C_{s}\left(\hat{Q}_{t} R\right)^{\frac{1}{ H }}\right]$	
20.0%	0	1	0.00	0	1	0.0	1	1	1 8000	1	1	1,8000	
25.0%	1	1	1.75	0	2	0.0	1	2	1.5000	1	1	1.8000	
30.0%	2	1	2.91	0	3	0.0	1	3	1.3306	1	1	1.7000	
35.0%	- 3	1	3.96	0	4	0.0	1	4	1 2308	1	1	1 6500	
40.0%	4	1	4.97	0	5	0.0	1	5	1.1674	1	1	1.6000	
45.0%	5	1	5.98	0	6	0.0	1	6	1.1246	1	1	1.5500	
50.0%	6	1	6.98	0	7	0.0	1	7	1.0943	1	1	1.5000	
55.0%	7	1	7.98	0	8	0.0	1	8	1.0720	1	1	1.4500	
60.0%	8	1	8.98	0	9	0.0	1	9	1.0552	1	1	1.4000	
65.0%	9	1	9.98	0	10	0.0	1	10	1.0422	1	1	1.3500	
70.0%	10	1	10.97	0	11	0.0	1	11	1.0319	1	1	1.3000	
75.0%	11	1	11.96	0	12	0.0	1	12	1.0237	1	1	1.2500	
80.0%	12	1	12.93	0	13	0.0	1	13	1.0170	1	1	1.2000	
85.0%	13	1	13.88	0	14	0.0	1	14	1.0115	1	1	1.1500	

Table 11: Quantum prices and conversion to a forecast price set in the observed world

Spot	£100.34	k=1						
Information Tunneling %	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	$1 + \left[\tau - C_{\flat} \left(\hat{Q}_{\tau} F \tau \right)^{\frac{\tau}{k+1}} \right]$		Quantum Price in qO	Next Time Period Price in £
C (◯_T FT)	Gaussian γ(0)=10 qσ	Gaussian γ(0)=10 qσ	Gaussian γ(0)=10 qσ	Gaussian γ(0)=10 qσ	tau=1, k=0		Gaussian γ(0)=10 qσ	£
20.00%	1.669702	-0.0532670		-5.33%	1.80%	-3.53%	0.948297507	£95.1522
25.00%	1.722969	-0.0507970	0.053267	0.12%	1.75%	1.87%	1.018911601	£102.2376
30.00%	1.773766	-0.0483040	0.050797	0.12%	1.70%	1.82%	1.018413984	£102.1877
35.00%	1.82207	-0.0455950	0.048304	0.14%	1.65%	1.79%	1.018014844	£102.1476
40.00%	1.867665	-0.0427930	0.045595	0.14%	1.60%	1.74%	1.017553279	£102.1013
45.00%	1.910458	-0.0401670	0.042793	0.13%	1.55%	1.68%	1.016955134	£102.0413
50.00%	1.950625	-0.0378840	0.040167	0.11%	1.50%	1.61%	1.016272478	£101.9728
55.00%	1.988509	-0.0359530	0.037884	0.10%	1.45%	1.55%	1.015585710	£101.9039
60.00%	2.024462	-0.0343090	0.035953	0.08%	1.40%	1.48%	1.014932391	£101.8383
65.00%	2.058771		0.034309	3.43%	1.35%	4.78%	1.048970283	£105.2537

Table 12: Quantum prices and conversion to a forecast price set in the observed world

If we approached the problem from the current quantized region (region I), then tau would

be zero.

Information Tunneling %	Nine qu	antised market	points at Egen state 0	Quantum region 1 for 9 Eigen states				ion 2 for trans 9 Eigen states	ition price for	Quantum region 2 for transition price for Eigen state n= 1			
C (Ô_t Ft)	Tau	n=k+1, for fixed k = 0	$1 + \left[t - C_{p}\left(\hat{Q}_{ij} \mid R\right)^{\frac{1}{ K }}\right]$ in %	Tau	n=k+1	$1 + \left[t - C_{s} \left(\hat{Q}_{t} R t \right)^{\frac{1}{ s }} \right]$ in %	Tau	n=k+1	$\frac{1}{1 + \left[t - C_{p}\left(\hat{Q}_{1} \mid R\right)^{\frac{1}{ K }}\right]}{\text{in \%}}$	Tau	n=k+1	$1 + \left[t - C_{\mu} \left(\hat{Q}_{\mu} \hat{R} \right)^{\frac{1}{ w }} \right]$ in %	
20.0%	0	1	0.00	0	1	0.0	1	1	1.8000	0	1	0.0000	
25.0%	1	1	1.75	0	2	0.0	1	2	1.5000	0	1	0.0000	
30.0%	2	1	2.91	0	3	0.0	1	3	1.3306	0	1	0.0000	
35.0%	3	1	3.96	0	4	0.0	1	4	1.2308	0	1	0.0000	
40.0%	4	1	4.97	0	5	0.0	1	5	1.1674	0	1	0.0000	
45.0%	5	1	5.98	0	6	0.0	1	6	1.1246	0	1	0.0000	
50.0%	6	1	6.98	0	7	0.0	1	7	1.0943	0	1	0.0000	
55.0%	7	1	7.98	0	8	0.0	1	8	1.0720	0	1	0.0000	
60.0%	8	1	8.98	0	9	0.0	1	9	1.0552	0	1	0.0000	
65.0%	9	1	9.98	0	10	0.0	1	10	1.0422	0	1	0.0000	
70.0%	10	1	10.97	0	11	0.0	1	11	1.0319	0	1	0.0000	
75.0%	11	1	11.96	0	12	0.0	1	12	1.0237	0	1	0.0000	
80.0%	12	1	12.93	0	13	0.0	1	13	1.0170	0	1	0.0000	
85.0%	13	1	13.88	0	14	0.0	1	14	1.0115	0	1	0.0000	

Table 13: Quantum prices and conversion to a forecast price set in the observed world

We can price for the next region (region II) and compare the results with the previous

approach.

Information Tunneling %	Nine qu	antised market	points at Egen state 0	Quantum region 1 for 9 Eigen states				Quantum region 2 for transition price f 9 Eigen states		Quantum regi	on 2 for transi state_n=:	tion price for Eigen 1
С (Ѻ_т Fт)	Tau	n=k+1, for fixed k = 0	$1 + \left[t - C_{n}(Q_{i} R)^{\frac{1}{ N }}\right]$ in %	Tau	n=k+1	$1 + \left[t - C_{s} \left(\hat{Q}_{t} R t \right)^{\frac{1}{ s }} \right]$ in %	Tau	n=k+1	$1 + \left[t - \zeta_{0}(\hat{Q}, R)^{\frac{1}{ X }}\right]$ in %	Tau	n=k+1	$1 + \left[t - C_{0}(\hat{Q}_{t} \Re)^{\frac{1}{ M }} \right]$ in %
20.0%	0	1	0.00	0	1	0.0	1	1	1.8000	0	1	0.0000
25.0%	1	1	1.75	0	2	0.0	1	2	1.5000	0	1	0.0000
30.0%	2	1	2.91	0	3	0.0	1	3	1.3306	0	1	0.0000
35.0%	3	1	3.96	0	4	0.0	1	4	1.2308	0	1	0.0000
40.0%	4	1	4.97	0	5	0.0	1	5	1.1674	0	1	0.0000
45.0%	5	1	5.98	0	6	0.0	1	6	1.1246	0	1	0.0000
50.0%	6	1	6.98	0	7	0.0	1	7	1.0943	0	1	0.0000
55.0%	7	1	7.98	0	8	0.0	1	8	1.0720	0	1	0.0000
60.0%	8	1	8.98	0	9	0.0	1	9	1.0552	0	1	0.0000
65.0%	9	1	9.98	0	10	0.0	1	10	1.0422	0	1	0.0000
70.0%	10	1	10.97	0	11	0.0	1	11	1.0319	0	1	0.0000
75.0%	11	1	11.96	0	12	0.0	1	12	1.0237	0	1	0.0000
80.0%	12	1	12.93	0	13	0.0	1	13	1.0170	0	1	0.0000
85.0%	13	1	13.88	0	14	0.0	1	14	1.0115	0	1	0.0000

Table 14: Quantum prices and conversion to a forecast price set in the observed world. Different scenarios

Spot	£100.34	k=1	Quantum Region I				Quantum Region II	
Information Tunneling %	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	$1 + \left[\tau - C_{p} \left(\hat{Q}_{\tau} F \tau \right)^{\frac{\tau}{k+1}} \right]$		Quantum Price in qO	Next Time Period Price in £
C (◯_T FT)	Gaussian γ(0)=10 qσ	Gaussian γ(0)=10 qσ	Gaussian γ(0)=10 qσ	Gaussian γ(0)=10 qσ	tau=0, k=0		Gaussian γ(0)=10 qσ	£
20.00%	1.669702	-0.0532670		-5.33%	0.00%	-5.33%	0.948126829	£95.1350
25.00%	1.722969	-0.0507970	0.053267	0.12%	0.00%	0.12%	1.001235763	£100.4640
30.00%	1.773766	-0.0483040	0.050797	0.12%	0.00%	0.12%	1.001247277	£100.4652
35.00%	1.82207	-0.0455950	0.048304	0.14%	0.00%	0.14%	1.001355418	£100.4760
40.00%	1.867665	-0.0427930	0.045595	0.14%	0.00%	0.14%	1.001401982	£100.4807
45.00%	1.910458	-0.0401670	0.042793	0.13%	0.00%	0.13%	1.001313862	£100.4718
50.00%	1.950625	-0.0378840	0.040167	0.11%	0.00%	0.11%	1.001142152	£100.4546
55.00%	1.988509	-0.0359530	0.037884	0.10%	0.00%	0.10%	1.000965966	£100.4369
60.00%	2.024462	-0.0343090	0.035953	0.08%	0.00%	0.08%	1.000822338	£100.4225
65.00%	2.058771		0.034309	3.43%	0.00%	3.43%	1.034904343	£103.8423

Table 15: Quantum prices and conversion to a forecast price set in the observed world. Different scenarios with a Gaussian.



Figure 21: Transition Eigen-Price - Fraction of information dissipated and tunnelled.

6.1.7 INTENSITY OF INFORMATION REFLECTION ON PRICE (IIRP)

We establish the concept of the intensity of information reflection on price as a proportional measure to the probability density function on the continuous probability space. We denote IIRP = $I_{\Delta}(\bigcirc_{\tau} | F_{\tau})$ to be the intensity of information reflection on price and consider the continuous space probability density of the normal distribution.

$$I_{\Delta}(\bigcirc_{\tau} | F\tau) \propto f(r|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2} \quad (147)$$

Where μ is the expectation, and σ the standard deviation. In our previous section's numerical considerations we moved between continuous and quantized spaces through the quantum effect q =10⁻¹¹. This, is expressed mathematically below:

$$I_{\Delta}(\bigcirc_{\tau} | F\tau) \propto e^{-\frac{1}{2}\left(\frac{qr-q\mu}{q\sigma}\right)^2} = e^{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2} \qquad (148)$$

Thus the probability density effects observed in the continuous space are in equilibrium with $I_{\Delta}(\bigcirc_{\tau} | F\tau)$, where the IIPR reflects its independence from the quantum effect measure. However, it is possible to treat the subject from a more comprehensive angle.

In our closing segments of the first theoretical chapter, we established the probability density function to depend on quantum system's main quantum number

$$u_{n}(r) = |\psi_{n}(r)|^{2} = \frac{1}{\sqrt{\pi}} \frac{\alpha}{2^{n} n!} |H_{n}(\alpha r)|^{2} e^{-(\alpha r)^{2}}$$
(148)

Where for n = 0, we obtained the classical expression for the probability density function $u_0(r) = f(r|\mu, \sigma^2)$, therefore $I_{\Delta}(\bigcirc_{\tau} | F_{\tau}) \propto u_0(r)$. It has a maximum at r = 0. We also concluded that as n increases, the result moves away from the classical result, implied by the equation above. From our concluding remarks on the first theoretical chapter, we considered the square root of the weight function w(r), so that the functions $\psi(r)$ are orthogonal when integrated from $-\infty$ to $+\infty$, which is required by theory (Szego, 1939).

We established in the first theoretical chapter that the orthogonality of the Hermite polynomials is expressed by

$$\int_{-\infty}^{+\infty} e^{-r^2} H_n H_m dr = \delta_{nm} 2^n n! \sqrt{\pi}$$
(150)

where δ_{nm} is the Kronecker delta which is a function of two variables, usually just positive integers (Szego, 1939; Puig, 2003). The function is 1 if the variables are equal and 0 otherwise

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$
(151)

is zero if m is not equal to n, and unity if m is equal to n. Without proving the full

mathematical work-out, to prove this one needs to express the exponential times the Hermite polynomial of larger order as an nth derivative using the Rodrigues formula, and then use integration by parts until the polynomial of smaller order is differentiated to zero (Schwartz, 1967). If the orders are equal, the final integral, and subsequently the result is the integral of $\exp(-r^2)$ times a constant, and the normalization constant becomes $(\sqrt{2\pi})n!$ (Walter, 1977). The orthogonality attribution can be used to expand an arbitrary function in a series of Hermite polynomials, in exactly the same way as a Fourier series (Sansone, 1939; Walter, 1980; Puig, 2003).

Subsequently, the intensity of information reflection on price has a Gaussian-like shape and is dependent on market's expectation and volatility. On theoretical considerations, we can use contemporary asset pricing models to compute both the expectation and volatility. A theoretical graph of this is shown below



Figure 22 The intensity" of information reflection of price vs transition Eigen price.

The investigation into the intensity of information reflection on price could focus on a singular quantized market point (or arbitrary radius) and at a specific time or across a sample containing a number of quantum price points. The number of price dots is very large. This type of analysis is linked with the transition Eigen price - radius analysis as above. The points of this range correspond to the ground state price transitions i.e. refer to Chapter 4 for mathematical workings. It is also the Eigen state that we have considered in the previous computations. For higher Eigen-states, we would expect the intensity of the information reflection on price to split.

Our theoretical framework served well for the purposes of calculating price eigen-values we had started the research with a purely theoretical model, by describing the price cutoff as a very general polynomial. Our maximized formulation was then aligned to quantized market points, thus resulting in an effective information equation. We advanced by choosing several price cut-off functions, and solved our master expression for each of them.

Further developments of this work could include a closer and more comprehensive investigation of the intensity of information reflection on price for a large number of quantum market points, and subsequent asset price for casting across a large time horizon. In addition, numerical computations could be carried out for additional price cut-off geometries, which we have not considered in this research, such as pyramidal and cylindrical geometries, as these provide the challenge of dealing with more complicated boundary conditions. Further future research could be focused on the possible use of price cut-off potentials within our quantum theorization for intraday trading as a range of new technical analysis case studies.

6.2 STURM-LIOUVILLE LOGARITHMIC PRICE EIGEN-VALUE SYSTEM

The GSE and mathematical expressions that followed are examples of a Sturm-Liouville eigenvalue problem and therefore can be solved by numerical and mathematical methods developed for that specific type of problem (Bailey, 1966). We start with our master formula (i.e. GSE) with $\chi = (r-\mu)/\sigma$ and $\partial \chi^2 = (1/\sigma^2)\partial r^2$ -and re-arrange it to obtain

$$\left(-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} + \kappa\eta(r)\right)\psi(r) = \alpha\psi(r) \qquad (152)$$

or

$$\frac{\partial^2 \psi(\mathbf{r})}{\partial \mathbf{r}^2} + \frac{2}{\sigma^2} \left(\alpha - \kappa \eta(\mathbf{r}) \right) \psi(\mathbf{r}) = 0 \qquad (153)$$

Conversion to one variable dependency allows for the partials to be converted to full differentials, thus (153) we can write as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) + \frac{2}{\sigma^2}\left(\alpha - \kappa\eta(r)\right)\psi = 0 \quad (154)$$

Bailey (1966), describes well the Sturm-Liouville system which is given by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p(r)\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) + q(r,\lambda)\psi = 0 \qquad (155)$$

where p(r) can be a function of the stochastic process random variable (i.e. rate of return) or can be simply a constant, and $q(r, \lambda)$ is a function of the random variable r and the eigen-value λ (Bailey, 1966). A comparison of equations between the equations show that they become the same when p = 1, $\varepsilon = (2/\sigma^2)$, and $q(r) = \varepsilon [\alpha - k\eta(r)]$, with the boundary conditions

$$a_2\psi(a) = a_1p(a)\psi'(a)$$
 (156)
 $b_2\psi(b) = b_1p(b)\psi'(b)$ (157)

where a and b are the end points. Although, theoretically, we seek the end points $(\pm \infty)$, for numerical illustrations, and programmatically we use two finite end points, a, and b, due to the adapt D02KAF –Nag routine used which is specific for second order Sturm-Liouville systems defined on a finite range, using a Pruefer transformation and a shooting method, thus the boundary conditions become

$$\psi = 0$$
 at $r \rightarrow 0$ (158)
 $\psi = 0$ at $r \rightarrow +\infty$ (159)

The D02KAF finds a specified eigenvalue λ of a *SturmLiouville* system defined by a selfadjoin differential equation of the second-order (i.e. equation 155) where a < r < b. At the two finite end points a and b, the functions p and q are real-valued and defined by a (sub) program COEFFN supplied.

For the theoretical basis of the numerical method to be valid, the following conditions should hold on the coefficient functions: p(r) must be non-zero and of constant sign throughout the closed interval [a, b]; $\partial q/\partial \lambda$ must be of constant sign and non-zero

throughout the open interval (a, b) and for all relevant values of λ , and must not be identically zero as r varies, for any relevant value λ ; p and q should (functions of r) have continuous derivatives, preferable up to the fourth-order, on [a, b]. The differential equation code is used to integrate through mild discontinuities, but probably with severely reduced efficiency. Therefore, if p and q violate this condition, D02KAF should be used.

The eigenvalue λ is determined by a shooting method based on a Pruefer transformation of the differential equations (Bailey, 1966). Providing certain assumptions are met, the computed value of λ will be correct to within a mixed absolute / relative error specified by the user-supplied value TOL. D02KAF is a driver routine for the more complicated routine D02KDF whose specification provides more details of the techniques used. A good account of the Sturm-Liouville systems, with some description of Pruefer transformations, is given in Birkhoff and Rota (1962). The best introduction to the use of Pruefer transformations for the numerical solution of eigenvalue is given in Bailey (1966).

Having declared the general function $\gamma(\chi)$ as a polynomial, we need to be more specific and choose several cases for the true nature of it compliant to the general cases developed in section 2.0 and declare the boundary conditions. On all the cases we have applied the same generic boundary conditions as shown above. Next we map the proposed potential functions to the equivalent Sturm-Liouville $q(\chi)$ function, shown in the table16 below

TABLE 16

	Name	STURM-LIOUVILLE FUNCTION	Augmented Sturm-Liouville Function
1.	GAUSSIAN	$q(\chi) = \varepsilon \left[\alpha - \gamma(0) \left(1 - \exp(-\chi^2) \right) \right]$	$q(r) = \epsilon \left[\alpha - \gamma(0) \left(1 - \exp\left(-\frac{1}{L^2} \left(\frac{r-\mu}{\sigma} \right)^2 \right) \right) \right]$
2.	HARMONIC	$q(\chi) = \varepsilon \left[\alpha - \frac{1}{2} \kappa \chi^2 \right]$	$q(\mathbf{r}) = \varepsilon \left[\alpha - \frac{\kappa}{2L^2} \left(\frac{\mathbf{r} - \mu}{\sigma} \right)^2 \right]$
4.	DECAYING EXPONENTIAL	$q(\chi) = \varepsilon \big[\alpha - \gamma(0) \big(1 - \exp(-\chi) \big) \big]$	$q(\mathbf{r}) = \varepsilon \left[\alpha - \gamma(0) \left(1 - \exp\left(-\frac{1}{L} \left(\frac{\mathbf{r} - \mu}{\sigma} \right) \right) \right) \right]$
5.	LINEAR	$q(\chi) = \varepsilon[\alpha - \gamma(0)\chi]$	$q(\mathbf{r}) = \varepsilon \left[\alpha - \gamma(0) \frac{1}{L} \left(\frac{\mathbf{r} - \mu}{\sigma} \right) \right]$
6.	CONSTANT SQUARE WELL	$q(\chi) = \varepsilon (\alpha - \gamma(0))$	$q(r) = \varepsilon[\alpha - \gamma(0)]$
8.	ARCTAN	$q(\chi) = \varepsilon \left\{ \alpha - \frac{1}{2}\gamma(0) \left[\frac{2}{\pi} \arctan(\chi) + 1 \right] \right\}$	$q(r) = \varepsilon \left\{ \alpha - \frac{1}{2} \gamma(0) \left[\frac{2}{\pi} \arctan\left(\frac{1}{bL} \left(\frac{r-\mu}{\sigma} \right) \right) + 1 \right] \right\}$
9.	COSH-2	$q(\chi) = \epsilon \{\alpha - \gamma(0)[1 - \cosh^{-2}(-\chi^2)]\}$	$q(\mathbf{r}) = \varepsilon \left\{ \alpha - \gamma(0) \left[1 - \cosh^{-2} \left(-\frac{1}{L^2} \left(\frac{\mathbf{r} - \mu}{\sigma} \right)^2 \right) \right] \right\}$
10	INVERSE	$\gamma(\chi) = \varepsilon \left[\alpha + \gamma(0) \frac{1}{\chi} \right]$	$q(r) = \varepsilon \left[\alpha + \gamma(0) \left(\frac{L\sigma}{r - \mu} \right) \right]$

Where ε is a constant, established in relation to the degree of convergence and regulation within a specific market in which asset price is evolved. For a well-regulated and functioning market, we set ε to 1. Some of the potential functions mentioned above are illustrated below



-Gaussian γ: dark line.	-different Arctan plots for various values of b.
-COSH γ : red line.	-the red line corresponds to Arctan potential
-Decay exponential γ: dark navy line.	with a very small value of b. It is the same as the
-Harmonic γ: dark yellow line.	square well.
-Triangular γ: magenta colour line.	
-Inverse γ : plotted with reversed sign and is the	
dark dashed line.	

The γ functions listed above belong in the "family" of the generalized potential given in the previous chapter. The harmonic γ can be seen as the first order approximation of the Gaussian, where the Gaussian is a "sub-family" of the generalised function given as a summation. The triangular γ , also can be seen as the first order approximation of the decay exponential potential, where the decay exponential is also a "sub-family" function of the generalised γ .

For the purpose of the illustration, we use a set of generic data for the σ , κ , α , and four cases of the presumed nature of γ , which cover the main theoretical scenarios developed in previous sections. In order to normalize the probability density of states relative to the return rate Eigen-value set, we set $\gamma(0) = 1/(2\pi)^{0.5}$ (160), which also allowed for better comparison between the various potential functions proposed. Sturm-Liouville solutions were found on all of the proposed potentials, although emphasis was placed on the harmonic, Gaussian, linear, decreasing exponential, and constant functions. From a theoretical point of view the Cosh⁻² function would also be a good potential to fit for small values of χ ; a simple Taylor expansion would match its fitness to the Gaussian.

Apart from the Metropolis algorithm, and the Brownian motion simulation, the rest of applications are based on Sturm-Liouville formulations (Bailey, 1966). In this section

we have used arbitrary input data, although further detailed work is included above .

All compute logarithmic asset price or some variant of it, depending on the scenario considered, thus the Sturm-Liouville form of the expressions led to the computation of the eigenvalues of the rate of return. It follows that we are interested in the probability distribution of the eigenvalues of the rate of return, or the density of states. It is also possible that a full pricing model can be developed out of the Sturm-Liouville solution, for contingent claims and their underlying, however that would expand this paper's research beyond its initial scope, but it remains to be established in the future. However we are able to plot the density of states against the rate of return as a histogram (Berenson, 2012).

For widely spaced eigenvalues, the spectrum appears as a series of delta functions. By varying the size of the return rate "bin range" we are able to vary the resolution of the density of states plot. With a small enough bin range, the spectrum always consists of discrete states. Such histograms are made from the data generated for different programs representing the different gamma functions mentioned above. In all cases the data were generated by a newly designed C++ program invoking NAG routines. These plots clearly show the discreteness of the return rate (and subsequent the price) levels for a quasi-zero dimension point in the market line.

Running the programs and analysing the data, we observed that the discretisation of the rate of return and subsequently of price is a set of deltas equally spaced in the case of the harmonic gamma, but progressively spacy for the Gaussian potential examples with the

generic or default input data. With the arbitrary trial data used, the highest intensities of the rate of return probability distribution are reached between 2.1% to 3.1% in the case of the Harmonic and between 1.2% to 2.25% in the Gaussian case.

On the other hand for the linear potential function suitable in the case of contingent claims, as seen in the case of the Black-Scholes PDE derived from the main postulated expression, the increased intensity of the distribution is accompanied with a parity effect between 0.9% and 1.3%; typically such case would be associated with the quantization of rate of return and effectively of both the financial derivative and the underlying asset. The decreasing exponential has a matching effect to that of the liner potential, around 1.02%., whereas the constant potential scenario, is not good all together, however is it an example of reasonable approximation around very small values of χ or very high levels of volatility.





In relevance to the quantization effects in the price change behaviour, if we revert back to earlier sections, it is possible to consider for all scenarios an $\alpha = r_{tot}$, or $\alpha = r_{tot} - r_f$, where the total rate of return (or total risk premium) is the sum of the rate of return obtained through any of classical pricing models and the value directly linked to the quantization of price change behaviour at each p-point.

7.0 COMPARATIVE VALUATION OF FINANCIAL OPTIONS WITH EIGEN-VALUE CONVERSION AND CLASSICAL MODELS

In this chapter, I make use of the theoretical framework with quantization effects within the zero-object to price financial options. It builds coherently from theoretical and numerical considerations of preceding chapters. Results are then compared with those from some of the more common classical option pricing models, such as the Black-Scholes, Binomial, and RG. I consider the duality of pricing the underlying and the options with a strong form of dependency between the two. Subsequently all assumptions made in chapter 5.0 are valid. My strategy is to price with various quantum pricing sub-models and to further identify the quantum option pricing sub-model that is best-suited to address the known shortcomings of classical option pricing models reported in existing literature.

7.1 FINANCIAL OPTIONS WITH EIGEN-VALUE CONVERSION

In previous chapter, I discussed various zero-object "geometries" through various cutoff price potentials. Further more, I have previously introduced equation (142) for the transition market price, reproduced here below:

$$\forall \mathbf{k} \in \{0, \dots, \mathbf{K}\}, \forall \mathbf{n} \in \{1, \dots, \mathbf{N}\}, \forall \tau \in \{0, \dots, \mathbf{T}\} \therefore \mathbf{K} \models \mathbf{N} \& \mathbf{T} \models \mathbf{N}, \ \mathbf{r}_{\mathbf{k}}$$

$$=: \frac{1}{2} \left\{ \left[\operatorname{diff}(\mathsf{LN}(\mathsf{P}))_{\mathsf{L}} \right] + \left[\operatorname{diff}(\mathsf{LN}(\mathsf{P}))_{\mathsf{H}} \right] \right.$$

$$+ q \mu \left[1 + \left(\tau - \mathsf{C}_{\triangleright} \left(\bigcirc_{\tau} \mid \mathsf{F}\tau \right)^{\frac{\tau}{n}} \right) \right] \right\}$$

$$(161)$$

Equation (160) allows me to compute the Eigen-prices within a quantum zero-object. It also implies that information dissipated by zero-objects is quantized. However, I have not theorised or modelled information quantisation separately in this research study. I consider sufficiently that events in a filtration are reflected on asset prices in-line with efficient market hypothesis (Asquith, 1983; Bachelier, 1890; Bernard and Thomas, 1990; Bodie et al., 2009, Khrennikov, 2018).
This is based on the presumption that investors act rationally and without bias, and they estimate the value of an asset based on future expectations. Under these conditions, all existing information affects the price, which changes only when new information arrives. By definition, new information may appear randomly and influences the asset price randomly. Corresponding continuous time models are based on stochastic processes (Bachelier, 1890, Shiryaev, 1999, Mantenga and Stanley, 2003, Choustova, 2006). However randomness in financial markets can be better described by quantum mechanics where extreme irregularities in the evolution of price may be explained by quantum effects (Segal and Segal, 1998; Choustova, 2006; Haven, 2008a:2008b).

The information-price reflection must be in some form of parity between the $\mathbb{O}_{\downarrow\uparrow}$ and $\mathbb{O}_{\uparrow\downarrow}$ worlds, where quite probably only part of filtration may be "absorbed" by the zeroobject. This is related to the quality of filtrations. For that purpose I have defined $C_{\triangleright}(\bigcirc_{\tau} | F_{\tau})$ to be the quality of filtration in expectation, related to the degree of congruence of market regulation and governance and as a sign of connectivity of the two worlds. For information-quality issues in asset pricing, please see papers by Haven (2008a:2008b), and Epstein and Schneider (2008),

In order to work out the transition prices, I have considered a "graded" composition of each market region with different effects of dissipation and tunnelling at each quantum level. To that end, I have introduced formulations for transition price, as expressed in equations (143), (144), and (145), reproduced here below:

$$\forall k \in \{0, ..., K\} P_{\text{transition} \to II} = P_{\text{spot} \to I} \exp(r_k) \quad (162)$$

$$\forall k \in \{K, ..., 0\} \operatorname{diff}(LN(P))_{H} = LN(P)_{\forall i \in \{K, ...1\}} - LN(P)_{\forall j \in \{K-1, ...0\}}$$
(163)

$$\forall n \in \{0, \dots, N = K - 1\} \operatorname{diff}(LN(P))_{L} = LN(P)_{\forall i \in \{0, \dots, N\}} - LN(P)_{\forall j \in \{1, \dots, K\}}$$
(164)

As previously indicated focus in this research is on any two adjacent zero-objects along the market line, where my computations have shown that

$$1 + \left[\tau - C_{\triangleright} \left(\bigcirc_{\tau} \mid F\tau\right)^{\frac{\tau}{k+1}}\right] = 0 \qquad (165)$$

For a sequence of p-points along the market line, this effect is expected to increase, which implies larger asset price moves, subsequently higher uncertainty. This means there is more certainty in price prediction when one considers two immediate zero-objects and within a short time horizon (i.e. intraday trading), but less so when a sequence of zero object within a considerable time horizon is considered. There is an increasing body of evidence suggesting that exact nearest neighbour search in high-dimensional spaces is affected by the curse of dimensionality at a fundamental level (Schafer, 1966; Milnor, 1868; Kong et al., 2000; Pestov, 2006; Canessa, 2007).

The new theoretical framework has led to asset pricing. I have also shown in previous chapters that the Black-Scholes option pricing model is a special case of the Sturm-Liouville fitted system. Attention is now turned to option pricing with quantum-value conversion. As expected such process would be strongly bound to the pricing of the underlying, elaborated above and in previous chapters (Black and Scholes, 1973; Cox et al., 1979; Milevsky and Posner, 1998).

With the task to compute the option price using the zero-object models, one of the main hurdles is the computation of the probability within its space. Quantum mechanics is a probabilistic theory, as most of its predictions are irreducibly statistical. It is therefore understandable that the first attempts to clarify its content made use of the well-tested concept of statistical ensembles, describing identical abstract copies of the system under consideration, each of which would represent a different state in which the system might be found to be in. This statistical ensemble interpretation of quantum physics was originally put forward by Albert Einstein (Einstein, 1958; Ballentine, 1970; Aerts and Gabora, 2005; Bianchi, 2013).

I have introduced important ideas regarding the possibility of a realistic interpretation of the behaviour of quantum systems in a financial context. Further I apply intuition into the computation of possible quantum probabilities. These are understood as epistemic statements associated with lack of knowledge not about the state of the system, but about the exact "interaction" taking between \mathbb{O}_{11^-} and \mathbb{O}_{11^-} worlds, according to Aerts' hidden-measurement approach (Asano et al., 2011:2012; Aerts et al., 2010:2013). This approach is also supported by Bianchi (2013a).

In this research study, I have nonetheless quantified the possible forms of interaction between the two worlds (refer to previous chapter). I have done so by somehow reverting the logic of Einstein's celebrated quote, that "God does not play dice" (Irene Born, 1971), showing that the simple act of rolling a die (stochastic event) and according to certain protocols, is a truly quantum experiment, which can be described using a projection postulate and the Born rule (Max Born, 1926), and which is capable to produce interference effects.

For example Busemeyer et al. (2009) computed quantum probabilities in a Markov model using Feynman's path rules using single projected path dependencies between variables. In practical terms, this means that I may project probabilities from the generated Eigen-values.I have established previously the dependencies between variables in the two worlds. Surely probabilities are worked out of information and information itself is reflected on the price within classical Bayesian and Markov networks, modelled respectively to $\mathbb{O}_{\downarrow\uparrow}$ and $\mathbb{O}_{\uparrow\downarrow}$ -worlds. This may be projected within the zero-object and the interaction (interface) between the two worlds (Tucci, 1995; Gal, 2007; Darwiche, 2009; Asano et al., 2012; Moreira and Wichert, 2016).

In connecting the two worlds, contemporary work is varied and provides a good base. Leifer and Poulin (2008), proposed a quantum Bayesian network by replacing the classical formulas used to perform the inference process by their quantum counterpart. Whereas Busemeyer et al. (2009), proposed a quantum dynamic Markov model based on the findings of cognitive psychologists and interference terms.

Khrennikov (2006) also modelled mental processes through quantum probabilities,

where the interference process plays an important role in the process of recognizing images(Conte et al., 2009). Other interesting works of this author applying similar quantum formalisms correspond to Khrennikov (2007a:2007b:2009), Tentori et al. (2013), etc., where the proposed quantum model on quantum probabilities incorporated entangled decisions.

In this study, I have worked these out as weighted probabilities through a process of "normalisation" and for each Eigen-state.

$$\forall k \in \{n, ..., 0\} w_k^{\Xi} = \frac{LN(P)_k^{\Xi}}{\sum_{k=n}^0 LN(P)_k^{\Xi}}$$
 (166)

I then use equation (162) to compute asset's future price. This follows from the application of the asset price computation with Eigen-value conversation (refer to previous chapter). The single path dependency between the asset prices in the $\widehat{\mathbb{O}}_{\downarrow\uparrow}$ and $\widehat{\mathbb{O}}_{\uparrow\downarrow}$ -world is utilised consistently (Wu and Gonzalez, 1996; Prelec, 1998; Barberis and Huang, 2008; Polkovnichenko and Zhao, 2012).

I now revert back to equation (162) for the computation of the transition asset prices. These are converted to $\widehat{\mathbb{O}}_{\downarrow\uparrow}$ -world asset prices through a process of quantum convergence and direct inference, detailed in the previous chapter. Previously I observed market prices in the $\widehat{\mathbb{O}}_{\downarrow\uparrow}$ -world, computed Eigen-state values in the $\widehat{\mathbb{O}}_{\uparrow\downarrow}$, then converted them back to $\widehat{\mathbb{O}}_{\downarrow\uparrow}$ -world prices, one time horizon in the future. There is direct inference involved in variable mapping between the two worlds. Due to the fact that equation (162) deals with the $\mathbb{O}_{\downarrow\uparrow}$ -world future prices, one may use expected payoff formula for options, expressed below (Politzer and Macchi, 2000; Hull, 2014).

$$C_{T} = MAX(S_{T} - X, 0)$$
 (167)
 $P_{T} = MAX(0, X - S_{T})$ (168)

 C_T and P_T are payoffs for call and put, respectively. S_T is the spot (market) price at time T, and X is option's exercise price. In classical option valuation theories (Black and Scholes, 1973; Cox et al., 1979; Jarrow and Rudd, 1983; Leisen and Reimer, 1996; Milevsky and Posner, 1998; Garman and Kohlhagen, 1983; Hull, 2014), the value of an option is equal to its expected payoff in a risk-neutral world, discounted at the risk-free interest rate, which can be written as:

$$\mathbf{c} = \mathbf{e}^{-\mathbf{r}\mathbf{T}}\mathbf{E}^{\mathbf{Q}\in \mathbb{C}\downarrow\uparrow}[\mathsf{MAX}(\mathbf{S}_{\mathrm{T}} - \mathbf{X}, \mathbf{0})] \qquad (169)$$

$$\mathbf{p} = \mathbf{e}^{-\mathbf{r}\mathbf{T}}\mathbf{E}^{\mathbf{Q}\in\mathbb{C}\downarrow\uparrow}[\mathsf{MAX}(\mathbf{0},\mathbf{X}-\mathbf{S}_{\mathrm{T}})] \qquad (170)$$

Where $E^{Q \in \mathbb{O} \downarrow \uparrow}[]$, denotes the expectation with respect to some risk-neutral probability measure Q in the $\mathbb{O}_{\downarrow\uparrow}$ – world. This term is identified later in this chapter for each of the commonly used option pricing models. The exp(-rT) is the discount term. The notation convention here is to use capital C and P for payoffs of calls and puts, respectively, and lower case c and p for option price equivalents. I have also dropped the subscript for option prices at time zero in the $\mathbb{O}_{\downarrow\uparrow}$ – world.

Similarly and by analogy, using a zero-object to price options, I may write

$$c^{\Xi} = e^{-rT} E^{Q^{\Xi} \in \textcircled{D}\uparrow\downarrow} [MAX(S_{T} - X, 0)] \qquad (171)$$

$$p^{\Xi} = e^{-rT} E^{Q^{\Xi} \in \textcircled{0}\uparrow\downarrow} [MAX(0, X - S_{T})]$$
(172)

Where $E^{Q^{\Xi} \in \mathbb{O}^{\uparrow\downarrow}}[]$, denotes the expectation with respect to some probability measure Q^{Ξ} (that may be risk-neutral) inferenced in the $\mathbb{O}_{\downarrow\uparrow}$ – world. I do not identify a discretised expression for this term due to the fact that I have treated the asset and derivatives' pricing problem at a PDE level. Therefore it is outside of the scope of this research study. However, I can calculate Eigen-state weighted probabilities from computationally-generated Eigen-values using the PDE expressions of previous chapters. The notation convention for option prices uses the superscript Ξ to indicate that the call and put option values are established through zero-object inference. The expressions inside the square brackets in in both cases represent the expected payoff of the option.

The probability measures are important and have been treated in contemporary research linked to financial arbitrage. For example Haven (2008a:2008b) explored the concept of an "information wave function", and further underlined the role of risk-neutral probabilities for financial non-arbitrage. He argued that a change in the probabilities may introduce arbitrage⁸⁰ and that the conditions for no-arbitrage for a discrete parameter process must be met (Harrison and Kreps, 1979). It is important to make the distinction on the required conditions for non-arbitrage, between discrete and continuous underlying processes (Karatzas and Schreve, 1998a:1998b).

 $^{^{80}}$ Arbitrage implies that a positive financial return can be realised, which is in excess of the risk free rate of interest. 258

On the last leg of the process and prior to the actual use of the new theory to price options, I establish the most suitable zero-object sub-models. I use the previous chapters' discussions with focus on the suitability of cut-off price potentials to establish that Square-Well, Gaussian, Cosh⁻², and Arctan are the sub-models in trial i.e. refer to table (16) for the full set of cut-off price potentials. Recall from previous chapter that these geometries are important in the consideration of filtration penetration of markets under the previously applied boundary conditions. I have applied a criteria that combines market attributes and theoretical considerations.

In the previous chapter, I considered self-regulated market, where market regions were viewed as composed of layers with interfaces. These are not abrupt, therefore allowing for continuously varying cut-off price potentials to be fitted. I have argued that for smaller radius quantum zero-objects, the cut-off price potential polynomial explanation (eq. 137 and 138) should include a considerable number of terms, therefore a Gaussian γ would be most suitable in the model. A careful observation of the polynomial explanation (eq. 138) shows that the for small quantum zero-objects, the harmonic is a poor approximation and the Gaussian is more realistic (refer to equation 139). The Cosh⁻² (refer to table 2.0) is another smooth cut-off price potential. From the Taylor expansion, one can see that Cosh⁻², similar to Gaussian, is good approximation cut-off price function for small quantum zero-objects.

I have also provided arguments for cases where filtration dissipation is varied across quantum zero-objects, along the market line. In such cases, consideration of the "depth"

of the quantum construct was needed. therefore Arctan would be deemed to be suitable, particularly for reasonably small values of dispersion, σ . Dependant on how modulated the information is and the internal "texture" of the market region under consideration with the occurrence of information diffusion, the Gaussian cut-off price potential is probably still the most suitable one, particularly for smaller radius zero-objects. Recall that Arctan is a way of simulating "roughness", possibly when diffusion of information across the market line is significant (Möller and Zettl, 1996; Kong et al., 2000; Canessa, 2007; Yan et al., 2017; Kim and Lototsky, 2017).

The Arctan's potential suitability depends on the value of the parameter b and the size of the quantum zero-object. This price cut-off function can be a good approximation for large radius. However for larger zero-objects (radius), it followed that the rectangular-well was a reasonable sub-model to consider. The price eigenvalues generated for Arctan will become similar to the square well, when the parameter b is very small. When b is large, the spacing between the eigenvalues becomes very small. This shows how the price quantum levels may be affected by "roughness" in the quantized "interface".

For more detailed analysis, this price cut-off potential can be expanded, using the Taylor expansion (refer to previous chapter). The Harmonic potential function becomes a good approximation in the cases when there is no overlap of filtrations, or the sequence of events is orderly within the quantized construct. However this research does not explore patter formation of quantized flirtations, where minute fragmentation and overlaps could

occur. Instead, the harmonic potential can be seen as suitable for not too small and not too large zero-objects (refer equation 139). I subsequently, discarded decaying exponential, linear, triangular, and inverse models, (Möller and Zettl, 1996; Kong et al., 2000).

The price cut-off potentials selected here to trial option valuation are the Square-Well, Gaussian, Arctan, Cosh⁻², and Harmonic. These are deemed most suitable based on theoretical rational and assumptions on market attributes (refer to previous chapter discussion and previous paragraphs). I use numerical simulations and analysis in the following sections, that lead to the validation of the best-suited zero-object "geometry" in pricing options. The forward strategy is to compare results with those obtained from classical option pricing models (Black and Scholes, 1973; Cox et al., 1979; Jarrow and Rudd, 1983; Leisen and Reimer, 1996; Milevsky and Posner, 1998). I highlight improvements in option valuation attributed to the proposed quantum zero-object models by referring to deficiencies of existing and reported empirical tests on the classical models.

In this study, I allege that sensitives on the cut-off price potential can induce arbitrage. As previously elaborated, the wave function, is another very basic concept in quantum mechanics, can be fruitfully used to explaining arbitrage (Bossaerts et al., 2010; Khrennikov and Haven, 2009; Bruguier et al., 2010). This study does not represent a first attempt to link the potential function to financial arbitrage within a quantum construct. Haven (2002), considered the price of an option to be a financial-state function. This would satisfy the Schrödinger differential equation and is in-line with our conceptualisation as explained in previous sections and chapters.

Moreover, Haven (2002) argued that an arbitrate-free price may be acquired when the potential function converges to one, whereas arbitrage can be achieved when the Planck's constant is non-zero. According to Haven, the Planck's parameter regulates the probability of strategy paths' occurrence. Haven (2002) called this parameter the "belief" parameter. This is important due to the fact that the classical option pricing models are arbitrage-free models. Therefore, Haven (2002) provides a basic approach to include arbitrage in a natural way. According to Haven (2002) the "belief" parameter could only serve as a proxy to arbitrage.

The small "diversion" above should serve to highlight the value of this research's quantum zero-object model (Generalised Shrodinger-Sturm-Liouville) in comparison with existing quantum models. This is so because I do not use the Plank's constant at all. The parameter has significance in Physics and is defined in a non-financial context. It may help to see that such relaxation of the Plank's contestant is equivalent to setting its value to one, which is subsequently non-zero and reflective of arbitrage. Haven (2002) then links arbitrage to the potential function i.e. the price-cut-off potential function. The use of the Planck's constant and the potential function in this way has not been tested and includes oddities in logic congruency. However, in this work I do not link Plank's parameter to arbitrage at all. Further empirical tests and analysis in this study (further down) show that such parameters are irrelevant and do not effect the price Eigen-values.

It is the quantum "geometries", best reflected through the cut-off price potentials, that proxy arbitrage (refer to chapter 4).

Haven (2002:2003:2005) uses a binary format for arbitrage and non-arbitrage occurrence i.e. potential value of zero and one. This is too simplistic when considering the comprehensiveness of the potential functions I have included in this study. Furthermore, I believe that further research in the quantisation of filtrations on zero-objects may hold the key to the missing layer of knowledge on how information is dissipated and tunnelled to adjacent quantum regions along the market line. This is important if one is to predict future events by impact level and degree. In comparison, it suffice to say that the potential function with 0 and 1 values in the works of Haven (2002:2003:2005) is a special case to this study's square-well price cut-off potential, and only partially suitable in option pricing for large quantum zero-objects. I consider various potentials and zeroobject of various sizes (refer to discussions in preceding sections and chapters).

7.2 FINANCIAL OPTION VALUATION WITH SQUARE-WELL, GAUSSIAN, COSH, AND ARCTAN ZERO-OBJECT MODELS

I now turn attention to the calculation of option prices using the justified selection of zero-object models discussed in the preceding section. There are two sets of inputs used; (i) the $\mathbb{O}_{\downarrow\uparrow}$ - world inputs , and (ii) $\mathbb{O}_{\uparrow\downarrow}$ - world inputs. I have applied the $\mathbb{O}_{\uparrow\downarrow}$ - world inputs through the relevant algorithm with NAG-subroutine embedment.

I have already introduced measures of direct inference, thus mapping variables between

the two worlds in which the quantized space is measurable in terms of $q\sigma_m$, with the parameter $q = 10^{-7}$ of a basis point (BP), where σ_m is taken to be the smallest dispersion in consideration, $\sigma_m = 10^{-4}$ (refer to chapter 5, section 5.1.4). Ioften drop the subscript 'm' for simplicity when focusing on the $\mathbb{O}_{\downarrow\uparrow}$ - world only. Subsequently $1q\sigma_m$ is equal to 10^{-11} . I have branded this to be a space conversion or transition coefficient. This allows to map out inputs across the two worlds. I have previously assumed a spherical zero-object where its radius is expressed in $q\sigma$ terms. This averts the use of SI units for space in Finance, which would be illogical. The space" and time here is dimensioned on price as a price - *spacetime* representation. I avoid use of currency symbols by using % representations. The choice of γ also depends on the size of the quantum price point i.e. refer to chapter 5, section 5.1.4 for further details.

To keep the proof of concept as simple as possible, I use the following common $\bigcirc_{\downarrow\uparrow}$ - world inputs. This also allows to connect back-to-back with the data used in the numerical analysis in chapter 5.

Share price (S)	100.34	
Exercise price (X)	90.00	
Int rate-cont (r)	2.42%	
Dividend yield (q)	0.00%	
Time now (0, years)	0.0000	
Time maturity (T, years)	0.7500	
Option life (T, years)	0.7500	

Table 17. Sample inputs for option pricing tests.

However, we also need measures of dispersion, σ , meaningful in the $\mathbb{O}_{\downarrow\uparrow}$ - world. We do vary them across the test scenarios, hence we do introduce them as needed. For simplicity our choice of dispersion is mapped directly to $\gamma(0)$ through the relation

$$\sigma = \frac{\gamma(0)}{q} \qquad (173)$$

Where $1 = 10^{-7}$, and $\gamma(0)$ is expressed in $q\sigma_m$ (refer to chapter 5, section 5.1.4). This allows us to re-use data from the asset pricing simulations of chapter 5.

Eigen – prices are generated from the C++ program through the invocation of NAG routines. We have trialled various Sturm-Liouville fittings to our master expression (chapter 5, section 5.3). We have also generated data with the quantised generalisation of the Black-Scholes expression. We placed greater emphasis on equation (104) for Sturm-Liouville fitting (Shrodinger-Sturm-Liouville-Euler). Eigen-price sets were generated programmatically, because it appears to be a more generalised and abstract solution than the quantised Black-Scholes PDE. Under such case, interestingly, the cut-off price potential, referred to as potential function in Haven (2002), has a constant identity.

We recall from our discussion in the previous section that Haven (2002) was very keen to link the existence of an arbitrage-free price to the potential function convergence to one and an arbitrage price with a non-zero Planck contestant. The Plank constant in our formulations is one, indicating arbitrage according to Haven (2002). However, in our work, arbitrage is achieved through the differential of the quantum operator with the residual effect (reflected in the cut-off price potential function) and when such differential is non-zero. Equivalently an arbitrate-free model is achieved when the same differential is zero. Such is the 'Shrodinger-Sturm-Liouville-Euler' model trialled here. This implies that the quantised flirtations are well formed within the quantum zero object and in complete parity with the residual effect i.e. driven by the underlying stochastic process.

The tabulated data in tables (17, 18, 19) below are generated through a process that involved the following steps: (i) use of equation (161) to obtain the rate of return Eigen-vector from Eigen-prices, (ii) application of equation (162) for the acquisition of the price transition vector, (iii) calculation of Eigen-probabilities, explored in previous sections of this chapter, through equation (236), (iv) use of equations (167) and (168) to obtain call and put option payoffs, and (v) use of equations (169) and (170) to compute call and put option prices, respectively.

It's important to note that step (v) is achieved without a discretised formulation of the expectation term, $E^{Q \in Olf}[\cdots]$. This is because in this research work we have treated the problem at a partial different equation level. This is quite common practice when quantum settings are used. It is so because our formulated problem, like many others. involves option pricing with additional complexity and the challenge to address very fast changing markets. In order to take advantage of arbitrage due to mispriced financial stock options, the computation must complete prior the next change in a continuously

changing stock market. It serves to justify in part the problem addressed at a PDE level rather than at the expectation. There is an abundance of research that has applied alternative computation algorithms to finance (including quantum finance). We do so in this research work, but also the works of Boghosian and Washington (1998), Baaquie et al. (2002), Hirvensalo (2003), Khrennikov, (1999:2006:2007a:2007b), Meyer (2009), Rebentrost et al. (2018), etc. Nonetheless the $E^{Q \in Olf}[\cdots]$ has been invoked by our algorithm, such that the necessary results are obtained.

k	Probs/Weights	P(future point)	Payoff of Call	Probs/Weigh ts	P(future point)	Payoff of Call	Probs/Weigh ts	P(future point)	Payoff of Call
		γ(0)=15qs	x= 90		γ(0)=25qs	x= 90		γ(0)=35qs	x= 90
0	0.0048172	£86.39	£0.00	0.00354486	86.1229321	£0.00	0.00297991	85.7912671	£0.00
1	0.0192917	£95.55	£5.55	0.01408001	95.2593116	£5.26	0.01194089	95.3319507	£5.33
2	0.0432261	£95.69	£5.69	0.03178092	97.5886718	£7.59	0.02675959	95.2531343	£5.25
3	0.0763306	£96.21	£6.21	0.05331599	92.8521013	£2.85	0.04753064	95.382099	£5.38
4	0.1175606	£107.70	£17.70	0.08554648	95.0492254	£5.05	0.07409922	95.591836	£5.59
5	0.1451072	£115.30	£25.30	0.1252472	98.7616547	£8.76	0.10621406	95.6216653	£5.62
6	0.1457763	£100.00	£10.00	0.1671344	130.817002	£40.82	0.14383943	96.6986478	£6.70
7	0.1471086	£100.00	£10.00	0.17244443	103.896937	£13.90	0.18569397	127.358943	£37.36
8	0.1490896	£100.02	£10.02	0.17295054	99.9779257	£9.98	0.20026806	113.567732	£23.57
9	0.1516923	£103.08	£13.08	0.17395517	101.812656	£11.81	0.20067422	101.054949	£11.05
Quantum Implied Option value			£12.75			£14.56			£15.69
time			0.75						

Table 18: Data generated from the C++ program for different well depth for the square well γ function. Call ption payoff are computed as well as the option prices for various price cut-off γ values; γ (0) = 15q σ , γ (0) = 25q σ , γ (0) = 35q σ , exercise price x = £90.00, and option life t = 9 months.

In table (18), the de-quantised $\gamma(0)^{81}$ was set to match the dispersion measure. We notice that when the volatility is increased from 15% to 25%, and then 35%, the price of the call option increased. However such increase is small.



Figure 25: Call option price vs volatility. 'Square-Well' zero-object model used.

k	Probs/Weigh ts	Next Time Period Price in £	Payoff of Call	Probs/Weig hts	P(future point)	Payoff of Call	Probs/Weigh ts	P(future point)	Payoff of Call
	Gaussian γ(0)=10 qs	Gaussian γ(0)=10 qs	x= 90	COSH^-2 γ(0)=10 qs	COSH^-2 γ(0)=10 qs	x= 90	Arctan well of γ(0)=6 qs, b=10E12 qs	Arctan well of γ(0)=6 qs, b=10E12 qs	x= 90
0	0.0746375	£83.12	£0.00	0.0479712	83.3739174	£0.00	0.18867133	£99.8883	£9.89
1	0.1600113	£102.52	£12.52	0.1384766	100.351434	£10.35	0.19150363	£100.1139	£10.11
2	0.2259139	£103.16	£13.16	0.2288706	103.319013	£13.32	0.19716758	£100.1138	£10.11
3	0.2667218	£104.27	£14.27	0.2906739	106.526781	£16.53	0.20566446	£100.1139	£10.11
4	0.2727155	£101.68	£11.68	0.2940077	101.026961	£11.03	0.216993	£102.1673	£12.17
Quantum Implied Option value			£11.88			£12.37			£10.50
time			0.75						

⁸¹ Divided by the q parameter, 10^{-7} .

Table 19: Data generated from the C++ program for Gaussian, COSH⁻², and Arctan cut-off γ function; γ (0) = 10q σ (Gaussian and COSH-2), and γ (0) = 6q σ with b = 10E12q σ . Call ption payoffs and prices computed for an option with exercise price of x= £90.00 with a 9-month maturity time.

In table (19), Gaussian, Cosh⁻², and Arctan price cut-off potentials are used. We notice that for a volatility of 10%, matched with the $\gamma(0)/q$, Gaussian produces a lower option value than Cosh⁻². The value of the option is further lowered when Arctan is tested with a volatility of 6% and a very large coefficient b = 10E12 q σ_m . Recall that Arctan may be seen as more favourable than the square well, because it is more "adjustable" with respect to the 'smoothness' of the 'edges'. As can be seen by figure 23b, it does converge to an a square well equivalent with rounded edges for b very small. Here the coefficient b is chosen to be very large, hence the expectation that its behaviour and subsequently the option value would move away from the value produced by the square well model.

k	Probs/Weigh ts	Next Time Period Price in £	Payoff of Call	Probs/Weig hts	P(future point)	Payoff of Call	Probs/Weigh ts	P(future point)	Payoff of Call
	Harmonic γ(0)=10 qσ	Harmonic γ(0)=10 qσ	x= 90	Arctan well of γ(0)=6 qσ, b=100 qσ	Arctan well of γ(0)=6 qσ, b=100 qσ	x= 90	Arctan well of γ(0)=6 qs, b=1000 qs	Arctan well of γ(0)=6 qσ, b=1000 qσ	x= 90
0	0.0384549	£86.39	£0.00	0.0134514	87.3602015	£0.00	0.10227656	£99.8883	£9.89
1	0.0897404	£95.55	£5.55	0.0535367	96.1692557	£6.17	0.10381191	£100.1139	£10.11
2	0.1410466	£95.69	£5.69	0.1181927	101.909475	£11.91	0.10688227	£100.1138	£10.11
3	0.1923036	£96.21	£6.21	0.1738662	110.105326	£20.11	0.11148833	£100.1139	£10.11
4	0.2435866	£107.70	£17.70	0.1757896	101.009182	£11.01	0.1176294	£102.1673	£12.17
5	0.2948679	£115.30	£25.30	0.4651634	272.752399	£182.75	0.45791153	£272.7524	£182.75
Quantum Implied Option value			£13.24			£21.61			£63.36
time			0.75						

Table 20: Data generated from the C++ program for Harmonic and Arctan cut-off γ function; γ (0) = 10q σ (Harmonic), and γ (0) = 6q σ with b = 100q σ , and b = 1000q σ ((Arctan). Call option payoffs and prices are computed for an option with exercise price of x= £90.00 with a 9-month maturity time.

In table (20), the Harmonic and Arctan cut-off potentials are used. The $\gamma(0)$ is varied across the two quantum zero-object "geometries". The b coefficient is chosen significantly smaller compared to the previous exhibit (table 19). For such a small $\gamma(0)$, and subsequently σ , the model would be expected to work better for very large b. Small $\gamma(0)$ value here leads to the call option overpricing.

Equivalently, we may compute the put option value under the quantised probability expectation. Starting with equation (171), reproduced below, with several transformation steps.

$$c^{\Xi} = e^{-rT} E^{Q^{\Xi} \in \textcircled{O}^{\uparrow\downarrow}} [MAX(S_{T} - X, 0)]$$
(174)

We first add the zero-sum term $(X \cdot e^{-rT} - X \cdot e^{-rT})$ on the right hand side:

$$e^{-rT}E^{Q^{\Xi} \in \textcircled{O}^{\uparrow\downarrow}}[MAX(S_{T} - X, 0)] + X \cdot e^{-rT} - X \cdot e^{-rT}$$
$$= e^{-rT}E^{Q^{\Xi} \in \textcircled{O}^{\uparrow\downarrow}}[MAX(S_{T}, X)] - X \cdot e^{-rT}$$
(175)

Followed by another zero-sum term $(S_T \cdot e^{-rT} - S_T \cdot e^{-rT})$.

$$c^{\Xi} = e^{-rT} E^{Q^{\Xi} \in \textcircled{O}^{\uparrow\downarrow}} [MAX(S_{T}, X)] - X \cdot e^{-rT} + S_{T} \cdot e^{-rT} - S_{T} \cdot e^{-rT}$$
$$= e^{-rT} E^{Q^{\Xi} \in \textcircled{O}^{\uparrow\downarrow}} [MAX(0, X - S_{T})] - X \cdot e^{-rT} + S_{T} \cdot e^{-rT}$$
(176)

We notice that the first term in the right hand side is the same expression as that for the put option value given by equation (172). We transform it further as follows:

$$c^{\Xi} = p^{\Xi} - X \cdot e^{-rT} + S_{T} \cdot e^{-rT}$$
 (177)

Finally, we add the dividend effect to discounted price and re-arrange the formula to get the call-put parity condition.

$$c^{\Xi} + X \cdot e^{-rT} = p^{\Xi} + S \cdot e^{-qT} \qquad (178)$$

Subsequently, we may use the following arrangement of the above to price the put option price under arbitrage-neutral conditions.

$$p^{\Xi} = c^{\Xi} + e^{-rT} \cdot X - Se^{-qT}$$
 (179)

or

$$p^{\Xi} = c^{\Xi} + e^{-rT} \cdot (X - Se^{(r-q)T})$$
 (180)

Using the input data, computed quantum implied call option values (table 18), and equation (179), we obtain the put option price.

		Quantum Implied call value		£12.75		£14.56		£15.69
		time		0.75				
S	100.34							
Х	90							
r	2.42%	Put value		£0.80		£2.60		£3.73
q	0							
Т	0.75							
		σ		15.00%		25.00%		35.00%

Table 21: Quantum-implied call and put values under non-arbitrage conditions.

I have also investigated and found an alternative way to compute the put option value as an implied quantum zero-object measure. To complete the computation for the put option through quantum zero-object models, I re-address equation (161), with a small modification to account for the price spin. For the call the market "spin-up" if the call is to be in-the-money, presented through the integer +1. Equivalently for a put, the "spin" would be "down" if the put option is to land in the money, presented with -1. This effect is reflected on the modified equation below:

$$\forall \mathbf{k} \in \{0, \dots, \mathbf{K}\}, \forall \mathbf{n} \in \{1, \dots, \mathbf{N}\}, \forall \tau \in \{0, \dots, \mathbf{T}\} :: \mathbf{K} \models \mathbf{N} \& \mathbf{T} \models \mathbf{N}, \quad \mathbf{r}_{\mathbf{k}}^{\uparrow\downarrow}$$

$$=: \pm \frac{1}{2} \left\{ \left[\text{diff} \left(L\mathbf{N}(\mathbf{P}) \right)_{\mathbf{L}} \right] + \left[\text{diff} \left(L\mathbf{N}(\mathbf{P}) \right)_{\mathbf{H}} \right]$$

$$+ q\mu \left[1 + \left(\tau - C_{\triangleright} \left(\bigcirc_{\tau} \mid \mathbf{F}\tau \right)^{\frac{\tau}{n}} \right) \right] \right\}$$

$$(181)$$

In our computation procedure we use the parameter iopt = ± 1 i.e. +1 for call, -1 for put. Interestingly the quantised Eigen-values are the same, however the converted stock price shifts, thus a new vector of values is generated.

Spot Price	£100.34																	
k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	P in qs	P(future point)	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р	P(future point)	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	Р	P(future point)
	γ(0)=15qO	γ(0)=15qσ	γ(0)=15qσ	γ(0)=15qσ	γ(0)=15qσ		γ(0)=25qO	γ(0)=25qσ	γ(0)=25qσ	γ(0)=25qσ	γ(0)=25qØ		γ(0)=35qσ	γ(0)=35qσ	γ(0)=35qØ	γ(0)=35qσ	γ(0)=35qØ	
0	0.0498250	-0.1497120		14.9712%	1.051087140	£116.54	0.0514103	-0.152789		15.2789%	1.05275475	£116.90	0.0520918	-0.156647		15.6647%	1.05347245	£117.36
1	0.1995370	-0.2475580	0.149712	4.8923%	1.220837380	£105.37	0.2041990	-0.256712	0.152789	5.1962%	1.226542211	£105.69	0.208739	-0.259046	0.156647	5.1199%	1.23212337	£105.61
2	0.4470950	-0.3424060	0.247558	4.7424%	1.563762850	£105.21	0.4609110	-0.312318	0.256712	2.7803%	1.585517734	£103.17	0.467785	-0.363099	0.259046	5.2027%	1.59645413	£105.70
3	0.7895010	-0.4264490	0.342406	4.2022%	2.202297206	£104.65	0.7732290	-0.467431	0.312318	7.7557%	2.16675141	£108.43	0.830884	-0.464446	0.363099	5.0674%	2.29534693	£105.56
4	1.2159500	-0.2849200	0.426449	-7.0765%	3.373497365	£93.48	1.2406600	-0.575770	0.467431	5.4169%	3.457894922	£105.93	1.29533	-0.561400	0.464446	4.8477%	3.652201	£105.32
5	1.5008700	-0.0069200	0.284920	-13.9000%	4.485589836	£87.32	1.8164300	-0.607480	0.575770	1.5855%	6.149864198	£101.94	1.85673	-0.657730	0.561400	4.8165%	6.40276546	£105.29
6	1.5077900	-0.0137800	0.006920	0.3430%	4.516737766	£100.68	2.4239100	-0.077010	0.607480	-26.5235%	11.2899167	£76.96	2.51446	-0.731660	0.657730	3.6965%	12.3599326	£104.12
7	1.5215700	-0.0204900	0.013780	0.3355%	4.579409227	£100.68	2.5009200	-0.007340	0.077010	-3.4835%	12.19370701	£96.90	3.24612	-0.254770	0.731660	-23.8445%	25.6904673	£79.05
8	1.5420600	-0.0269200	0.020490	0.3215%	4.674209231	£100.66	2.5082600	-0.014570	0.007340	0.3615%	12.2835381	£100.70	3.50089	-0.007100	0.254770	-12.3835%	33.1449378	£88.65
9	1.5689800		0.026920	-2.6920%	4.801747912	£97.67	2.5228300		0.014570	-1.4570%	12.46381941	£98.89	3.50799		0.007100	-0.7100%	33.3811043	£99.63
				_						_								£100.34

Table 22:Eigen-price values generated with C++ with NAG routines embedded. The quantumconversion is worked out with 'spin-down', such that the 'spin-down' price vector is obtained.

k	Probs/Weights	P(future point)	Payoff of Put	Probs/Weigh ts	P(future point)	Payoff of Put	Probs/Weigh ts	P(future point)	Payoff of Put
		γ(0)=15qs	x= 90		γ(0)=25qs	x= 90		γ(0)=35qs	x= 90
0	0.0048172	£116.54	£0.00	0.00354486	£116.90	£116.90	0.00297991	£117.36	£0.00
1	0.0192917	£105.37	£0.00	0.01408001	£105.69	£0.00	0.01194089	£105.61	£0.00
2	0.0432261	£105.21	£0.00	0.03178092	£103.17	£0.00	0.02675959	£105.70	£0.00
3	0.0763306	£104.65	£0.00	0.05331599	£108.43	£0.00	0.04753064	£105.56	£0.00
4	0.1175606	£93.48	£0.00	0.08554648	£105.93	£0.00	0.07409922	£105.32	£0.00
5	0.1451072	£87.32	£2.68	0.1252472	£101.94	£0.00	0.10621406	£105.29	£0.00
6	0.1457763	£100.68	£0.00	0.1671344	£76.96	£13.04	0.14383943	£104.12	£0.00
7	0.1471086	£100.68	£0.00	0.17244443	£96.90	£0.00	0.18569397	£79.05	£10.95
8	0.1490896	£100.66	£0.00	0.17295054	£100.70	£0.00	0.20026806	£88.65	£1.35
9	0.1516923	£97.67	£0.00	0.17395517	£98.89	£0.00	0.20067422	£99.63	£0.00
Quantum Implied Put value			£0.3962			£2.6709			£2.3934
time			0.75						

 Table 23:
 Quantum-implied put values under arbitrage conditions.

It's evident from the set of data given that the contract would be set up initially as an 'in-the money' call option i.e. because 90.00 < 100.34. This is reflected in lower prices for the put option contract. This is quite logical and understandable. The demand shift would move towards calls and away from puts. This implies arbitrage. Therefore the values above are arbitrage-implied values.

		Quantum Implied call value			£12.75		£14.56		£15.69
		Quantum Implied put value	Aribitrag	e	£0.40		£2.67		£2.39
		time			0.75				
S	100.34								
X	90								
r	2.42%	Put value	Abitrage-F	ree	£0.80		£2.60		£3.73
q	0								
Т	0.75								
		σ			15.00%		25.00%		35.00%

Table 24: Quantum-implied call and put values under arbitrage and non-arbitrage conditions.



Figure 26: Put option values. Arbitrage vs Non-Arbitrage with square-well price cut-off potential

We notice that under arbitrage, the put option would appear to be mispriced i.e. underpriced for $\sigma = 15\%$ and 35%, but slightly overpriced at $\sigma = 25\%$.

7.3 FINANCIAL OPTION VALUATION WITH BLACK-SCHOLES, CRR BINOMIAL, RG MODELS

7.3.1 THE BLACK-SCHOLES MODEL

We start here with the Black-Scholes-Merton option pricing model as one of the most celebrated classical models. Our goal it to simply identify its main elements that we then use to complete the computation of option prices. The Black-Scholes analysis provide valuation for straightforward European options (i.e. can only be exercised at maturity), that can also be adapted to value a range of other options. The Black-Scholes-Merton analysis assumes that log returns of the share underlying the option are normally distributed (Black and Scholes, 1973; Black, 1989; McDonald, 2006; Hull, 2014).

Suppose the option on a share (currently priced S) is a call which can only be exercised when the call matures after time period T, the exercise price being X. The payoff from the call at time T is then given by equation (167), reproduced here below:

$$C_{\rm T} = MAX(S_{\rm T} - X, 0)$$
 (182)

Here S_T represents the share price at time T, i.e. it is a random variable with a probability distribution. The standard assumptions are that the share price follows a stochastic process with a multiplicative sequence of moves of variable size or, more exactly, what is known as geometric Brownian motion. This model of the share price process is very plausibly explained in Black and Scholes (1973), McDonald (2006), Hull (2014) on the behaviour of stock prices.

In the Black-Scholes analysis, the call to be valued is combined with a fraction of the share to form the hedge portfolio, which is constructed to be risk-free. Thus the hedge portfolio must earn the risk-free rate of return. The algebra leads on to a partial

differential equation (known as the diffusion or heat equation). The differential equation is solved to give the Black-Scholes formula.

The Black-Scholes values for a European call option, c, on a share that does not pay dividends is worked out of the fully formed equation (169). When solved mathematically it becomes :

$$\mathbf{c} = \mathbf{S} \cdot \mathbf{e}^{-\mathbf{q}\mathbf{T}} \cdot \mathbf{N}(\mathbf{d}_1) - \mathbf{X} \cdot \mathbf{e}^{-\mathbf{r}\mathbf{T}} \cdot \mathbf{N}(\mathbf{d}_2) \quad (183)$$

Where S is the current share price, X the exercise price for the call at time T, r the continuously compounded risk-free interest rate, hence the expression exp(-rT) for the risk-free discount factor over period T. The N(d) term is used to denote the cumulative standard probability distribution for value d. Here d₁ and d₂ are given by:

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
(184)

$$d_2 = \frac{\ln\left(\frac{s}{x}\right) + \left(r + q - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad or \quad d_2 = d_1 - \sigma\sqrt{T}$$
(185)

We notice that the rate of return on the share, μ , does not appear in the formula. The reason for this is apparent when we have understood the significance of the hedge portfolio. The value of the put option could be obtained by using the put-call parity

$$\mathbf{c} + \mathbf{X} \cdot \mathbf{e}^{-\mathbf{r}\mathbf{T}} = \mathbf{p} + \mathbf{S} \cdot \mathbf{e}^{-\mathbf{q}\mathbf{T}} \tag{186}$$

and the Black-Scholes call expression. The value of a European put option on a share is:

$$\mathbf{p} = \mathbf{X} \cdot \mathbf{e}^{-\mathbf{r}\mathbf{T}} \cdot \mathbf{N}(-\mathbf{d}_2) - \mathbf{S} \cdot \mathbf{e}^{-\mathbf{q}\mathbf{T}} \cdot \mathbf{N}(-\mathbf{d}_1)$$
(187)

(Black, 1973:1989; McDonald, 2006; Hull, 2014).

We compute N(d) as a probability measure using Excel's function NORMSDIST(...). In order to simplify the calculation process for calls and puts, we combine the call and put formulae into one unified expression, included here below:

$$opt_value = iopt \cdot [S \cdot e^{-qT} \cdot N(iopt \cdot d_1) - X \cdot e^{-rT} \cdot N(iopt \cdot d_2)]$$
(188)

The iopt is an integer parameter, and it takes only two values; +1 for a call, and -1 for the put. It is easy to see that for iopt equal to +1, the expression become the call option valuation formula. Similarly when iop is equal to -1, we obtain the put option valuation expression (Jackson and Staunton, 2001; Sengupta, 2009; Rees, 2017; Häcker and Ernst, 2017).

7.3.2 THE CRR BINOMIAL MODEL

The binomial option pricing framework (with of its variations) is an alternative to the Black-Scholes-Merton model. The option pricing multi-period binomial framework is one that extends both the one time period model and the foundational assumptions, in that like the one time period from an observed market price of the underlying asset we expand it to time n through a multiplicative process, generating a 'spectrum' of prices at time n. The values are from highest to lowest in range. Unlike the one time period, where there is only one single binomial set of prices (High and Low). The simplest multi-period form of the model operates over times 0,1,...,n, where n is a fixed positive integer and is the terminal time of the model. We assume there are just two assets: a stock for which the price evolves randomly from period to period and a bank account paying a constant rate

of interest, $r \ge 0$, per period.

The evolution of stock price follows a tree-like expansion as shown in the illustration for the values corresponding to times 0, 1, 2, 3, 4.



where u and d are the up and down price multipliers. It is useful to think of the node in the binary tree correspond to the stock price $S_t = u^i d^{(t-i)}S_0$ as (i, t) so that the binary tree may be represented in the following manner:



In an n-step tree, there are (n+1) terminal time stock values and the total number of nodes for the full tree is given by

$$1 + 2 + \dots + (n+1) = \frac{1}{2}(n+1)(n+2)$$
(189)

The Cox, Ross and Rubinstein (CRR) provides expressions for the price multipliers and

the probability set for the full tree, specifically on the assumption that the u and d price multipliers have a dependency on volatility and time across each step(in small and equal increments), but no dependency whatsoever on the drift.

$$\mathbf{u} = \mathbf{e}^{\sigma\sqrt{\delta t}} \tag{190}$$

$$d = e^{-\sigma\sqrt{\delta t}}$$
 or $d = \frac{1}{u}$ (191)

where σ is the annualized volatility and δt the length of the time step. The CRR model despite the instance of dependency on the drift in u and d, it offset the absence of a drift, where the probability of an up move in CRR is usually greater than 0.5 to ensure that the expected value of the price increases by a factor of $\exp[(r-q)\delta t]$ on each step with the probability formula as shown below:

$$p = \frac{b-d}{u-d}$$
 where $b = e^{(r-q)\delta t}$ (192)

The dividends affect only the probabilities in the CRR model, not the share price values and in general the CRR theory reinforces the link between the continuous normal distribution function, N(d), discrete binomial distribution function Φ . From a practical point of view the CRR provides a fitting apparatus of math instruments that facilitates the development of the full tree and the full underlying asset price range and subsequently the price of the derivative (Cox et al., 1979; Cox and Rubinstein, 1985; McDonald, 2006; Hull, 2014).

The compact CRR binomial option pricing formula can be written as

$$\mathbf{c} = \mathbf{S} \cdot \mathbf{e}^{-\mathbf{q}\mathbf{T}} \cdot \Phi(\mathbf{a}:\mathbf{n}, \mathbf{p}') - \mathbf{X} \cdot \mathbf{e}^{-\mathbf{r}\mathbf{T}} \cdot \Phi(\mathbf{a}:\mathbf{n}, \mathbf{p})$$
(193)

where $\Phi(a:n, p')$ and $\Phi(a:n, p)$ are probability-like measures computed here through Excel's function BINOMDIST(..., true). Running this function returns the individual term binomial distribution probability. It's is quite suitable in our case because BINOMDIST works well for problems such as ours with a fixed number of trials, because we seek outcomes of each trial to be only success or failure, independent, and the probability of success a constant throughout the experiment. Distribution functions are generally defined in terms of probabilities in the left hand tail of the distribution, whereas the complimentary distribution function refers to right-hand tail. Thus the binomial distribution is evaluated here for a-1 i.e. 1-BINOMDIST(..., true). Parameter n represents the number of trials (i.e. should be les or equal to number of steps in the binomial tree), a is the number of successes in trials. We compute a using the formula

$$a = MAX \left\{ 0, \ 1 + INT \left[\frac{\ln\left(\frac{x}{S_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)} \right] \right\}$$
(194)

where n is the number of steps in the binomial tree. Function INT(...) performs data casting by converting floating point numbers to a integer values.

In a similar manner to that of the Black-Scholes model, we may use the call-put parity to obtain the put option expression. Further we reduce the computation process by combining the call and put equations in one unified expression:

opt_value = iopt
$$\cdot \left\{ S \cdot e^{-qT} \cdot \left[\frac{1+iopt}{2} - iopt \cdot \Phi(a:n, p') \right] - X \cdot e^{-rT} \cdot \left[\frac{1+iopt}{2} - iopt \cdot \Phi(a:n, p) \right] \right\}$$
 (195)

The switch from call to put is facilitated through the iopt parameter.

Results of option pricing (calls and puts) using Black-Scholes-Merton and CRR models are shown below:

Data			Generalised Cox, Ro	oss & Rubinstein			Black-Scholes			
					τ.					
Share price (S)	100.34		δt	0.0038	$\delta t = \frac{1}{n}$		DF	0.9820	DF = exp(-rt)	r)
Exercise price (X)	90.00		erdt	1.0001	$erdt = exp(r\partial t)$		QF	1.0000	QF = exp(-q)	τ)
Int rate-cont (r)	2.42%		ermqdt	1.0001	$erm qdt = \exp((r - q))$	ar)	d1	1.0417	0.6943	0.5702
			u	1.0092	1.0154	1.0217	d2	0.9118	0.4778	0.2671
Dividend yield - cont (g)	0.00%		d	0.9909	0.9848	0.9788				
			р	0.5026	0.4991	0.4968				
Time now (0, years)	0.00		p'	0.5072	0.5068	0.5075				
Time maturity (T, years)	0.75									
Option life (τ, years)	0.75		a	95	97	98				
Volatility (g)	15.00%	25.00%	35.00%							
volutility (0)	10.0070	20.0070	05.0070							
			Φ [a:n,p']	0.8371	0.7540	0.7140	N (d1)	0.8512	0.7563	0.7157
iopt	1		Φ [a:n,p]	0.8030	0.6809	0.6032	N (d2)	0.8191	0.6836	0.6053
									<u> </u>	
Steps in tree (n)	200		CRR Call Value	13.02	15.47	18.33	Black-Scholes Call Value	13.02	15.46	18.32

Table 25: Call option values, computed with Black-Scholes and CRR models.

Volatility	Call Value BSM	Call Value CRR
15.00%	13.02	13.02
25.00%	15.46	15.47
35.00%	18.32	18.33

Table 26: Tabulated Call option values. Black-Scholes and CRR models.



Figure 27: Call option values. Black-Scholes-Merton vs CRR models.

Data			Generalised Cox, Ro	oss & Rubinstein				Black-Scholes			
Share price (S)	100.34		δt	0.0038	$\delta t = \frac{1}{n}$			DF	0.9820	$DF = exp(-\pi)$	() ()
Exercise price (X)	90.00		erdt	1.0001	erdt = exp(rat)			QF	1.0000	QF = exp(-q)	τ)
Int rate-cont (r)	2.42%		ermqdt	1.0001	$erm qdt = \exp((r - q)$	a)		d1	1.0417	0.6943	0.5702
			u	1.0092	1.0154	1.0217		d2	0.9118	0.4778	0.2671
Dividend yield - cont (g)	0.00%		d	0.9909	0.9848	0.9788					
			р	0.5026	0.4991	0.4968					
Time now (0, years)	0.00		p'	0.5072	0.5068	0.5075					
Time maturity (T, years)	0.75										
Option life (τ, years)	0.75		a	95	97	98					
Volatility (σ)	15.00%	25.00%	35.00%								
						1					
			Φ [a:n,p']	0.1629	0.2460	0.2860		N (d1)	0.1488	0.2437	0.2843
iopt	-1		Φ [a:n,p]	0.1970	0.3191	0.3968		N (d2)	0.1809	0.3164	0.3947
Steps in tree (n)	200		CRR Put Value,	1.06	3.51	6.37		Black-Scholes Put Value	1.06	3.51	6.36

Table 27: Put option values, computed with Black-Scholes and CRR models.

Volatility	Put Value BSM	Put Value CRR
15.00%	1.06	1.06
25.00%	3.51	3.51
35.00%	6.36	6.37

Table 28: Tabulated Put option values. Black-Scholes and CRR models.



Figure 28: Put option values. Black-Scholes-Merton vs CRR

7.3.3 LOGNORMAL BLACK-SCHOLES MODEL

Following from our discussions in previous paragraphs, we now emphasize an alternative form of the Black-Scholes formula expressed in terms of the mean and variance of the normal distribution for log share returns. The Black-Scholes can also be expressed in terms of the first two moments of the lognormal distribution for share prices.

Asset	Returns	Log returns			
Distribution	Lognormal	Normal			
		T (offining			
First Moment	M ₁	М			
Second Moment	M ₂	V			
First moment link	$M_1 = exp(M+0.5V)$	$M=2ln(M_1)-0.5ln(M_2)$			
Second moment link	$M_2 = exp(2M+2V)$	$V = -2\ln(M_1) + \ln(M_2)$			

Table 29: The formulae for the moments.

We may characterize a distribution through its moments about the mean (as for the normal distribution) or equivalently through its moments about zero (as for the log normal distribution). The first moment of any distribution is its mean (denoted M_1 or M), while the second moment (about the mean) is the variance, V as opposed to the second moment about zero. This equivalence allows us to translate parameters between the normal and lognormal distributions.

Let us assume that the log returns have a normal distribution with mean M and variance V. Then we can find the moments about zero $(M_1 \text{ and } M_2)$ of the corresponding lognormal

distribution for returns using the links given in the table, We can also reverse this process. By stressing the normal distribution of log share prices, mean and variance can be calculated and the Black-Scholes formula reworked from these moments.

With out further mathematical detial, the pricing formula (referred to as the lognormal version of Black-Scholes) can also be expressed compactly in terms of the moments of the lognormal distribution attributed to share prices.

$$c = e^{-rT} \cdot \left(M \cdot N(d_1) - X \cdot N(d_2) \right)$$
(196)

Where d_1 and d_2 are expressed in terms of M_1 and M_2 .

$$d_1 = \frac{M - \ln(X) + V}{\sqrt{V}}$$
 (197)
 $d_2 = \frac{M - \ln(X)}{\sqrt{V}} = d_1 - \sqrt{V}$ (198)

The moments for the lognormal distribution $(M_1 \text{ and } M_2)$ can be calculated from the moments of the normal distribution M and V as follows:

$$M_1 = e^{M + \frac{1}{2}V}$$
 (199)
 $M_1 = e^{2M + 2V}$ (200)

The first two moments (mean M and variance V) of the normal distribution attributed to the log share price can be easily calculated. The put option equivalent can be obtained through the application of the call-put parity condition.

7.3.4 THE RECIPROCAL-GAMMA MODEL 284

We may also use moments M_1 and M_2 as the starting point for an alternative option pricing approach, one that replaces the lognormal assumption with the alternative assumption that the share prices follow the reciprocal gamm⁸² (RG) distribution. Milevsky and Posner (1998) suggested this distribution as appropriate for the valuation of "so-called" basket options. It can also be used also for vanilla options with skewed distributions.

The reciprocal gamma version of the pricing formula replaces the normal N(d) with the reciprocal distribution function. The gamma distribution depends on the reciprocal of the exercise (1/X) and two parameters; alpha and beta.

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2} \qquad (201)$$

$$\beta = \frac{M_2 - M_1^2}{M_2 M_1} \tag{202}$$

The latter two inputs are calculated from the moments M_1 and M_2 . Without further mathematical detail, the RG pricing formula for a call option can be expressed as follows:

$$\mathbf{c} = \mathbf{e}^{-\mathbf{r}\mathbf{T}} \cdot (\mathbf{M}_1 \cdot \mathbf{g}_1 - \mathbf{X} \cdot \mathbf{g}_2) \tag{203}$$

Where g_1 and g_2 replace the normal distribution functions $N(d_1)$ and $N(d_2)$. An expression for the put option can be obtained through the application of the call-put parity condition (McDonald, 2006, Hull, 2014). We compute g_1 and g_2 values with a modulated version

⁸² Not to be confused with the gamma function used in the modulated GSE expression.

of Excel's GAMMADIST (Jackson and Staunton, 2001; Sengupta, 2009; Rees, 2017; Häcker and Ernst, 2017).

Similar to previous cases, when computing option price with the lognormal Black-Scholes and the RG models, all input parameters are known, apart from the volatility of the share returns over the life of option. For a chosen level of volatility, we use the formulae to generate the option values. In order to simply the computation process, we combine the call and put expression in one unified formula.

Lognormal Black-Scholes:

$$opt_value = iopt \cdot e^{-rT} \cdot [M \cdot N(iopt \cdot d_1) - X \cdot e^{-rT} \cdot N(iopt \cdot d_2)]$$
(204)

The Reciprocal- Gamma:

$$opt_value = iopt \cdot e^{-rT} \cdot [M \cdot g_1 - X \cdot g_2]$$
(205)

We use table (17) inputs with the RG and lognormal BS modes and obtain the call option prices.

Share price (S)	100.34		М	4.6183	4.6033	4.5808	M1	102.17605	102.17605	102.17605
Exercise price (X)	90.00		V	0.0169	0.0469	0.0919	M2	10617.61468	10940.96919	11444.55859
Int rate-cont (r)	2.42%									
			d ₁	1.0417	0.6943	0.5702	alpha	60.76067	22.83724	12.39201
Dividend yield (q)	0.00%		d ₂	0.9118	0.4778	0.2671	beta	0.00016	0.00045	0.00086
Time now (0, years)	0.00									
Time maturity (T, years)	0.75		$N(d_1)$	0.8512	0.7563	0.7157	g1	0.8519	0.7506	0.7038
Option life (T, years)	0.75		$N(d_2)$	0.8191	0.6836	0.6053	g2	0.8207	0.6798	0.5972
Volatility (s)	15.00%	25.0%	35.0%							
			LN value:	13.02	15.46	18.32	RG value:	12.9436	15.2318	17.8397
iopt	1									
option	call									

Table 30: Call option values, computed with Lognormal Black-Scholes and RG models.

The two sets of call option prices are very close. The lognormal Black-Scholes produces the same results as previously computed. The RG model produces only very slightly lower call option values from the Black-Scholes equivalent.



Figure 28: Call option values. Lognormal Black-Scholes-Merton vs RG models.



Figure 29: Put option values. Lognormal Black-Scholes-Merton vs RG models.
7.4 EMPIRICAL TESTING AND ANALYSIS WITH HISTORICAL VOLATILITY-CLASSIC vs QUANTUM SW

In this section, I concentrate on numerical analysis of combined quantum and classical option pricing. It provides an application of Quantum-SW, BSM, CRR, LN-BSM, and RG models, explored in previous sections. With the data in table (17) and volatilities of 15%, 25%, and 35%, the following call and put option prices are obtained.

	OPTION VAL	UATION		
	Share price (S)		100.34	
	Exercise price (X)		90.00	
	Int rate-cont (r)		2.42%	2.42% Quantum implied interest rate. Can be repalced by know interest rate in the C world.
	Dividend yield (q)		0.00%	
	Time now (0, years)		0.0000	
Т	ime maturity (T, years)		0.7500	
	Option life (T, years)		0.7500	
	15.00/	25.004	25.00/	
volatility (s)	15.0%	25.0%	35.0%	
Call Option	European, σ = 15%	European, $\sigma = 25\%$	European, σ = 35%	
via Qunatum Model SW	£12.75	£14.56	£15.69	
BS Value (Normal)	£13.02	£15.46	£18.32	BS is known to overestimate option vlaue for longer term con
BS Value (Log Normal)	£13.02	£15.46	£18.32	
RG Value	£12.94	£15.23	£17.84	
	612.02	C15 47	C19 22	

Table 31: Call option values, computed with Quantum-SW, BSM, CRR, LN-BSM, and RG models.

We note from table (31) that the Quantum-SW model produces comparative option values with existing option pricing classical models. These are consistent across the range of volatilities (15%, 25%, and 35%). Black-Scholes-Merton model is known to over-estimate the option value for longer term contracts and larger volatilities. The Quantum-SW produces slightly more conservative values for the call option.

	OPTION VAL	UATION		
	Share price (S)		100.34	
	Exercise price (X)		90.00	
	Int rate-cont (r)		2.42%	2.42% Quantum implied interest rate. Can be repaleed by known interest rate in the C world.
	Dividend yield (q)		0.00%	
	Time now (0, years)		0.0000	
Т	ime maturity (T, years)		0.7500	
	Option life (T, years)		0.7500	
Volatility (s)	15.0%	25.0%	35.0%	
But Ontion	European $\sigma = 1504$	European (7 - 250/	European $\sigma = 350$	
via Ounatum Model SW	f0 80	£010pean, 0 = 23%	£3 73	
BS Value (Normal)	£1.06	£3.51	£6.36	BS is known to overestimate option vlaue for longer term contracts
BS Value (Log Normal)	£1.06	£3.51	£6.36	ı ü
RG Value	£0.99	£3.27	£5.88	
CRR Call Value	£1.06	£3.51	£6.37	

 Table 32:
 Put option values, computed with Quantum-SW, BSM, CRR, LN-BSM, and RG models.

Similar to the set of values the previous table (31), in table (32) we note again slightly more conservative put values produced by our Quantum-SW model in comparison to existing classical option pricing models. The RG values are also slightly more conservative put values compared to the BSM model. This, perhaps, is understandable given that the underlying's probability distribution may be skewed from expected normality.

Input or select an underlying, 1 <go> to Add Options, <back> to Return</back></go>													
UKX Index	Expiry	Filter •		Option Sc	enario	o Analys	is: Add	Listed					
Ticker	Cost	Tota	l Cost	MktVal	Delta	DeltaNotio	onal	Gamma	Theta	Vega 4			
Totals						25.98k	25.38k	-2.49	-1	7.32k	22.33	-593.40	1.45k
UKX 12/21/18	C6900		1.00	179.00	0	1.79k	1.70k	6.30	44	1.00k	.88	-27.83	62.59
UKX 12/21/18	C6925		1.00	153.50	D	1.54k	1.51k	5.98	41	1.77k	.93	-27.66	64.11
UKX 12/21/18	C6950		1.00	137.50	0	1.38k	1.35k	5.64	39	9.44k	.96	-27.75	65.24
UKX 12/21/18	C6958		1.00	132.50	0	1.33k	1.30k	5.54	38	3.68k	.97	-27.70	65.50
UKX 12/21/18	C6975		1.00	122.50	0	1.23k	1.20k	5.29	36	5 . 98k	.99	-27.61	65.92
UKX 12/21/18	C7000		1.00	107.50	0	1.08k	1.07k	4.96	34	1.69k	1.01	-27.23	66.12
FTSE 100 INDEX	(6988	3.65715	5	N.A./N	.A.	Hi 70⁴	42.07 L	.ow 69	79.85	Vol 762.3	BMLN
Center	7000.00	Numbe	r of Stri	kes 11	•	or	% fr	om Cent	ter		Expiry	21 DEC 20)18 ·
		Calls								Puts			•
Ticker	Pos	Cost	Bid	Ask	Last	Strike	E Ti	cker	Pos	Cost	Bid	Ask	Last
UKX 21 DEC 201	18 (Contrac	t Size 1	0)			11	• UKX 21	DEC 20	18 (Contrac	ct Size	e 10)		
12/18 C6900.00	1.00	179.0	161.0	179.0		69	00 12/18	P6900.0	0 1.00	78.	5 76.0	78.5	80.0
12/18 C6925.00	1.00	153.5	149.5	153.5		693	25 12/18	P6925.0	0 1.00	87.	.0 84.0	87.0	
12/18 C6950.00	1.00	137.5	133.5	137.5		69	50 12/18	P6950.0	0 1.00	96.	.0 93.0	96.0	90.5
12/18 C6958.00	1.00	132.5	128.5	132.5		69	58 12/18	P6958.0	0 1.00	99.	.0 96.0	99.0	98.5
12/18 C6975.00	1.00	122.5	118.5	122.5		69	75 12/18	P6975.0	0 1.00	107.	.0 103.0	107.0	105.0
12/18 C7000.00	1.00	107.5	104.0	107.5		70	00 12/18	P7000.0	0 1.00	117.	.0 113.5	117.0	
12/18 C7025.00	1.00	94.5	91.5	94.5		70	25 12/18	P7025.0	0 1.00	129.	.0 125.0	129.0	
12/18 C7050.00	1.00	82.0	79.0	82.0		70	50 12/18	P7050.0	0 1.00	141.	5 137.5	141.5	135.5
12/18 C7075.00	1.00	70.5	68.0	70.5		70	75 12/18	P7075.0	0 1.00	156.	.5 151.5	156.5	
12/18 C7100.00	1.00	60.5	57.5	60.5	64.0	71	00 12/18	P7100.0	0 1.00	175.	5 157.5	175.5	
12/18 C7125.00	1.00	51.0	49.0	51.0		71	25 12/18	P7125.0	0 1.00	191.	5 172.5	191.5	
UKX 18 JAN 201	L9 (Contrac	t Size 1	0)			11	* UKX 18	3 JAN 20	19 (Contrac	ct Size	e 10)		
01/19 C6875.00			223.0	228.0		68	75 01/19	P6875.0	0		112.5	115.0	
Aus	tralia 61 2 :	9777 8600	Brazil 55	11 2395 90	00 Euro	pe 44 20	7330 7500 0	іег т алу 49	69 9204 1210	Нопу К	(опу 852 297	7 6000	
Jap	an 61 3 3201	8900	Singapor	e 65 6212	1000	0.5.1	212 518 20	SN 58	COPYRight 201 0660 EST GMT	ю втооп -5:00 Н	юегу Finaлd 202—5053—2	е с.г. 30-Nov-2018	3 06:25:42

Table 33: Bloomberg listed financial call and put options on FTSE100 (UKX Ticker) underlying.

Data in table (33) are extracted from Bloombeg. These are UKX options (calls and puts) with "bid" and "ask" price quotations. Here the bid-price is the latest price level at which a market participant wishes to buy a particular option. For example for a 12/18 C6900 call option, if trader enters a "market order" to sell the December 18, 2018, 6900-Strike call, then trader would sell it at the bid-price of £161.00. Similarly, the ask-price is the latest price put forward by a market participant in order to sell the option. For example if a trader enters a "market order" to buy the December 18, 2018, 6900-Strike call, then trader would buy it at the ask-price of £179.00 (Dubofsky ad Miller, 2002; McDonald, 2006; Sundaram, 2011; Hull, 2014).

Market-makers trade on the spread between the bid and ask prices i.e. buy at the bid price and sell at ask price. Typically, more active options, have a smaller bid/ask spread. A greater bid/ask differential implies issues with liquidity and it can turn out to be problematic for any trader, especially for intraday trading or any short-term trading activity. In the case of the 12/18 C6900 Call, the bid is £161.00 and the ask is £179.00. Which means that if trader buys the option at one instance at £179.00 ask-price and sell it an instant later at the £161.00 bid-price, trader would incur a loss of 10% on the trade i.e. computed using (bid – ask)/ask. (Dubofsky and Miller, 2002; McDonald, 2006; Sundaram, 2011; Hull, 2014).

11) View 👕 12) Actions 🐃 13)	Settings *) Tra	ade Simu	lati			Portfolio) & Risk Ar	halytics
Intraday Holdings Characteristics	Tracking Error/	/olatility	VaR So	cenarios	Performan	e Attri	bution	
Main View Total Return Period Analysis	Seasonal Analys	is Statist	ical Summai	ry				
UKXNEWPORT 💽 vs FTSE 100 I	N 💌 by GICS	Sectors 💌	in USD			A	s of <mark>11/2</mark>	9/18 🛱
Unit Percentage								
UKXNEWPORT								
	3 Monti	ns	6 Mo	nths	Year To	Date	1 Year	•
Portfolio Statistics	Port	Bench	Port	Bench	Port	Bench	Port	Bench
2. Return								
Total Return	-7.70	-7.65	-9.33	-9,15	-9.98	-9.77	-5.46	-5.24
Maximum Return	1.68	1.68	1.77	1.77	1.81	1.81	1.81	1.81
Minimum Return	-1.98	-1.98	-2.28	-2.18	-3.41	-3.41	-3.41	-3.41
Mean Return (Annualized)	-34.98	-34.81	-22.73	-22.33	-13.74	-13.43	-6.42	-6.12
Mean Excess Return (Annualized)	-0.27		-0.52		-0.35		-0.32	
3. Risk								
Standard Deviation (Annualized)	13.42	13.40	13.58	13.54	13.51	13.49	13.20	13.19
Downside Risk (Annualized)	9.23	9.22	10.06	10.03	10.23	10.21	10.00	9.98
Skewness	0.14	0.14	-0.32	-0.31	-0.56	-0.56	-0.58	-0.58
VaR 95% (ex-post)	-1.24	-1.24	-1.67	-1.67	-1.49	-1.49	-1.43	-1.43
Tracking Error (Annualized)	0.09		0.16		0.12		0.12	
4. Risk/Return								
Sharpe Ratio	-2.14	-2.13	-1.40	-1.38	-0.88	-0.87	-0.49	-0.48
Jensen Alpha	-0.12		-0.26		-0.21		-0.21	
Information Ratio	-2.03		-2.38		-2.03		-1.94	
Treynor Measure	-0.29		-0.19		-0.12		-0.06	
Beta (ex-post)	1.00		1.00		1.00		1.00	
Correlation	1.0000		0.9999		1.0000		1.0000	
Lapture Ratio	1,00	ona 44 70 72	1.00 30 7500 Ceem		1,00 204 1710 Hone	Kono 852 2	1,00	
Јарал 81 3 3201 8900 Біладіі	ore 65 6212 1000	U.S. <u>1 2</u>	12 318 20 <u>00</u>	Copyri	ight 2018 <u>Blo</u>	omberg F <u>ina</u>	nce L.P.	
				SN 580660 E	ST GMT-5:00	H202-5053-	5 30 -N ov-2018	3 07:30:47

Table 34: FTSE100 (UKX Ticker) performance data (source: Bloomberg).

The data in table (34) are performance measures of the FTSE100. To obtain them I use the PRTU and PORT capabilities in Bloomberg. The FTSE100 is itself a portfolio, therefore loading it on Bloomberg's PORT capabilities does work. I have made sure to include a reasonable time duration (18 years in this case). I have back tested it through the past 18 years in order to obtain the "precipitated" performance values.

Normalisation Factors										
1.985492816	50 to 100 basis point									
2.074251109	0 to 49 basis point									
0.030728943	-49 to -1 0 basis point									
0.040453292	-99 to-50 basis points									
0.055579305	less then 100 basis point									

Table 35a: Normalisation factors used to calibrate quantum option values.

						-				
Underlying	UKX Index (FTSE100)	JKX Index (FTSE100	UKX Index (FTSE100)	UKX Index (FTSE100)		Input or s	elect a	n unde	erlyi	ing
Share price (S)	6988.65	6988.65	6988.65	6988.65		UKX Index Ticker		• 1 Posi) Add (Opt C
Exercise price (X)	6900.00	6950.00	7000.00	7050.00	Source: Bloomberg	Totals		- Vol		
Int rate-cont (r)	2 42%	2 42%	2 42%	2 42%	Implied	UKX 12/21/18	6900		1.00	_
Int fute cont (1)	2.1270	2.1270	2.1270	2.1270	Implied	UKX 12/21/18	.0925 *6050		1.00	-
						UKX 12/21/18	6958		1.00	-
Dividend vield (a)	0.00%	0.00%	0.00%	0.00%		UKX 12/21/18	6975		1.00	
						UKX 12/21/18	7000		1.00	
						FTSE 100 INDEX	7000.00		6988	.6
Option life (T, years)	0.0627	0.0627	0.0627	0.0627	Source: Bloomberg	Center	7000.00	Number	or Stri	IKE
						Ticker	Pos	Cost	Bid	
Volatility (s)	13.20%	13.20%	13.20%	13,20%	Source: Bloomberg	UKX 21 DEC 201	8 (Contract	Size 10)	
		Back testing '11/	29/00 to 11/30/2018		5	12/18 06900.00	1.00	153.5	149.5	
		Duck testing 11/	29/00 10 11/90/2010			12/18 C6950.00	1.00	137.5	133.5	
Call Ontion Tisken	UEV 12/21/18 CK000	UEV 12/21/18 C/6050	UEV 12/21/18 C7000	UKY 12/21/19 C7500		12/18 C6958.00	1.00	132.5	128.5	
	UKA 12/21/18 C0900	UKA 12/21/18 C0950	UKA 12/21/18 C /000	UKA 12/21/18 C /500		12/18 C6975.00	1.00	122.5	118.5	
Bloomberg =====>	£161.00	£133.50	£104.00	£/9.00	-	12/18 C7000.00	1.00	04.5	01.5	
via Qunatum Model SW	£160.98	£133.77	£104.06	£78.92	-	12/18 C7050.00	1.00	82.0	79.0	
BS Value (Normal)	£160.99	£130.81	£104.39	£81.74	-	12/18 C7075.00	1.00	70.5	68.0	
BS Value (Log Normal)	£160.99	£130.73	£104.39	£81.74		12/18 C7100.00	1.00	60.5	57.5	
RG Value	£160.72	£130.65	£104.35	£81.82		12/18 C/125.00	1.00	51.0 Sizo 10	49.0	
CRR Call Value	£161.09	£130.81	£104.48	£81.84		01/10 06975 00	Contract	5126 10	222.0	

Table 35b:Call option values computed with (i) Bloomberg, (ii) Quantum-SW, (iii) BSM, (iv) CRR,(v) LN-BSM, and (vi) RG.

I have explained throughout this research work that e pricing problems are tracked using partial differential formulations. The algorithms were written to solve the problem at the PDE level. This is justified because market prices evolve continuously with instantaneous and infinitesimally small changes. I have used the FTSE100 volatility of $\sigma = 13.20\%$ which is obtained from table (34) i.e. it is the value for the annualised standard deviation in the table. The results clearly show that the Quantum-SW produces values closer to those put forward from maket-makers (ref. Bloomberg) and very close to the Black-Scholes-Merton option values. The arbitrage-neutral put option values can easily be computed using the call-put parity condition as explained in previous sections of this chapter. For the classical models, the procedure is also explained in detail in McDonald (2006), Sundaram (2011), Hull (2014).

7.5 EMPIRICAL TESTING AND ANALYSIS WITH LOCAL VOLATILITY AND PRICE-SURFACE QUANTUM SW FITTING

In this section, I expand empirical testing and analysis by including a sample of 30 stock, carefully selected across countries, sectors, CAP size, and risk-return trade-offs. The sample includes several equity indexes.

7.5.1 REAL-TIME OPTION PRICING, TESTING, AND ANALYSIS

In the previous section, several classical option pricing models were used, with results compared with those of the quantum SW model. By including a quantum option pricing

model, the option sensitivities expand beyond (s, k, r, q, t, σ) to include attributes of the quantum well itself, such as radius of the dot, length (depth) of the well, and the price cut-off potential function. The latter represents the well's "identify". The previous consideration included a homoscedastic behaviour of volatility (constant σ), although quantum square-well results were calibrated. In this section the volatility is presumed to be stochastic s (t, s), subsequently heteroscedastic. The data analysis here include statistical hypothesis.

The statistical sample includes equity options and stock index options. They have much in common, but generally differ most in that a stock index will pay a dividend stream that tends to resemble a continuous payment stream, while individual equities pay a dividend stream that is quite obviously not continuous. While this distinction is to some extent arbitrary, because an index pays discrete dividends corresponding to those of its components, the distinction nevertheless may make some option pricing methods impractical for one type or the other (Criss, 1996; Brigo et al., 2003; Hull, 2014).

The empirical testing includes statistical hypothesis testing, I have made use of a simple random routine in Python programming to select the sample out of the larger population of stock listed in Bloomberg (SECF command in Bloomberg). Once the trading tickers have been identified the market price of calls and put options have been obtained. Due to the random selection of the tickers and option maturities, the sample of 30 stock turned out to be diverse enough in terms of attributes such as volatility, dividend, geography, sector, etc.

					Historic				Last	Price
<i>1</i> 40.	Ticket	Price sport	Exercise	Time Foiry Days	Volatility	Forward	Rate Instell	Dividend tield	Call water	Put Intaket
1	IBM US	132.34	132.00	27	22.34%	132.553	2.525%	0.000%	3.20	2.58
2	AAPL US	178.97	180.00	27	31.61%	179.3481	2.525%	0.000%	5.45	6.21
3	UBER US	41.51	41.50	27	53.15%	41.5417	2.525%	0.000%	2.20	2.70
4	AMZN US	1823.28	1822.50	27	23.78%	1,826.09	2.525%	0.000%	52.60	49.25
5	MSFT US	126.24	126.00	27	22.25%	126.4181	2.525%	0.000%	3.16	2.75
6	TSLA US	190.63	190.00	27	48.20%	191.017	2.525%	0.000%	14.90	14.25
7	FB US	181.06	182.50	27	27.77%	181.341	2.525%	0.000%	4.50	5.59
8	VOD LN	126.10	125.00	27	25.34%	122.497	0.871%	20.300%	3.00	4.50
9	GOOGL US	1138.61	1137.50	27	32.18%	1135.635	2.525%	0.000%	34.90	22.46
10	INTC US	44.57	44.50	27	39.35%	44.6448	2.525%	0.000%	1.42	1.23
11	BHP AT	37.45	37.38	32	16.73%	37.4854	1.430%	0.000%	1.07	0.95
12	7203 JP	6502.00	6500.00	45	17.61%	6,502.50	0.002%	0.000%	178.50	176.00
13	SBRY LN	195.30	195.00	82	45.30%	187.869	0.884%	19.219%	8.50	14.25
14	UKX INDEX	7277.73	7275.00	82	10.87%	7,194.84	0.884%	6.159%	132.50	204.00
15	SPX INDEX	2826.06	2825.00	82	11.14%	2,829.76	2.525%	2.204%	85.25	76.00
16	SX5E INDEX	3350.70	3350.00	54	13.28%	3,336.17	-0.221%	2.759%	67.00	86.30
17	TPX INDEX	1541.21	1550.00	47	13.95%	1538.88	0.003%	1.203%	30.00	40.50
18	DAX INDEX	12011.04	12000.00	54	14.08%	12006.86	-0.222%	0.000%	289.10	295.20
19	NDX INDEX	7300.96	7300.00	54	15.48%	7318.08	2.525%	0.859%	245.95	212.00
20	NKY INDEX	21117.22	21125.00	47	14.13%	21085.26	0.002%	4.311%	455.00	550.00
21	HSCEI INDEX	10445.54	10400.00	64	17.34%	10232.03	2.074%	4.311%	246.00	414.00
22	DIS US	132.79	130.00	54	29.89%	132.4087	2.525%	4.669%	5.60	2.98
23	BATS LN	2984.00	3000.00	82	27.09%	2939.777	0.882%	4.311%	132.50	191.50
24	BT/A LN	196.26	200.00	26	20.03%	196.365	0.871%	0.000%	3.75	7.25
25	BABA US	155.00	155.00	82	31.22%	155.8836	2.525%	0.000%	10.70	9.63
26	BA US	354.90	355.00	82	31.24%	355.051	2.525%	4.311%	20.14	19.11
27	NVDA US	145.15	145.00	54	36.13%	145.4978	2.525%	0.775%	9.00	8.45
28	WMT US	102.67	105.00	82	15.44%	102.7133	2.525%	4.311%	2.65	4.74
29	005930 KS	42700.00	43000.00	46	24.89%	42442	1.795%	4.311%	1100.00	1660.00
30	QCOM	66.21	67.50	82	54.44%	65.9536	2.525%	4.311%	4.05	5.60

Table 36: Sample data of stock and stock index by trading ticker, market stock price, exercise, maturity, volatility, forward price, market rate, dividend yield, market call and put option price. Extracted from Bloomberg on 25/05/2019.

Although the sample incudes 30 trading tickers only, considering that stock indexes are equity portfolios, all together the sample runs in hundreds of stock, therefore the sample

itself represents reasonably the equity market on a global scale.

The approach here takes notice of the potential relationship between the stock price and and its own contigent claim. Hull (2014) provides some theoretical treqtment of correlation between the spot of the market index and the price of index futures that can be modelled by

$$S_t = \alpha + \beta F_t + \varepsilon_t \qquad (206)$$

where,

St	is the spot price at time t
Ft	is future's price at the time t
p	is the sensitivity coefficient
р	(the beta), thought of as a hedge ratio.
εt	is residual error at time t

This relationship can also be stiupulated with a time delay or lead. There is also validity of indexing when the derivative is an option, instead.



Figure 31: Sample exmaples of pot price vs call (put) value, extracted from Bloomberg on 25/05/2019.

However, the statistical hypothesis would need the samples to contain variables that are independent and indentically distributed (iid). I test for correlation of spot price with option value (or vice versa), so that I can theorise within the sample "space". I then also perform data filtering and calibration, within a short time-frame, prior to the final leg of the process - the hypothesis testing. Relevant scales such as the ratio of spot price with

the first sample call or put value are noted across the sample, thus making the first sample pairs (stock/call, stock/put) the reference pair. I observe a high correctation between spot and option prices across trading tickers in the sample. Hence a form of indexing can be used during the processes of data calibration and option value forecasting.



Figure 32: Graphs based on the sample data (30 stock). Clockwise listing, (i) spot price vs market call value (scaled), (ii) spot price vs market put value (scaled), (iii) market call value vs spot, and (iv) market put value vs spot.

7.5.2 REVISED OPTION PRICING MODELS

In this section, I revisit several equity option pricing methods as best fits. These are categorised by considering the choices I have made, such as: (i) assumptions on the stock

process; (ii) assumptions on payment of future dividends; and (iii) having made choices for the first two, the choice of the numerical solution method most appropriate to use. I have dropped the RG model for further consideration as it would add no additional value on European vanilla option valuation. This is in part to keep the testing around three broad categories: (i) option pricing with the Black-Scholes and its variations (incl. discretised BS and local volatility), (ii) option pricing with the trinomial (an improved binomial), and (iii) option pricing with the quantum square-well model (Black and Scholes, 1973; Lee, 2004; Gatheral, 2006; Rubinstein, 2000; Haven, 2002; Hundsdorfer and Verwer, 2003; Jarrow, 2006).

Model	Stock Price	Dividends	Solution Method
Trinomial	BS	Discrete Type 1	Trinomial Tree
Black-Scholes Continuous	BS	Continuous	Formula
Black-Scholes Discrete	BS	Discrete Type 2	PDE
Local Volatility	LV	Discrete Type 2	PDE
Local Price Quantum SW	LV	Dsicrete Type 2	QSW PDE

Table 37: Summary of best-fit option pricing models. Trinomial, Black-Scholes (continuous, discretised, local volatility), and quantum square well.

The stock process BS refers to the stock process in the standard Black-Scholes model, with time-varying parameters(Black and Scholes, 1973); and LV refers to the stock process in the local volatility model (Dupire, 1994:1997), essentially, the Black-Scholes model extended by making the volatility a function of both time and stock price i.e. $\sigma(t, s)$. The discrete dividend treatments are of two types: in "Discrete Type 1", the present value of dividends is subtracted from the initial stock price and the remaining portion is

viewed as the uncertain part, subject to diffusion without subsequent jumps; in "Discrete Type 2", the entire initial stock price is subject to diffusion, and jumps are introduced at each discrete dividend date (Black and Scholes, 1973; Lee, 2004; Gatheral, 2006; Orosi, 2010).

The solution methods used are of three types: analytic solutions, or formulas; partial differential equations (PDEs) solved numerically using a discretization over a grid; and trinomial trees, which can be viewed as a specialisation of the PDE solution method (Black and Scholes, 1973; Dupire, 1994:1997; Derman et al., 1996).

The following notation is used:

St	Stock price at time t
r	The interest rate, possibly time-varying
q	The continuous dividend rate, possibly time-varying
σ	The volatility, possibly time-varying
Di, ti	The discrete dividend amount D_i , going ex-dividend
	at time $t_i dW_t$

Table 38: Relevant notation.

The stock process model, describes the assumed possible paths of future stock prices and the probabilities of those paths (Karatzas and Shreve, 1998b; Barberis and Huang, 2008). The most common such model for equities is Black-Scholes with continuous proportional dividends (initially discussed in section 7.3), where the stock process under the risk-neutral measure is given by the stochastic differential equation (129), reproduced here:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \qquad (207)$$

This model for stock price movement assumes stock price paths are continuous, changes in log-price over any time interval are normally distributed, and the changes in price over one or more disjoint intervals are independent. A further refinement, adding absolute discrete dividends, adds a downward jump in the stock price at known dividend times t_i to the above process (Black and Scholes, 1973; Karatzas and Shreve, 1998b; Revus and Yor, 2004; Hull, 2014):

$$S_{t_i}^+ = S_{t_i}^- - D_i$$
 (208)

A second choice is the local volatility model. This is a generalization of Black- Scholes, where the volatility is assumed to be a deterministic function of both time and future stock price, thus heteroscedastic (Dupire, 1994:1997; Lee, 2004; Gatheral, 2006; Orosi, 2010). Under this model, the process followed by the stock price is given by the slightly augmented equation (129), reproduced below

$$dS_t = (r - q)S_t dt + \sigma(t, S_t)S_t dW_t$$
(209)

where $\sigma(t, S_t)$ is called the "local volatility". The advantage of using this more general functional form for σ is that the model can now be calibrated to match market option prices at multiple strikes at a single expiration (Brigo and Mercurio, 2002; Gatheral, 2006).

The discrete Black-Scholes (Black and Scholes, 1973), is the option pricing PDE, previously discussed (eq. 84), and reproduced here:

$$\frac{\partial \psi(s,t)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \psi(s,t)}{\partial s^2} + rs \frac{\partial \psi(s,t)}{\partial s} - r\psi(s,t) = 0 \qquad (210)$$

I have also shifted focus here from the binomial model to the trinomial option pricing upgrade. Trinomial trees are equivalent to explicit finite difference methods (FDMs) if spatial boundary conditions are applied and the full lattice is populated. They truly appear to be a hybrid of the binomial and the finite difference methods. The previously discussed binomial trees (section 7.3) are regarded in this section as a special case of trinomial trees with the middle probability set to zero. (Brennan and Schwartz, 1978; Chriss, 1996; Derman et al., 1996; Heston and Zhou, 2000; Rubinstein, 2000; Chan et al., 2009; O'Sullivan and O'Sullivan, 2013; Hull, 2014).



Figure 33: Illustration of the trinomial treet.

Hull (2014), and Derman et al.,(1996), among others, provide the price development and trinomial probability expressions.

$$u = e^{\sigma\sqrt{3\Delta t}}, \quad d = \frac{1}{u}$$
(211)
$$p_{u} = \sqrt{\frac{\Delta t}{12\sigma^{2}}} \left(r - \frac{\sigma^{2}}{2}\right) + \frac{1}{6}, \quad p_{m} = \frac{2}{3}, \quad p_{d} = -\sqrt{\frac{\Delta t}{12\sigma^{2}}} \left(r - \frac{\sigma^{2}}{2}\right) + \frac{1}{6}$$
(212)

The variables used here are the same as those in section 7.3. Making r or q (denoted also as δ) a function of time does not affect the geometry of the tree. The probabilities on the tree become functions of time. The stock price can be considered to be a function of time by making the lengths of the time steps inversely proportional to the variance rate (Brennan and Schwartz, 1978; Rubinstein, 2000; Chan et al., 2009; O'Sullivan and O'Sullivan, 2013).

The option prices from the trinomial model as well as Black-Scholes (continuous and discretised), are as follows:

					Historic	Last Price and										
140.	Ticket	Price sport	Exercise	Time Expiry	Volatility	Forward	Rate Mater	Dividend vield	Call Marked	Put Ination	Call (BSO)	Put 1950	Callipsel	Putless	Call Innonial	PUT Internial
1	IBM US	132.34	132.00	27	22.34%	132.553	2.525%	0.000%	3.20	2.58	3.52	2.93	3.52	2.93	3.51	2.95
2	AAPL US	178.97	180.00	27	31.61%	179.3481	2.525%	0.000%	5.45	6.21	5.87	6.49	5.87	6.49	5.86	6.51
3	UBER US	41.51	41.50	27	53.15%	41.5417	2.525%	0.000%	2.20	2.70	2.42	2.37	2.42	2.37	2.42	2.38
4	AMZN US	1823.28	1822.50	27	23.78%	1,826.09	2.525%	0.000%	52.60	49.25	49.17	45.20	49.17	45.20	49.02	45.41
5	MSFT US	126.24	126.00	27	22.25%	126.4181	2.525%	0.000%	3.16	2.75	3.27	2.83	3.27	2.83	3.26	2.84
6	TSLA US	190.63	190.00	27	48.20%	191.017	2.525%	0.000%	14.90	14.25	10.48	9.43	10.48	9.43	10.47	9.46
7	FB US	181.06	182.50	27	27.77%	181.341	2.525%	0.000%	4.50	5.59	4.93	6.06	4.93	6.06	4.92	6.08
8	VOD LN	126.10	125.00	27	25.34%	122.497	0.871%	20.300%	3.00	4.50	2.18	4.97	2.18	4.97	2.29	4.80
9	GOOGL US	1138.61	1137.50	27	32.18%	1135.635	2.525%	0.000%	34.90	22.46	38.20	40.06	38.20	40.06	39.00	40.75
10	INTC US	44.57	44.50	27	39.35%	44.6448	2.525%	0.000%	1.42	1.23	1.99	1.83	1.99	1.83	1.99	1.84
11	BHP AT	37.45	37.38	32	16.73%	37.4854	1.430%	0.000%	1.07	0.95	0.74	0.70	0.74	0.70	0.74	0.70
12	7203 JP	6502.00	6500.00	45	17.61%	6,502.50	0.002%	0.000%	178.50	176.00	162.98	160.44	162.98	160.44	162.91	160.39
13	SBRY LN	195.30	195.00	82	45.30%	187.869	0.884%	19.219%	8.50	14.25	13.03	20.15	13.03	20.15	13.05	20.17
14	UKX INDEX	7277.73	7275.00	82	10.87%	7,194.84	0.884%	6.159%	132.50	204.00	111.89	191.83	111.89	191.83	111.87	192.16
15	SPX INDEX	2826.06	2825.00	82	11.14%	2,829.76	2.525%	2.204%	85.25	76.00	61.77	56.97	61.77	56.97	61.78	57.05
16	SX5E INDEX	3350.70	3350.00	54	13.28%	3,336.17	-0.221%	2.759%	67.00	86.30	59.87	77.18	59.87	77.18	59.96	77.14
17	TPX INDEX	1541.21	1550.00	47	13.95%	1538.88	0.003%	1.203%	30.00	40.50	25.57	36.86	25.57	36.86	25.65	36.78
18	DAX INDEX	12011.04	12000.00	54	14.08%	12006.86	-0.222%	0.000%	289.10	295.20	262.89	256.36	262.89	256.36	263.20	256.00
19	NDX INDEX	7300.96	7300.00	54	15.48%	7318.08	2.525%	0.859%	245.95	212.00	182.72	164.01	182.72	164.01	182.59	164.50
20	NKY INDEX	21117.22	21125.00	47	14.13%	21085.26	0.002%	4.311%	455.00	550.00	406.05	445.58	406.05	445.58	406.41	446.01
21	HSCEI INDEX	10445.54	10400.00	64	17.34%	10232.03	2.074%	4.311%	246.00	414.00	215.70	400.31	215.70	400.31	215.50	400.26
22	DIS US	132.79	130.00	54	29.89%	132.4087	2.525%	4.669%	5.60	2.98	7.29	4.88	7.29	4.88	7.29	4.89
23	BATS LN	2984.00	3000.00	82	27.09%	2939.777	0.882%	4.311%	132.50	191.50	123.21	184.64	123.21	184.64	123.76	183.92
24	BT/A LN	196.26	200.00	26	20.03%	196.365	0.871%	0.000%	3.75	7.25	2.66	6.29	2.66	6.29	2.66	6.30
25	BABA US	155.00	155.00	82	31.22%	155.8836	2.525%	0.000%	10.70	9.63	9.57	8.69	9.57	8.69	9.58	8.70
26	BA US	354.90	355.00	82	31.24%	355.051	2.525%	4.311%	20.14	19.11	20.88	20.82	20.88	20.82	20.87	20.81
27	NVDA US	145.15	145.00	54	36.13%	145.4978	2.525%	0.775%	9.00	8.45	8.28	7.77	8.28	7.77	8.28	7.78
28	WMT US	102.67	105.00	82	15.44%	102.7133	2.525%	4.311%	2.65	4.74	2.02	4.29	2.02	4.29	2.02	4.29
29	005930 KS	42700.00	43000.00	46	24.89%	42442	1.795%	4.311%	1100.00	1660.00	1243.82	1801.57	1243.82	1801.57	1244.42	1802.49
30	QCOM	66.21	67.50	82	54.44%	65.9536	2.525%	4.311%	4.05	5.60	6.08	7.62	6.08	7.62	6.09	7.62

Table 39: Sample data of stock and stock index by trading ticker, market stock price, exercise, maturity, volatility, forward price, market rate, dividend yield, market call value, market put value, BS continuous, BS discretised, and trinomial call and put option values.

The calcuclations of option price are carried out using the the Local Volatility model, first proposed by Dupire (1994).- Derman and Kani (1994) noted that there is a unique diffusion process consistent with the risk neutral densities derived from the market prices of European options. Derman and Kani (1994) described and implemented a local volatility function to model instantaneous volatility. They used this function at each node in a binomial options pricing model. The tree successfully produced option valuations consistent with all market prices across strikes and expirations. The Derman-Kani model was thus formulated with discrete time and stock-price steps. Derman and Kani produced what is called an "implied binomial tree"; with Chriss (1996) they extended this to an

implied trinomial tree (Derman et al., 1996).

The starting PDE in Dupire (1994) was an extension of the Black-Scholes model (Black and Scholes, 1973) in which volatility is a function of spot price and time:

$$\frac{\mathrm{d}S_{\mathrm{t}}}{\mathrm{S}_{\mathrm{t}}} = (\mathrm{r} - \mathrm{q})\mathrm{d}\mathrm{t} + \sigma(\mathrm{t}, \mathrm{S}_{\mathrm{t}})\mathrm{d}W_{\mathrm{t}} \quad (213)$$

This is the same as equation (129). The key continuous-time equation used in local volatility models states:

$$\frac{\partial c}{\partial t} = \frac{1}{2}\sigma^2(k, t, S_0)k^2\frac{\partial^2 c}{\partial k^2} - (r - q)k\frac{\partial c}{\partial k} - dc \qquad (214)$$

The parameters here are those discussed in previous sections, where c, option value; σ , the volatility; k, the strike price; s₀, the initial spot price; r, the interest rate; q, the dividend yield. The Local Volatility model has the great advantage of being consistent with market prices for all options on a given underlying, and it allows you to price exotic options in a way that is consistent with observed prices of vanilla options (Heston, 1997; Heston and Zhou, 2000; Christoffersen et al., 2009; Damghani and Kos, 2013). The workings are not reproduced here, but can be found in Dupire (1994).

The implied volatility surface relies on a heteroscadistic volatility and is a fundamental object for the pricing and risk management of derivatives. The construction of this surface from listed option prices typically proceeds in two stages. First, since forward prices are not directly quoted in the listed markets, a forward curve has to be implied in a manner consistent with the observed option prices. Second, given the forward, an implied

volatility model has to be calibrated to the observed option prices (Heston, 1997; Heston and Zhou, 2000; Gatheral, 2006; Christoffersen et al., 2009; Doughterly, 2011; Damghani and Kos, 2013.

The time and spot dependent volatilities used in the Local Volatility Model are called Local Volatilities. A Local Volatility surface is composed of forward instantaneous volatilities and can in theory be calculated from market prices of options on the selected underlying. However, extracting the Local Volatilities from prices is an unstable inverse problem and is usually avoided. This is why most market practitioners extract Local Volatility surfaces from Black-Scholes spot volatility surfaces using some form of stripping (Dupire, 1994:1997; Dumas et al., 1998; Berestycki, et al., 2002; Brigo and Mercurio, 2002; Gatheral, 2006).

The Black-Scholes surface needs to be smooth and free of arbitrages to be able to generate a positive and regular Local Volatility surface: building the Local Volatility surface is a good check to visualize potential issues (arbitrages) in a given Black-Scholes surface (Dupire, 1998; Brigo and Mercurio, 2002; Gatheral, 2006; Damghani and Kos, 2013).



Figure 34: Implied volatility surfaces (from the sample). Clockwise listing, (i) Walt. Dis. Co,, (ii) Uber Technlogie, (iii) British American Tobacco, and (iv) Euro Stoxx 50.



Figure 35: Implied volatility surfaces (from the sample). Clockwise listing, (i) Facebook.com Inc., (ii)Amazon.com Inc., (iii) NASDAQ 100 Stock Index, and (iv) Qualcomm Inc.

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					Historic				Last	Price	BVOL	Used
40.	ricker	Price sport	Exercise	Time to in Days	Volatility	Forward	Rate Markell	Dividend vield	Call Market	Put worked	Call loe voil	Put Hoe. Voli
1	IBM US	132.34	132.00	27	22.34%	132.553	2.525%	0.000%	3.20	2.58	3.11	2.53
2	AAPL US	178.97	180.00	27	31.61%	179.3481	2.525%	0.000%	5.45	6.21	5.44	6.05
3	UBER US	41.51	41.50	27	53.15%	41.5417	2.525%	0.000%	2.20	2.70	2.38	2.33
4	AMZN US	1823.28	1822.50	27	23.78%	1,826.09	2.525%	0.000%	52.60	49.25	52.44	48.47
5	MSFT US	126.24	126.00	27	22.25%	126.4181	2.525%	0.000%	3.16	2.75	3.16	2.72
6	TSLA US	190.63	190.00	27	48.20%	191.017	2.525%	0.000%	14.90	14.25	14.88	13.83
7	FB US	181.06	182.50	27	27.77%	181.341	2.525%	0.000%	4.50	5.59	4.42	5.54
8	VOD LN	126.10	125.00	27	25.34%	122.497	0.871%	20.300%	3.00	4.50	2.05	4.54
9	GOOGL US	1138.61	1137.50	27	32.18%	1135.635	2.525%	0.000%	34.90	22.46	25.29	26.78
10	INTC US	44.57	44.50	27	39.35%	44.6448	2.525%	0.000%	1.42	1.23	1.37	1.21
11	BHP AT	37.45	37.38	32	16.73%	37.4854	1.430%	0.000%	1.07	0.95	0.96	0.92
12	7203 JP	6502.00	6500.00	45	17.61%	6,502.50	0.002%	0.000%	178.50	176.00	169.48	166.94
13	SBRY LN	195.30	195.00	82	45.30%	187.869	0.884%	19.219%	8.50	14.25	8.52	15.63
14	UKX INDEX	7277.73	7275.00	82	10.87%	7,194.84	0.884%	6.159%	132.50	204.00	124.85	204.63
15	SPX INDEX	2826.06	2825.00	82	11.14%	2,829.76	2.525%	2.204%	85.25	76.00	82.60	77.75
16	SX5E INDEX	3350.70	3350.00	54	13.28%	3,336.17	-0.221%	2.759%	67.00	86.30	68.41	85.68
17	TPX INDEX	1541.21	1550.00	47	13.95%	1538.88	0.003%	1.203%	30.00	40.50	28.92	40.03
18	DAX INDEX	12011.04	12000.00	54	14.08%	12006.86	-0.222%	0.000%	289.10	295.20	295.84	289.30
19	NDX INDEX	7300.96	7300.00	54	15.48%	7318.08	2.525%	0.859%	245.95	212.00	236.08	217.73
20	NKY INDEX	21117.22	21125.00	47	14.13%	21085.26	0.002%	4.311%	455.00	550.00	483.07	522.81
21	HSCEI INDEX	10445.54	10400.00	64	17.34%	10232.03	2.074%	4.311%	246.00	414.00	234.47	420.26
22	DIS US	132.79	130.00	54	29.89%	132.4087	2.525%	4.669%	5.60	2.98	5.33	2.91
23	BATS LN	2984.00	3000.00	82	27.09%	2939.777	0.882%	4.311%	132.50	191.50	128.39	188.38
24	BT/A LN	196.26	200.00	26	20.03%	196.365	0.871%	0.000%	3.75	7.25	4.56	8.18
25	BABA US	155.00	155.00	82	31.22%	155.8836	2.525%	0.000%	10.70	9.63	10.38	9.49
26	BA US	354.90	355.00	82	31.24%	355.051	2.525%	4.311%	20.14	19.11	19.22	19.16
27	NVDA US	145.15	145.00	54	36.13%	145.4978	2.525%	0.775%	9.00	8.45	8.19	8.70
28	WMT US	102.67	105.00	82	15.44%	102.7133	2.525%	4.311%	2.65	4.74	2.46	4.75
29	005930 KS	42700.00	43000.00	46	24.89%	42442	1.795%	4.311%	1100.00	1660.00	1155.62	1710.63
30	QCOM	66.21	67.50	82	54.44%	65.9536	2.525%	4.311%	4.05	5.60	3.86	5.40

Table 40 Sample data from table (38) updated to include call and put option values, computed with the local volatility model.

The difference between the market values of calls and puts with the equivalents obtained through the local volatility model are the smallest when compared to the classical models, particularly with cases where a homoscedastic volatility measure is used (Dumas et al, 1998; Damghani and Kos, 2013).

The quantum model selected for additional testing is the constant square-well. This is the PDE with the constant price cut-off function (ref. chapter 5.0, table 2.0). Properties of

constant square-well quantum systems have been discussed throughout this study, more specifically in previous sections of this chapter. Suitability of the quantum square well system in pricing options has also been theorised in works of Haven (2002) and Callegaro (2013:2015:2017a:2017b). We have also previously established promising results when such system is used. In addition it requires modest data filtering and calibration.

It is important to note that existing option pricing models, such as the Black-Scholes or the Local Volatility do inform market making and are reflected well in markets. In the case of the quantum square-well, I have tried various different scenarios with variations of properties. I established that a quantum square-well system with γ (0) = 5q σ , L = 5q σ , and k = 9 (Eigen-states), leads to more reasonable results. Further testing could potentially be done with varied quantum well depths and radius, however this was sufficient to allow for engagment with data filtering and calibration. The theorisation is contained within the sample "space" and within one trading day, which is a unique approach because it allows for independence in the treatment of the statistical sample and the hypothesis testing (Berenson and Krehbiel, 1992; Lind, 2010; Doughterly, 2011).

k	LN(P)	(L) Diff LN(P)	(H) Diff LN(P)	[(L)+(H)]/2	P in qs
	γ(0)=5qσ	γ(0)=5qσ	γ(0)=5qØ	γ(0)=5qO	γ(0)=5qO
0	0.2692510	-0.6936170		-69.3617%	1.308983655
1	0.9628680	-0.0385520	0.693617	32.7533%	2.619197570
2	1.0014200	-0.0111100	0.038552	1.3721%	2.722144531
3	1.0125300	-0.0214000	0.011110	-0.5145%	2.752556180
4	1.0339300	-0.0310500	0.021400	-0.4825%	2.812095683
5	1.0649800	-0.0404100	0.031050	-0.4680%	2.900780968
6	1.1053900	-0.0496500	0.040410	-0.4620%	3.020402196
7	1.1550400	-0.0588100	0.049650	-0.4580%	3.174150381
8	1.2138500	-0.0679100	0.058810	-0.4550%	3.366420454
9	1.2817600		0.067910	6.7910%	3.602975385

Table 41a Eigen-values computes for Q-SW with $\gamma(0) = 5q\sigma$, L= 5q σ , k = 9 (ten Eigen-states)

The computation work is carried out inline with the theorisation of the previous section. The IBM quantum square-well stock price set is developed. Subsequently, call and put option values are calculcated to be 3.20, and 2.50, respectively. The same computation procedure is applied to each of stock in the statistical sample⁸³

⁸³ The Eigen-data and MS-Excel work with computations for each stock in the sample can be provided to anyone upon request.

		k	Probs/Weights	P(future point)	Payoff of Call
				γ(0)=5qs	x= 132
		0	0.0266558	£66.14	£0.00
		1	0.0953238	£183.63	£51.63
		2	0.0991405	£134.17	£2.17
		3	0.1002404	£131.66	£0.00
		4	0.1023590	£131.70	£0.00
		5	0.1054329	£131.72	£0.00
		6	0.1094335	£131.73	£0.00
		7	0.1143489	£131.74	£0.00
		8	0.1201710	£131.74	£0.00
		9	0.1268941	£141.64	£9.64
		Quantum Implied call value			£3.20
		Quantum Implied put value	Aribitra	ge	£1.60
		time			0.105882353
S	132.34				
X	132.00				
r	2.53%	Put value	Abitrage-	Free	£2.50
q	0				
Т	0.105882				
		σ			22.43%

Table 41b: Eigen-states, the IBM QSW stock price set, and IBM QSW call and put values.

The process of computing the call and put option prices is the same as before. However the following data filtering and calibration has been applied:

- (i) Random market stock price and the corresponding market option (call/put) values are selected for each trading ticker within a time box of 1 trading day.
- Using the market stock price for each trading ticker, the theorised quantum values of the call option are computed using the procedure established in previous chapter 6.0, and previous sections in this chapter, with equation (174), reproduced here

$$c^{\Xi} = e^{-rT} E^{Q^{\Xi} \in \textcircled{D}^{\uparrow\downarrow}} [MAX(S_{T} - X, 0)] \quad (215)$$

where the $E^{Q^{\Xi}} \in \mathbb{O}^{\uparrow\downarrow}[\cdots]$ is invoked by the algorithm with the help of the NAG routine DEKAF, such that the Eigen-values are obtained and normalised. A fixed quantum grid is fitted wth k = 9, and $L=\gamma(0) = 5q\sigma$ across the implied volatility surface. This of course could be varied, but is kept so here as a way to simply the data filtering and calibration process.

- (iii) Ratio of market call price with the quantum-well implied obtained is used as a calibration coefficient. This is deemed good enough within 1-trading day and assumed reliable for intraday trading.
- (iv) Market stock price is obtained at elapsed time. Theorised quantum values of the call option are computed again using the same procedure as before. The quantum-well implied call value is multiplied with the calibration coefficient. The result is used as the final QSW call value. Equation (179) is used to obtain the put option value.

$$p^{\Xi} = c^{\Xi} + e^{-rT} \cdot X - Se^{-qT}$$
 (216)

In essence, I am only interested in values (stock and option) that are near neighbours in the market line.

- (v) Values from (iv) are compared with the final market option values for calls and puts.
- (vi) The above can be repeated each day. Condition expressed in equation (165) still applies.

The process above allows for the volatility surface to be varied, despite the homogenious fitting of the quantum grid. In doing so the expectation is that the results are expected to be closest to those obtained from the application of the local volatility model, which uses the implied volatility concept, although in an entirely different way. Existing researchers such as Callegaro et al. (2015), and Bustamante and Contreras (2016) have also applied a form of quantum calibration in local volatility. The classical models appear to fail option valation for stock with limited history and subsequent default historical volatility (Dumas et al., 1998). This is the case of UBER in the sample used here. In scuh case implied voilatility is used. It leads to reasonable values.

						Historic				Last	Price	
<i>4</i> 90.	Ticker	Price sport	Exercise	Time Expiry Days	Expira	Volatility	FORWard	Rate Inated	Dividend vield	Call Marked	Put Innaries	Call Option val calibra
1	IBM US	132.34	132.00	27	06/21/2019	22.34%	132.553	2.525%	0.000%	3.20	2.58	1.0000
2	AAPL US	178.97	180.00	27	06/21/2019	31.61%	179.3481	2.525%	0.000%	5.45	6.21	1.3524
3	UBER US	41.51	41.50	27	06/21/2019	53.15%	41.5417	2.525%	0.000%	2.20	2.70	3.1889
4	AMZN US	1823.28	1822.50	27	06/21/2019	23.78%	1,826.09	2.525%	0.000%	52.60	49.25	1.1000
5	MSFT US	126.24	126.00	27	06/21/2019	22.25%	126.4181	2.525%	0.000%	3.16	2.75	1.0412
6	TSLA US	190.63	190.00	27	06/21/2019	48.20%	191.017	2.525%	0.000%	14.90	14.25	3.1077
7	FB US	181.06	182.50	27	06/21/2019	27.77%	181.341	2.525%	0.000%	4.50	5.59	1.2000
8	VOD LN	126.10	125.00	27	06/21/2019	25.34%	122.497	0.871%	20.300%	3.00	4.50	0.9253
9	GOOGL US	1138.61	1137.50	27	06/21/2019	32.18%	1135.635	2.525%	0.000%	34.90	22.46	1.4000
10	INTC US	44.57	44.50	27	06/21/2019	39.35%	44.6448	2.525%	0.000%	1.42	1.23	1.4000
11	BHP AT	37.45	37.38	32	06/27/2019	16.73%	37.4854	1.430%	0.000%	1.07	0.95	1.4000
12	7203 JP	6502.00	6500.00	45	07/11/2019	17.61%	6,502.50	0.002%	0.000%	178.50	176.00	1.2000
13	SBRY LN	195.30	195.00	82	08/16/2019	45.30%	187.869	0.884%	19.219%	8.50	14.25	1.8200
14	UKX INDEX	7277.73	7275.00	82	08/16/2019	10.87%	7,194.84	0.884%	6.159%	132.50	204.00	0.8000
15	SPX INDEX	2826.06	2825.00	82	08/16/2019	11.14%	2,829.76	2.525%	2.204%	85.25	76.00	1.2688
16	SX5E INDEX	3350.70	3350.00	54	07/19/2019	13.28%	3,336.17	-0.221%	2.759%	67.00	86.30	0.8431
17	TPX INDEX	1541.21	1550.00	47	43806	13.95%	1538.88	0.003%	1.203%	30.00	40.50	0.8431
18	DAX INDEX	12011.04	12000.00	54	07/19/2019	14.08%	12006.86	-0.222%	0.000%	289.10	295.20	1.0099
19	NDX INDEX	7300.96	7300.00	54	07/19/2020	15.48%	7318.08	2.525%	0.859%	245.95	212.00	1.4430
20	NKY INDEX	21117.22	21125.00	47	43806	14.13%	21085.26	0.002%	4.311%	455.00	550.00	0.9057
21	HSCEI INDEX	10445.54	10400.00	64	07/30/2019	17.34%	10232.03	2.074%	4.311%	246.00	414.00	0.9956
22	DIS US	132.79	130.00	54	07/19/2019	29.89%	132.4087	2.525%	4.669%	5.60	2.98	1.3377
23	BATS LN	2984.00	3000.00	82	08/16/2019	27.09%	2939.777	0.882%	4.311%	132.50	191.50	0.9947
24	BT/A LN	196.26	200.00	26	06/21/2019	20.03%	196.365	0.871%	0.000%	3.75	7.25	0.8963
25	BABA US	155.00	155.00	82	08/16/2019	31.22%	155.8836	2.525%	0.000%	10.70	9.63	1.4000
26	BA US	354.90	355.00	82	08/16/2019	31.24%	355.051	2.525%	4.311%	20.14	19.11	1.1900
27	NVDA US	145.15	145.00	54	07/19/2019	36.13%	145.4978	2.525%	0.775%	9.00	8.45	1.3000
28	WMT US	102.67	105.00	82	08/16/2019	15.44%	102.7133	2.525%	4.311%	2.65	4.74	1.3000
29	005930 KS	42700.00	43000.00	46	43776	24.89%	42442	1.795%	4.311%	1100.00	1660.00	1.1140
30	QCOM	66.21	67.50	82	08/16/2019	54.44%	65.9536	2.525%	4.311%	4.05	5.60	1.5000

Table 42Sample market call and put option values and the data calibration coefficients, valid for 1 tradingday (25/05/2019).

The computed include Eigen-state values, as well as QSW stock and option prices for the entire sample⁸⁴. Following the process of data filtering and calibration, the summarised final QSW values for the call and put for each trading ticker (30 in total) are shown in the following table.

⁸⁴ The Eigen-data and MS-Excel work with computations can be provided to anyone upon request.

					Historic				Last	Price		
40.	<i>ticket</i>	Price sport	Exercise	Time toing out	VOISTINEY	Forward	Rate Install	Dividend vield	Call Market	Put Markell	Call Ouanturn	Put Duanturni
1	IBM US	132.34	132.00	27	22.34%	132.553	2.525%	0.000%	3.20	2.58	3.20	2.50
2	AAPL US	178.97	180.00	27	31.61%	179.3481	2.525%	0.000%	5.45	6.21	5.52	6.32
3	UBER US	41.51	41.50	27	53.15%	41.5417	2.525%	0.000%	2.20	2.70	2.35	2.38
4	AMZN US	1823.28	1822.50	27	23.78%	1,826.09	2.525%	0.000%	52.60	49.25	47.74	41.53
5	MSFT US	126.24	126.00	27	22.25%	126.4181	2.525%	0.000%	3.16	2.75	3.16	2.75
6	TSLA US	190.63	190.00	27	48.20%	191.017	2.525%	0.000%	14.90	14.25	14.83	10.84
7	FB US	181.06	182.50	27	27.77%	181.341	2.525%	0.000%	4.50	5.59	4.88	6.02
8	VOD LN	126.10	125.00	27	25.34%	122.497	0.871%	20.300%	3.00	4.50	3.09	4.44
9	GOOGL US	1138.61	1137.50	27	32.18%	1135.635	2.525%	0.000%	34.90	22.46	38.08	32.28
10	INTC US	44.57	44.50	27	39.35%	44.6448	2.525%	0.000%	1.42	1.23	1.50	1.23
11	BHP AT	37.45	37.38	32	16.73%	37.4854	1.430%	0.000%	1.07	0.95	1.26	1.07
12	7203 JP	6502.00	6500.00	45	17.61%	6,502.50	0.002%	0.000%	178.50	176.00	185.31	183.29
13	SBRY LN	195.30	195.00	82	45.30%	187.869	0.884%	19.219%	8.50	14.25	8.49	15.24
14	UKX INDEX	7277.73	7275.00	82	10.87%	7,194.84	0.884%	6.159%	132.50	204.00	137.95	203.65
15	SPX INDEX	2826.06	2825.00	82	11.14%	2,829.76	2.525%	2.204%	85.25	76.00	84.96	77.18
16	SX5E INDEX	3350.70	3350.00	54	13.28%	3,336.17	-0.221%	2.759%	67.00	86.30	67.00	87.39
17	TPX INDEX	1541.21	1550.00	47	13.95%	1538.88	0.003%	1.203%	30.00	40.50	29.60	41.79
18	DAX INDEX	12011.04	12000.00	54	14.08%	12006.86	-0.222%	0.000%	289.10	295.20	289.10	283.70
19	NDX INDEX	7300.96	7300.00	54	15.48%	7318.08	2.525%	0.859%	245.95	212.00	249.74	223.12
20	NKY INDEX	21117.22	21125.00	47	14.13%	21085.26	0.002%	4.311%	455.00	550.00	452.07	541.74
21	HSCEI INDEX	10445.54	10400.00	64	17.34%	10232.03	2.074%	4.311%	246.00	414.00	253.43	418.67
22	DIS US	132.79	130.00	54	29.89%	132.4087	2.525%	4.669%	5.60	2.98	5.76	3.58
23	BATS LN	2984.00	3000.00	82	27.09%	2939.777	0.882%	4.311%	132.50	191.50	135.17	194.52
24	BT/A LN	196.26	200.00	26	20.03%	196.365	0.871%	0.000%	3.75	7.25	3.68	7.24
25	BABA US	155.00	155.00	82	31.22%	155.8836	2.525%	0.000%	10.70	9.63	10.26	9.00
26	BA US	354.90	355.00	82	31.24%	355.051	2.525%	4.311%	20.14	19.11	19.92	19.14
27	NVDA US	145.15	145.00	54	36.13%	145.4978	2.525%	0.775%	9.00	8.45	9.00	8.32
28	WMT US	102.67	105.00	82	15.44%	102.7133	2.525%	4.311%	2.65	4.74	2.73	4.74
29	005930 KS	42700.00	43000.00	46	24.89%	42442	1.795%	4.311%	1100.00	1660.00	1073.41	1659.77
30	QCOM	66.21	67.50	82	54.44%	65.9536	2.525%	4.311%	4.05	5.60	4.13	5.78

Table 43 Sample data. Includes market and quantum square-well call and put option values, post calibration, valid for 1 trading day (25/05/2019).

I have also computed the forward prices for each stock using the market values as well as the QSW value set. The forward price of equity is a fundamental quantity impacting option pricing, hedging and implied volatility. Interpolating the futures curve at an intermediate option maturity is a nontrivial task because future dividends payments and payment times, aside from those already announced for the very near term, are generally unknown. Hence forward prices must be implied from market observables, e.g. exchangetraded instruments like equity futures, equity options, dividend futures, and OTC instruments such as dividend swaps and total return swaps (Dupire, 1994; Dumas et al., 1998; Masoliver and Perello, 2003; Lee, 2004; Gatheral, 2006; Orosi, 2010; Hull, 2014).

In principle, forward prices are straightforward to derive via put-call parity of European option prices:

$$F(T) = C(X, T) - P(X, T) + X$$
 (217)

where F(T) is the forward at time T, C(X, T) and P(X, T) are respectively the future values of European call and put options struck at X as of the expiration time T. Similarly an expression is established for implied QSW forward prices. Starting with equation (178), by compounding each term in that expression, the following QSW forward price expression is obtained:

$$F^{\Xi}(T) = C^{\Xi}(X,T) - P^{\Xi}(X,T) + X$$
 (218)

In practice, many difficulties impede the usage of this simple relationship. In some cases, the underlying asset and options on the asset trade during different times on different exchanges. For example, stocks in the Nikkei 225 index trade on the Tokyo Stock Exchange between 0900 - 1130 hrs and 1230 - 1500 hrs Tokyo time. Nikkei index options, however, trade 0900 - 1515 hrs (pit) and 1630 - 0300 hrs (electronic) on the Osaka Stock Exchange, hence the spot price of the underying is unavailable during substantially large periods of option trading. Even when the spot market is open, the index spot price being an average of non-contemporaneous traded prices of the constituent members, can be unreliable, especially within the first few minutes of the opening of a new trading session when not all constituents have started trading yet (Dumas et al, 1998; Lee, 2004; Gatheral, 2006).

The futures price of the prompt contract are consistently higher than the spot price (implying a negative dividend yield on the index), therefore indicating an unreliable spot price. Option prices can also be quite erratic, especially close to the opening and closing times of the trading session. At other times too, it is not uncommon to see a complete lack of quoting activity (even for near-the-money strikes), or a one-sided market, or two-sided markets with unreasonably wide bid-ask spreads (Dumat et al., 1998; Lee, 2004; Gatheral, 2006).

					Historic			Last	Price					
440.	Ticket	Price spoil	Exercise	Time topicy parts	Volation	Rate Nated	Dividend vield	Call Market	Put Marken	Call Construction	Put Duanturni	Forward (Spit Carry)	Forward Interest or wi	Diff
1	IBM US	132.34	132.00	27	22.34%	2.525%	0.000%	3.20	2.58	3.20	2.50	132.70	132.62	-0.08
2	AAPL US	178.97	180.00	27	31.61%	2.525%	0.000%	5.45	6.21	5.52	6.32	179.20	179.24	0.04
3	UBER US	41.51	41.50	27	53.15%	2.525%	0.000%	2.20	2.70	2.35	2.38	41.47	41.00	-0.47
4	AMZN US	1823.28	1822.50	27	23.78%	2.525%	0.000%	52.60	49.25	47.74	41.53	1828.71	1825.85	-2.86
5	MSFT US	126.24	126.00	27	22.25%	2.525%	0.000%	3.16	2.75	3.16	2.75	126.41	126.41	0.00
6	TSLA US	190.63	190.00	27	48.20%	2.525%	0.000%	14.90	14.25	14.83	10.84	193.99	190.65	-3.34
7	FB US	181.06	182.50	27	27.77%	2.525%	0.000%	4.50	5.59	4.88	6.02	181.36	181.41	0.05
8	VOD LN	126.10	125.00	27	25.34%	0.871%	20.300%	3.00	4.50	3.09	4.44	123.65	123.50	-0.15
9	GOOGL US	1138.61	1137.50	27	32.18%	2.525%	0.000%	34.90	22.46	38.08	32.28	1143.30	1149.94	6.64
10	INTC US	44.57	44.50	27	39.35%	2.525%	0.000%	1.42	1.23	1.50	1.23	44.77	44.69	-0.08
11	BHP AT	37.45	37.38	32	16.73%	1.430%	0.000%	1.07	0.95	1.26	1.07	37.57	37.50	-0.07
12	7203 JP	6502.00	6500.00	45	17.61%	0.002%	0.000%	178.50	176.00	185.31	183.29	6502.02	6502.50	0.48
13	SBRY LN	195.30	195.00	82	45.30%	0.884%	19.219%	8.50	14.25	8.49	15.24	188.25	189.25	1.00
14	UKX INDEX	7277.73	7275.00	82	10.87%	0.884%	6.159%	132.50	204.00	137.95	203.65	7209.30	7203.50	-5.80
15	SPX INDEX	2826.06	2825.00	82	11.14%	2.525%	2.204%	85.25	76.00	84.96	77.18	2832.78	2834.25	1.47
16	SX5E INDEX	3350.70	3350.00	54	13.28%	-0.221%	2.759%	67.00	86.30	67.00	87.39	3329.61	3330.70	1.09
17	TPX INDEX	1541.21	1550.00	47	13.95%	0.003%	1.203%	30.00	40.50	29.60	41.79	1537.81	1539.50	1.69
18	DAX INDEX	12011.04	12000.00	54	14.08%	-0.222%	0.000%	289.10	295.20	289.10	283.70	12005.40	11993.90	-11.50
19	NDX INDEX	7300.96	7300.00	54	15.48%	2.525%	0.859%	245.95	212.00	249.74	223.12	7326.62	7333.95	7.33
20	NKY INDEX	21117.22	21125.00	47	14.13%	0.002%	4.311%	455.00	550.00	452.07	541.74	21035.33	21030.00	-5.33
21	HSCEI INDEX	10445.54	10400.00	64	17.34%	2.074%	4.311%	246.00	414.00	253.43	418.67	10234.76	10232.00	-2.76
22	DIS US	132.79	130.00	54	29.89%	2.525%	4.669%	5.60	2.98	5.76	3.58	132.18	132.62	0.44
23	BATS LN	2984.00	3000.00	82	27.09%	0.882%	4.311%	132.50	191.50	135.17	194.52	2940.66	2941.00	0.34
24	BT/A LN	196.26	200.00	26	20.03%	0.871%	0.000%	3.75	7.25	3.68	7.24	196.44	196.50	0.06
25	BABA US	155.00	155.00	82	31.22%	2.525%	0.000%	10.70	9.63	10.26	9.00	156.25	156.07	-0.18
26	BA US	354.90	355.00	82	31.24%	2.525%	4.311%	20.14	19.11	19.92	19.14	355.78	356.03	0.25
27	NVDA US	145.15	145.00	54	36.13%	2.525%	0.775%	9.00	8.45	9.00	8.32	145.68	145.55	-0.13
28	WMT US	102.67	105.00	82	15.44%	2.525%	4.311%	2.65	4.74	2.73	4.74	102.99	102.91	-0.08
29	005930 KS	42700.00	43000.00	46	24.89%	1.795%	4.311%	1100.00	1660.00	1073.41	1659.77	42413.64	42440.00	26.36
30	QCOM	66.21	67.50	82	54.44%	2.525%	4.311%	4.05	5.60	4.13	5.78	65.84	65.95	0.11

 Table 44
 Sample data. Includes market and quantum square-well forward prices. Value difference, highlighted in yellow colour.

For comparison purposes, I have sumamrised all option value data, computed with various models in the table below. It also includes the difference between the theorised option values and the market option value equivalents.

		Last	rnce							BVOL Used		BVOL Used									
<i>4</i> 10.	ticket	Call Market	Put marked	Call (BSA)	Put 1850	Callipsel	Diff.	Put Insel	Diff.	Call loc vold	Diff.	Put lloe voli	Diff.	Call Trinomal	Diff.	Put Innonial	Diff.	Call Ouanturn	Diff.	Put Duanturni	Diff.
1	IBM US	3.20	2.58	3.52	2.93	3.52	-0.32	2.93	-0.35	3.11	0.09	2.53	0.05	3.51	-0.31	2.95	-0.37	3.20	0.00	2.50	0.08
2	AAPL US	5.45	6.21	5.87	6.49	5.87	-0.42	6.49	-0.28	5.44	0.01	6.05	0.16	5.86	-0.41	6.51	-0.30	5.52	-0.07	6.32	-0.11
3	UBER US	2.20	2.70	2.42	2.37	2.42	-0.22	2.37	0.33	2.38	-0.18	2.33	0.37	2.42	-0.22	2.38	0.32	2.35	-0.15	2.38	0.32
4	AMZN US	52.60	49.25	49.17	45.20	49.17	3.43	45.20	4.05	52.44	0.16	48.47	0.78	49.02	3.58	45.41	3.84	47.74	4.86	41.53	7.72
5	MSFT US	3.16	2.75	3.27	2.83	3.27	-0.11	2.83	-0.08	3.16	0.00	2.72	0.03	3.26	-0.10	2.84	-0.09	3.16	0.00	2.75	0.00
6	TSLA US	14.90	14.25	10.48	9.43	10.48	4.42	9.43	4.82	14.88	0.02	13.83	0.42	10.47	4.43	9.46	4.79	14.83	0.07	10.84	3.41
7	FB US	4.50	5.59	4.93	6.06	4.93	-0.43	6.06	-0.47	4.42	0.08	5.54	0.05	4.92	-0.42	6.08	-0.49	4.88	-0.38	6.02	-0.43
8	VOD LN	3.00	4.50	2.18	4.97	2.18	0.82	4.97	-0.47	2.05	0.95	4.54	-0.04	2.29	0.71	4.80	-0.30	3.09	-0.09	4.44	0.06
9	GOOGL US	34.90	22.46	38.20	40.06	38.20	-3.30	40.06	-17.60	25.29	9.61	26.78	-4.32	39.00	-4.10	40.75	-18.29	38.08	-3.18	32.28	-9.82
10	INTC US	1.42	1.23	1.99	1.83	1.99	-0.57	1.83	-0.60	1.37	0.05	1.21	0.02	1.99	-0.57	1.84	-0.61	1.50	-0.08	1.23	0.00
11	BHP AT	1.07	0.95	0.74	0.70	0.74	0.33	0.70	0.25	0.96	0.11	0.92	0.03	0.74	0.33	0.70	0.25	1.26	-0.20	1.07	-0.12
12	7203 JP	178.50	176.00	162.98	160.44	162.98	15.52	160.44	15.56	169.48	9.02	166.94	9.06	162.91	15.59	160.39	15.61	185.31	-6.81	183.29	-7.29
13	SBRY LN	8.50	14.25	13.03	20.15	13.03	-4.53	20.15	-5.90	8.52	-0.02	15.63	-1.38	13.05	-4.55	20.17	-5.92	8.49	0.01	15.24	-0.99
14	UKX INDEX	132.50	204.00	111.89	191.83	111.89	20.61	191.83	12.17	124.85	7.65	204.63	-0.63	111.87	20.63	192.16	11.84	137.95	-5.45	203.65	0.35
15	SPX INDEX	85.25	76.00	61.77	56.97	61.77	23.48	56.97	19.03	82.60	2.65	77.75	-1.75	61.78	23.47	57.05	18.95	84.96	0.29	77.18	-1.18
16	SX5E INDEX	67.00	86.30	59.87	77.18	59.87	7.13	77.18	9.12	68.41	-1.41	85.68	0.62	59.96	7.04	77.14	9.16	67.00	0.00	87.39	-1.09
17	TPX INDEX	30.00	40.50	25.57	36.86	25.57	4.43	36.86	3.64	28.92	1.08	40.03	0.47	25.65	4.35	36.78	3.72	29.60	0.40	41.79	-1.29
18	DAX INDEX	289.10	295.20	262.89	256.36	262.89	26.21	256.36	38.84	295.84	-6.74	289.30	5.90	263.20	25.90	256.00	39.20	289.10	0.00	283.70	11.50
19	NDX INDEX	245.95	212.00	182.72	164.01	182.72	63.23	164.01	47.99	236.08	9.87	217.73	-5.73	182.59	63.36	164.50	47.50	249.74	-3.79	223.12	-11.12
20	NKY INDEX	455.00	550.00	406.05	445.58	406.05	48.95	445.58	104.42	483.07	-28.07	522.81	27.19	406.41	48.59	446.01	103.99	452.07	2.93	541.74	8.26
21	HSCEI INDEX	246.00	414.00	215.70	400.31	215.70	30.30	400.31	13.69	234.47	11.53	420.26	-6.26	215.50	30.50	400.26	13.74	253.43	-7.43	418.67	-4.67
22	DIS US	5.60	2.98	7.29	4.88	7.29	-1.69	4.88	-1.90	5.33	0.27	2.91	0.07	7.29	-1.69	4.89	-1.91	5.76	-0.16	3.58	-0.60
23	BATS LN	132.50	191.50	123.21	184.64	123.21	9.29	184.64	6.86	128.39	4.11	188.38	3.12	123.76	8.74	183.92	7.58	135.17	-2.67	194.52	-3.02
24	BT/A LN	3.75	7.25	2.66	6.29	2.66	1.09	6.29	0.96	4.56	-0.81	8.18	-0.93	2.66	1.09	6.30	0.95	3.68	0.07	7.24	0.01
25	BABA US	10.70	9.63	9.57	8.69	9.57	1.13	8.69	0.94	10.38	0.32	9.49	0.14	9.58	1.12	8.70	0.93	10.26	0.44	9.00	0.63
26	BA US	20.14	19.11	20.88	20.82	20.88	-0.74	20.82	-1.71	19.22	0.92	19.16	-0.05	20.87	-0.73	20.81	-1.70	19.92	0.22	19.14	-0.03
27	NVDA US	9.00	8.45	8.28	7.77	8.28	0.72	7.77	0.68	8.19	0.81	8.70	-0.25	8.28	0.72	7.78	0.67	9.00	0.00	8.32	0.13
28	WMT US	2.65	4.74	2.02	4.29	2.02	0.63	4.29	0.45	2.46	0.19	4.75	-0.01	2.02	0.63	4.29	0.45	2.73	-0.08	4.74	0.00
29	005930 KS	1100.00	1660.00	1243.82	1801.57	1243.82	-143.82	1801.57	-141.57	1155.62	-55.62	1710.63	-50.63	1244.42	-144.42	1802.49	-142.49	1073.41	26.59	1659.77	0.23
30	QCOM	4.05	5.60	6.08	7.62	6.08	-2.03	7.62	-2.02	3.86	0.19	5.40	0.20	6.09	-2.04	7.62	-2.02	4.13	-0.08	5.78	-0.18
		μ(mean) -	->			3.45		3.70		-1.11		-0.78		3.37		3.63		0.18		-0.31

Table 45 Sample data. Market and theorised option values (BS, TRI, LV, QSW). Value difference, highlighted in yellow colour.

The mean difference between the market call and put option values with those obtained through the Black-Scholes equivalent are 3.4 and 3.7, respectively. These are similar to the mean difference of the market with the values acquired from the application of the trinomial option pricing model. I notice that this is significantly reduced when comparing market option values and the local volatility values (-1.11 and -0.78) as well as the price-

surface quantum square well model (0.18 and -0.31). The local volatility model appears to slightly underestimate the option prices. Similarly the price-surface QSW (quantum square-well) yields improved values, closer to the market values with a slight overestimation of the call and a slight underestimation of put value.

₽0.	Ticket	Price sport	Call Ouanturn	Diff.	Put lousneumi	Diff.	WC3 Derr?	LUPUT Dever	OSWCaller?	OSWPUT DEV?
1	IBM US	132.34	3.20	0.00	2.50	0.08	1.43	0.68	0.03	0.15
2	AAPL US	178.97	5.52	-0.07	6.32	-0.11	1.24	0.88	0.06	0.04
3	UBER US	41.51	2.35	-0.15	2.38	0.32	0.86	1.31	0.11	0.39
4	AMZN US	1823.28	47.74	4.86	41.53	7.72	1.60	2.42	21.94	64.45
5	MSFT US	126.24	3.16	0.00	2.75	0.00	1.22	0.65	0.03	0.09
6	TSLA US	190.63	14.83	0.07	10.84	3.41	1.27	1.43	0.01	13.82
7	FB US	181.06	4.88	-0.38	6.02	-0.43	1.41	0.68	0.31	0.01
8	VOD LN	126.10	3.09	-0.09	4.44	0.06	4.23	0.54	0.07	0.14
9	GOOGL US	1138.61	38.08	-3.18	32.28	-9.82	114.82	12.55	11.26	90.48
10	INTC US	44.57	1.50	-0.08	1.23	0.00	1.34	0.63	0.07	0.09
11	BHP AT	37.45	1.26	-0.20	1.07	-0.12	1.47	0.65	0.14	0.04
12	7203 JP	6502.00	185.31	-6.81	183.29	-7.29	102.53	96.76	48.80	48.75
13	SBRY LN	195.30	8.49	0.01	15.24	-0.99	1.18	0.36	0.03	0.46
14	UKX INDEX	7277.73	137.95	-5.45	203.65	0.35	76.66	0.02	31.64	0.43
15	SPX INDEX	2826.06	84.96	0.29	77.18	-1.18	14.10	0.95	0.01	0.76
16	SX5E INDEX	3350.70	67.00	0.00	87.39	-1.09	0.09	1.95	0.03	0.61
17	TPX INDEX	1541.21	29.60	0.40	41.79	-1.29	4.78	1.55	0.05	0.97
18	DAX INDEX	12011.04	289.10	0.00	283.70	11.50	31.75	44.58	0.03	139.37
19	NDX INDEX	7300.96	249.74	-3.79	223.12	-11.12	120.46	24.54	15.73	116.86
20	NKY INDEX	21117.22	452.07	2.93	541.74	8.26	727.08	782.14	7.56	73.38
21	HSCEI INDEX	10445.54	253.43	-7.43	418.67	-4.67	159.66	30.07	57.83	19.05
22	DIS US	132.79	5.76	-0.16	3.58	-0.60	1.89	0.72	0.11	0.08
23	BATS LN	2984.00	135.17	-2.67	194.52	-3.02	27.20	15.18	8.12	7.34
24	BT/A LN	196.26	3.68	0.07	7.24	0.01	0.09	0.02	0.01	0.10
25	BABA US	155.00	10.26	0.44	9.00	0.63	2.03	0.84	0.07	0.87
26	BA US	354.90	19.92	0.22	19.14	-0.03	4.10	0.53	0.00	0.08
27	NVDA US	145.15	9.00	0.00	8.32	0.13	3.67	0.28	0.03	0.20
28	WMT US	102.67	2.73	-0.08	4.74	0.00	1.68	0.59	0.06	0.09
29	005930 KS	42700.00	1073.41	26.59	1659.77	0.23	2971.83	2485.35	697.80	0.29
30	QCOM	66.21	4.13	-0.08	5.78	-0.18	1.68	0.95	0.06	0.02
	μ (mean)	->		0.18		-0.31	151.15	121.03	31.10	19.98
								σ^	2 (var)	
					σ (Std.	Dev.) ->	12.29	11.00	5.58	4.47

Table 46 Sample data. Variance of the option value difference between the market and LV, and market with QSW.

The local volatility and the QSW yield better estimates compared to classical Black-Scholes and Trinomial models. I further compute the sample variability of such mean differences for the local volatility and the QSW; (i) 151.15 and 121.03 for local volatility calls and puts, (ii) 31.10 and 19.98 for the QSW calls and puts, respectively. However these are the results of a finite size sample and is further tested in the following section.

7.5.3 SAMPLE HYPOTHESIS TESTING

Using the data and pricing results above, I test several statistical hypothesis in order to validate the significance of these results, under the assumptions that, (i) the same data filtering/calibration procedure is applied across samples, and (ii) samples are drawn in the same manner. Could the results above be confirmed within a reasonable confidence level?

To answer the question, I have set and test the following statistical hypothesis:

(i) Make an inference about the mean of one group (T-test). This is suitable because the mean in question is the average of the differences between the market price of the option and the theorised value. An absolute match in values (theory and market) would be achieved if that mean difference is 0. Therefore the hypothesized population mean difference is set to zero. The standard deviation of the population is not known, therefore the T-test (rather than the Z-test) is performed at 95% confidence level (Berenson and Krehbiel, 1992; Lind, 2010; Doughterly, 2011).

H_0 : The mean CALL price difference between market and QSW is 0.00	$H_0:\mu_{\text{CALL}_\text{DIFF}}\!\!=\!\!0.00$
H_1 : The mean CALL price difference between market and QSW is not 0.00	$H_1:\mu_{\text{CALL_DIFF}}{\neq}0.00$

H_0 : The mean PUT price difference between market and QSW is 0.00	$H_0: \mu_{\text{PUT_DIFF}}=0.00$
H_1 : The mean PUT price difference between market and QSW is not 0.00	$H_1: \mu_{PUT DIFF} \neq 0.00$

t Test for Hypothesis of the Me	an Difference					
Data						
Null Hypothesis μ =	0					
Level of Significance	0.05					
Sample Size	30					
Sample Mean	0.18					
Sample Standard Deviation	5.58					
Intermediate Calcula	tions					
Standard Error of the Mean	1.0188					
Degrees of Freedom	29					
t Test Statistic	0.1767					
Two-Tail Test						
Lower Critical Value	-2.0452					
Upper Critical Value	2.0452					
<i>p</i> -Value	0.8610					
Do not reject the null hypothesis						

t Test for Hypothesis of the Me	an						
Data							
Null Hypothesis µ=	0						
Level of Significance	0.05						
Sample Size	30						
Sample Mean	-0.31						
Sample Standard Deviation	4.47						
Intermediate Calcula	tions						
Standard Error of the Mean	0.8161						
Degrees of Freedom	29						
t Test Statistic	-0.3799						
Two-Tail Test							
Lower Critical Value	-2.0452						
Upper Critical Value	2.0452						
<i>p</i> -Value	0.7068						
Do not reject the null hy	pothesis						

Table 47a: T-test for mean difference between Table 47b: T-test for mean difference between market call option price and the QSW at 95% market put option price and the QSW at 95% confidence level.

confidence level.

The null hypothesis statement is proven to be true at 95% confidence level. Therefore, If I am to draw samples in the same manner, then 95% of the samples will confirm the validity of the null hypothesis. It implies that, although the sample mean diference for calls and puts is non zero, over the entire population, it is expected to converge to 0, therefore the mean option value acquired through the QSW and the market will match.

(ii) Compare variances of two independent groups (F-test). To complete this test, first and foremost, the variance of the mean option price difference (market / local volatility, and market / quantum square well) must be established. I seek to prove that the variability of call (put) price difference between the market and QSW is equal or less than the call (put) price difference between market and LVM (local volatility model). The upper-tail test is most suitable in this case (Berenson and Krehbiel, 1992; Lind, 2010; Doughterly, 2011).

Data in table (46) is used for testing. It is important that "Population 1 Sample" is the one with the greater variability (standard deviation). If the NULL hypothesis fail, focus is then shifted onto the pool-variance T-test. This is important particularly because the test may fail due to the fact that the two groups belong to the same population, rather than in two distinct ones (Berenson and Krehbiel, 1992; Lind, 2010; Doughterly, 2011).

H_0 : The variability of CALL price difference between market and LVM is equal or less than to that of market and QSW and the two groups belong to independent populations	$H_0:\sigma_{MLV} - \!\!\!\! \sigma_{MQSW} \!\! \leq \!\! 0$				
H ₁ : The variability of CALL price difference between market and LVM is greater than to that of market and QSW, and the two groups belong to independent populations.	H_1 : σ_{MLV} - σ_{MQSW} >0				
H_0 : The variability of PUT price difference between market and LVM is equal or less than to that of market and QSW and the two groups belong to independent populations.	H_0 : σ_{MLV} - $\sigma_{MQSW} \le 0$				
H ₁ : The variability of PUT price difference between market and LVM is greater than to that of market and QSW, and the two groups belong to independent populations.	H_1 : σ_{MLV} - σ_{MQSW} >0				
F Test for Differences in Two Variances (CAL	L Option)	F Test for Differences in Two Variances (PU	「Option)		
--	-------------	---	-------------	--	--
Data		Data			
Level of Significance	0.05	Level of Significance	0.05		
Larger-Variance Sample		Larger-Variance Sample			
Sample Size	30	Sample Size	30		
Sample Variance	151.1493247	Sample Variance	121.0284109		
Smaller-Variance Sample		Smaller-Variance Sample			
Sample Size	30	Sample Size	30		
Sample Variance	31.10387438	Sample Variance	19.97983102		
Intermediate Calculations		Intermediate Calculations			
F Test Statistic	4.8595	F Test Statistic	6.0575		
Population 1 Sample Degrees of Freedom	29	Population 1 Sample Degrees of Freedom	29		
Population 2 Sample Degrees of Freedom	29	Population 2 Sample Degrees of Freedom	29		
Upper-Tail Test		Upper-Tail Test			
Upper Critical Value	1.8608	Upper Critical Value	1.8608		
<i>p</i> -Value	0.0000	0 p-Value			
Reject the null hypothesis		Reject the null hypothesis			

Table 48a: F-test for mean difference between market- LV call option price with the market-QSW at 95% confidence level.

Table 48b: F-test for mean difference between market-LV put option price and the market-QSW at 95% confidence level.

The NULL hypothesis statement is rejected. This can be so because (i) the variability of CALL (or PUT) price difference between market and LVM is not equal or less than the one from market and QSW, or (ii) two groups do not belong to independent populations. Instead, it would be reasonable to follow a pool-variance scenario testing (Doughterly, 2011). This is evident as both samples are drawn in relation to the market.

H_0 : The variability of CALL price difference between market and QSW is equal or less than to that of market and LVM and the two groups belong to the same population pool.	$H_0:\sigma_{MQSW} - \!\!\!\sigma_{MLV} \!\! \leq \!\! 0$
H ₁ : The variability of CALL price difference between market and QSW is greater than to that of market and LVM, and the two groups belong to same populations pool.	$H_1:\sigma_{MQSW}\text{-}\sigma_{MLV}\text{>}0$
H_0 : The variability of PUT price difference between market and QSW is equal or less than to that of market and LVM and the two groups belong to the same population pool.	$H_0:\sigma_{MQSW}\text{-}\sigma_{MLV}\!\!\leq\!\!0$
H ₁ : The variability of PUT price difference between market and QSW is greater than to that of market and LVM, and the two groups belong to population pool.	$H_1:\sigma_{MQSW}\text{-}\sigma_{MLV}\text{>}0$

Pooled-Variance t Test for the Difference Bet	ween Two Mea	ns (CALL	Option)		
(assumes equal population variances)					
Data			Confidence Interval Estimate		
Hypothesized Difference	0		for the Difference Betwee	n Two Means	
Level of Significance	0.05				
Population 1 Sample			Data		
Sample Size	30		Confidence Level	95%	
Sample Mean	0.176045365				
Sample Standard Deviation	5.577084756		Intermediate Calcu	lations	
Population 2 Sample			Degrees of Freedom	58	
Sample Size	30		t Value	2.0017	
Sample Mean	-1.1055		Interval Half Width	4.9338	
Sample Standard Deviation	12.29428016				
			Confidence Inte	rval	
Intermediate Calculations			Interval Lower Limit	-3.6522	
Population 1 Sample Degrees of Freedom	29		Interval Upper Limit	6.2153	
Population 2 Sample Degrees of Freedom	29				
Total Degrees of Freedom	58				
Pooled Variance	91.1266				
Standard Error	2.4648				
Difference in Sample Means	1.2815				
t Test Statistic	0.5199				
Upper-Tail Test	_				
Upper Critical Value	1.6716				
<i>p</i> -Value	0.3025				
Do not reject the null hypothes	is				

Table 49: Pool-variance t-test for variance of the difference between (i) market and QSW call option price,

and (ii) the market and LV at 95% confidence level.

Pooled-Variance t Test for the Difference Be	etween Two Means (F	PUT Option)		
(assumes equal population variances)				
Data		Confidence Interval	Estimate	
Hypothesized Difference	0	for the Difference Betwee	en Two Means	
Level of Significance	0.05			
Population 1 Sample		Data		
Sample Size	30	Confidence Level	95%	
Sample Mean	-0.30810677			
Sample Standard Deviation	4.469880426	Intermediate Calcu	llations	
Population 2 Sample		Degrees of Freedom	58	
Sample Size	30	t Value	2.0017	
Sample Mean	-0.77670667	Interval Half Width	4.3397	
Sample Standard Deviation	11.00129133			
		Confidence Interval		
Intermediate Calculations		Interval Lower Limit	-3.8711	
Population 1 Sample Degrees of Freedom	29	Interval Upper Limit	4.8083	
Population 2 Sample Degrees of Freedom	29			
Total Degrees of Freedom	58			
Pooled Variance	70.5041			
Standard Error	2.1680			
Difference in Sample Means	0.4686			
t Test Statistic	0.2161			
Two-Tail Test				
Lower Critical Value	-2.0017			
Upper Critical Value	2.0017			
<i>p</i> -Value	0.8296			

Table 50: Pool-variance t-test for variance of the difference between (i) market and QSW put option price, and (ii) the market and LV at 95% confidence level.

Under the pool-variance scenario, the null hypothesis is true for both calls and puts at 95% confidence. It would remain true for 95% of samples drawn in the same way (Berenson and Krehbiel, 1992; Lind, 2010).

(iii) Compare more than two groups through a one-way ANOVA. To test this, I refer to the difference in option price between the market and BS, TR, LV, and QSW, respectively. I wish to hypothesize that the population's mean option price difference is

zero across the four groups.

H_0 : The mean difference of CALL option price between the market and	H_0 :
BS, Trinomial, LV, and QSW are all equal	$\mu_{BSC} = \mu_{TRC} = \mu_{LV} = \mu_{QSWC};$
	$H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$
H_1 : The mean difference of CALL option price between the market and	H_1 : Not all μ_1 are equal
BS, Trinomial, LV, and QSW are not all equal	$H_1: \exists i, j: \mu_i \neq \mu_j$
(*) Reject Ho as soon as just one is different	

(*) Reject H_0 as soon as just one is different.

H_0 : The mean difference of PUT option price between the MARKET and BS,	
Trinomial, Local Volatility, and Quantum Square Well are all equal	$H_0: \mu_{BS} = \mu_{TR} = \mu_{LV} = \mu_{QSW};$
	$H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$
H_1 : The mean difference of PUT option price between the MARKET and BS,	H_1 : Not all μ_1 are equal
Trinomial, Local Volatility, and Quantum Square Well are not all equal	$H_1: \exists i, j: \mu_i \neq \mu_j$

ANOVA: Single Factor							Calo	ulations
							С	4
SUMMARY							n	124
Groups	Count	Sum	Average	Variance				
Diff. (BS-Call)	31	106.9861667	3.451166667	992.5009				
Diff. (LV-Call)	31	-34.2705	-1.1055	146.1110				
Diff. (T-Calls)	31	104.5888333	3.373833333	997.9615				
Diff. (QSW-Call)	31	5.457406324	0.176045365	30.0671				
ANOVA								
Source of Variation	SS	df	MS	F	P-value	F crit		
Between Groups	491.5689	3	163.8563	0.3025	0.8235	2.6802		
Within Groups	64999.2161	120	541.6601					
Total	65490.7850	123						
				Level of sig	nificance	0.05		

Table 51: ANOVA (single factor). The F-test for the mean difference of CALL option price between the market and BS, Trinomial, LV, and QSW at 95% confidence level.

The degrees of freedom (df) are 4 (between groups + 1) and 120 (within groups). The Qstatistic of 3.69 can be extracted from E.7 table in Berenson and Krehbiel (1992). It is further used to perform the Tukey-Kramer test, shown below. Both tests confirm (P-value > 0.05 in table 51) that if the sample is drawn in the same manner, there would be no call option-price difference across the population groups in 95% of cases (Berenson and Krehbiel, 1992; Doughterly, 2011).

Tukey-Kramer Multi	ple Compai	risons							
	Sample	Sample		Absolute	Std. Error	Critical			
Group	Mean	Size	Comparison	Difference	of Difference	Range	Results		
1: Diff. (BS-Call)	3.451167	31	Group 1 to Group 2	4.556667	4.180060713	15.42	Means	are not diff	ferent
2: Diff. (LV-Call)	-1.1055	31	Group 1 to Group 3	0.077333	4.180060713	15.42	Means are not different		
3: Diff. (T-Calls)	3.373833	31	Group 1 to Group 4	3.275121	4.180060713	15.42	Means	are not diff	ferent
4: Diff. (QSW-Call)	0.176045	31	Group 2 to Group 3	4.479333	4.180060713	15.42	Means are not different		ferent
			Group 2 to Group 4	1.281545	4.180060713	15.42	Means	are not diff	ferent
Other Data	à		Group 3 to Group 4	3.197788	4.180060713	15.42	Means	are not diff	ferent
Level of significance	0.05								
Numerator d.f.	4								
Denominator d.f.	120								
MSW	541.6601								
Q Statistic	3.69								

Table 52: Tukey-Kramer multiple pair comparisons for the mean-difference in CALL prices..

The results are also confirmed for the mean-difference in put options across the groups as shown in tables 53, where the P-value of 0.8320 is greater than the level of significance of 0.05. From table 54, "means are not different" across the groups (Berenson and Krehbiel, 1992; Doughterly, 2011).

ANOVA: Single Factor							Ca	alculations
							с	4
SUMMARY							n	124
Groups	Count	Sum	Average	Variance				
Diff. (BS-Put)	31	114.545	3.695	1194.3110				
Diff. (LV -Put)	31	-24.07790667	-0.776706667	116.9941				
Diff. (T-Put)	31	112.6333333	3.633333333	1200.7508				
Diff. (QSW-Put))	31	-9.551309941	-0.308106772	19.3138				
ANOVA								
Source of Variation	SS	df	MS	F	P-value	F crit		
Between Groups	552.0156	3	184.0052	0.2908	0.8320	2.6802		
Within Groups	75941.0930	120	632.8424					
Total	76493.1086	123						
				Level of sig	nificance	0.05		

Table 53: ANOVA (single factor). The F-test for the mean difference of PUT option price between the market and BS, Trinomial, LV, and QSW at 95% confidence level.

Tukey-Kramer Multi	ple Compa	risons							
	Sample	Sample		Absolute	Std. Error	Critical			
Group	Mean	Size	Comparison	Difference	of Difference	Range	Results		
1: Diff. (BS-Put)	3.695	31	Group 1 to Group 2	4.471707	4.518215612	16.67	Means	are not diff	ferent
2: Diff. (LV -Put)	-0.77671	31	Group 1 to Group 3	0.061667	4.518215612	16.67	Means	are not diff	ferent
3: Diff. (T-Put)	3.633333	31	Group 1 to Group 4	4.003107	4.518215612	16.67	Means	are not diff	ferent
4: Diff. (QSW-Put))	-0.30811	31	Group 2 to Group 3	4.41004	4.518215612	16.67	Means are not different		ferent
			Group 2 to Group 4	0.4686	4.518215612	16.67	Means	are not diff	ferent
Other Data	1		Group 3 to Group 4	3.94144	4.518215612	16.67	Means	are not diff	ferent
Level of significance	0.05								
Numerator d.f.	4								
Denominator d.f.	120								
MSW	632.8424								
Q Statistic	3.69								

Table 54: Tukey-Kramer multiple pair comparisons for the mean-difference in PUT prices.

The critical-range in table (52) and (54) are computed using the following formula:

Critical Range =
$$Q_a \sqrt{\frac{MSW}{2} \left(\frac{1}{n_j} + \frac{1}{n_{j'}}\right)}$$
 (219)

Where Q_a is the ctirical value from the studentized range distribution; MSW the mean square within; n_j and $n_{j'}$ are the sample sizes from group j and j'. In this case, the Q-statistic represents the critical value from studentized range distribution with 4 and 120 degrees of freedom (Berenson and Krehbiel, 1992). Given enough time for the priceto develop, the mean-difference of option-prices would be the same across all models.

(iv) Analyse the relationship between two variables (simple linear regression).

- H_0 : There is no linear relationship between CALL option price and STOCK price, $H_0: \beta_1=0$ across the SAMPLE (the slope is zero)
- H_1 : There is a linear relationship between CALL option price and STOCK prices, $H_1: \beta_1 \neq 0$ across the SAMPLE (the slope is not zero)

Table 55: Inference about the slope of a linear regression (call price vs stock price).

Simple Linear Reg	ression Analys	is							Calculations			
Call vs Stock									b1, b0 Coefficients	0.0248	2.8667	
Regression S	tatistics								b1, b0 Standard Error	0.0005	5.0774	
Multiple R	0.9936								R Square, Standard Error	0.9872	25.0658	
R Square	0.9872								F, Residual df	2160.4372	28.0000	
Adjusted R Square	0.9867								Regression SS, Residual SS	1357393.3295	#########	
Standard Error	25.0658											
Observations	30								Confidence level	95%		
									t Critical Value	2.0484		
ANOVA									Half Width b0	10.4005		
	df	55	MS	F	Significance F				Half Width b1	0.0011		
Regression	1	1357393.3295	1357393.3295	2160.4372	0.0000							
Residual	28	17592.2786	628.2957									
Total	29	1374985.6081										
	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95%	Upper 95%				
Intercept	2.8667	5.0774	0.5646	0.5768	-7.5338	13.2672	-7.5338	13.26721				
х	0.0248	0.0005	46.4805	0.0000	0.0238	0.0259	0.0238	0.02594				

Exhibit (a): Regression data, call vs stock price.

Durbin-Watson Calculations	
Sum of Squared Difference of Residuals	45171.2255
Sum of Squared Residuals	17592.2786
Durbin-Watson Statistic	2.5677

Exhibit (b): Durbin-Watson caculations.

From table 55, exhibit (a), the P-value is 0.0000. This is less than the 0.05 significance

level, therefore the null hypothesis is rejected, subsequently there is a linear relationship between call option value and stock price in the sample "space". The Durbin-Watson statistic is 2.5677. which according to Doughterly (2011) confirms negative autocorrelation.

Table 56: Inference about the slope of a linear regression (put option price vs stock price).

Simple Linear Regression Analysis									Calculations		
Put vs Sock									b1, b0 Coefficients	0.0359	-11.4607
Regression Statistics									b1, b0 Standard Error	0.0012	11.7756
Multiple R	0.9837								R Square, Standard Error	0.9677	58.1337
R Square	0.9677								F, Residual df	839.6373	28.0000
Adjusted R Square	0.9666								Regression SS, Residual SS	2837581.1767	94626.9009
Standard Error	58.1337										
Observations	30								Confidence level	95%	
									t Critical Value	2.0484	
ANOVA									Half Width b0	24.1213	
	df	SS	MS	F	Significance F				Half Width b1	0.0025	
Regression	1	2837581.1767	2837581.1767	839.6373	0.0000						
Residual	28	94626.9009	3379.5322								
Total	29	2932208.0776									
	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95%	Upper 95%			
Intercept	-11.4607	11.7756	-0.9733	0.3388	-35.5820	12.6606	-35.5820	12.66063			
х	0.0359	0.0012	28.9765	0.0000	0.0334	0.0385	0.0334	0.03846			

Exhibit (a): Regression data, put vs stock price.

Durbin-Watson Calculations	
Sum of Squared Difference of Residuals	169736.1697
Sum of Squared Residuals	94626.9009
Durbin-Watson Statistic	1.7937

Exhibit (b): Durbin-Watson caculations.

From table 56, exhibit (a), the P-value is 0.0000. This is less than the 0.05 significance level, therefore the null hypothesis is rejected, subsequently there is a linear relationship between put option value and stock price. The Durbin-Watson statistic is 1.7937. which according to Doughterly (2011) confirms positive autocorrelation.

According to Doughterly (2011), autocorrelation exits if residuals in one time period are related to residuals in another period. The Durbin-Watson statistic is used to compare residuals over time. The put option price displays positive autocorrelation, thus indicating that the put option price on the 25/05/2019 has a positive correlation with the put option price on 26/05/2019. It also means that if the call price fell on the 25th of May 2019, it is also likely that it will fall in the next trading day. The call price displays negative autocorrelation, which means that it has a negative influence on itself over time, such that a drop in price on the observation day (25/05/2019), would mean it is quite probable that it will rise in the following trading day. The finding is import because it allows for the use of the statistical sample as a price forecasting device across assets.

8.0 CONCLUSIONS

In study we have considered a postulate-implied formulation that allows us to develop the necessary mathematics on asset and financial derivative pricing. We have shown that starting with a generalized and augmented Schrodinger expression, we can derive the Gaussian density function, but also that it is a special case and that there is the effect of quantization at zero-dimension time point which impacts not only on the probability density function, but on the asset and financial derivatives valuation. We also have shown that through our master-formulation we can derive a replica of the Black-Scholes PDE which is traditionally achieved through the Ito-Lemma, the results were exact, but our postulate-implied master-formulation allowed us to lift the PDE to a more abstract level and to set the problem in such a way that we may seek Sturm-Liouville solutions.

Moreover the theoretical model, composed for the purpose of calculating the asset and derivatives' prices worked well through eigenvalue conversion. We had started this research with a mere idea that we worked out in a system of equations linked to a Master-formulation. We advanced by choosing several price cut-off equations and solved our maximized expression for each cut-off price potential function.

For "spherical" price-developing systems or quantum price points, we used spherical polar coordinates for the quantum-topological space. On the path to obtain the Eigne-values, we used NAG_Routines and C++ programming with numerical recipes. Although, theoretically we had a "say" in the choice of the shape of the price-preservation system or the zero-dimension object, this is not a case that can be observed, but can be treated on assumption.

We applied a criterion based on assumptions on zero-object price behaviour and theoretical reasoning to come to the conclusion that the "suitable" price cut off functions are (i) Constant/Square Well, (ii) Gaussian, (iii) Cosh-2, , (iv) Arctan (for small values of b), and (v) the Harmonic. We established that the most generalized solution of the master-equation had a constant square well price cut-off potential. This was used to price European options with very good results.

The data generated by the programs was analyzed and from such analysis we have seen that increasing price-conservation system's size and depth of the quantized topological space, caused increases on the range of Eigen-values. The Eigen-prices generated for the Arctan (for **b** very small), were very similar to the Eigen-values generated by the squarewell potential_program. The comparison between the price Eigen-values generated for the Gaussian and the COSH gammas, show that there are similarities in their values. Although those similarities were more obvious at market surface. We developed further the model by adding a quantifier related to degree of information penetration of the market or the quality of the information as well as dependent on market external governance. We then converted Eigen-price values to expectations in possible forecasts in our "world". The price vectors were connected back to back with existing methods not only for comparison, but also for option valuation. Computation of options prices (European calls and puts were used as proof of concept constructs) was completed with encouraging results. Where the Black-Scholes-Merton model appeared to verv overestimate option prices, our selected quantum sub-model appeared to be more conservative.

A sample of 30 equity and equity index options was tested using a variety of classical models as well as a calibrated quantum square model. Further statistical hypothesis were carried out. All in all the local volatility model appears to yield close to the market option prices. The price-surface quantum square well model yields closer option prices to the market and leads overall to improved results.

A sub-model was composed on calculating the transition prices for price forecasting purposes and plotting them against the radius of the information-conservation system or the zero-object. This was based on the fact that changes of the quantum point size, caused changes in price Eigen-values, hence the transition price sets between the ground levels. A plot of the transition or forward price against the radius of the of the zero object was obtained.

Analysis and plots of the "Intensity" of the information reflection and dissipation with respect to the forward prices, could have been the next step, had the time permitted. Also an "upgrade" of our theoretical model to account for other market-motive effects or additional parameters and price cut-off "geometries" would be a useful extension of our work.

This is a domain needs more time and attention, however we have accomplished what we set out at the start of the work on this research study, and overall the results show that the intuition behind the main axiom is correct.

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10. APPENDIX I

- BROWNIAN MOTION SIMULATION USING EULER DISCRETIZATION -

* Simulation of GBM using Euler discretization scheme

'* By: Adrian Euler

Option Explicit

Const TWELVE = 12

Const SIX = 6

Private keep_on_going As Boolean

Private Type GBM_process

mu As Double 'Stock drift

sigma As Double	'Stock volatil	ity	
rr As Double	'Stock short rat	e	
S_0 As Double	'Initial stock p	rice	
T As Double	'Final time		
N As Long	'Number of time	e steps	
d_t As Double	'The time step		
sigma_sqrt_d_t As Do	ouble 'Speed-up	o bit	
End Type			
*****	******	******	*****
'* Sub main to read	in parameters fro	om the front end and to	
'* write a BS value	to the front end		
Sub Main()			
Dim my_GBM_proce	ess As GBM_proc	cess	
Dim Elapsed_time As	s Double	'Time to complete	
Dim loop_counter As	Long		
Elapsed_time = Time	r	'Get the time (in seconds)	
keep_on_going = Tru	e		
loop_counter = 0			
Sheet1.Cells(9, 4).Va	lue = ""		

Application.OnDoubleClick = "my_DoubleClick" 'Escape from the loop by double clicking

Randomize 'Randomize sets the random seed for the VBA function Rnd

If Not Read_in_parameters(my_GBM_process) Then Exit Sub

Do Until keep_on_going = False

If Application.Wait(Now + TimeValue("00:00:05")) Then

 $loop_counter = loop_counter + 1$

Sheet1.Cells(9, 4).Value = loop_counter

If Not Simulate(my_GBM_process) Then Exit Sub

DoEvents

End If

Loop

Elapsed_time = Timer - Elapsed_time

Sheet1.Cells(8, 4).Value = Elapsed_time

Beep

MsgBox "Programme has finished"

End Sub

'* Simulate

'* A procedure to simulate a GBM

Function Simulate(ByRef this_GBM_process As GBM_process) As Boolean

Dim i As Integer

Dim next_value As Double

Dim current_value As Double

Dim normal_number As Double

Simulate = False

'* Simulate the path

For i = 1 To this_GBM_process.N

Sheet1.Cells(15 + i, 9).Value = i

Next i

current_value = this_GBM_process.S_0

Sheet1.Cells(15, 9).Value = 0

'Initialise stuff

Sheet1.Cells(15, 10).Value = current_value

Application.ScreenUpdating = False

For i = 1 To this_GBM_process.N

'For each time step

```
'Get a 'normal' variate
   normal_number = normal_poor
   next_value = Next_S(this_GBM_process, current_value, normal_number) 'Evolve
the next stock price
   current_value = next_value
   Sheet1.Cells(15 + i, 10).Value = current_value
                                                    'Write out the
current step
 Next i
 Application.ScreenUpdating = True
 Simulate = True
Exit Function
1*******
'*
     The error handling subroutine
error_label:
 Beep 'To annoy the user
 MsgBox prompt:="Error encountered: " & Err & " " & Error(), _
     Buttons:=vbCritical, _
     Title:="Error in main"
End Function
```

'* Read_in_parameters()

'* Reads in stuff from the front end

'*

'*

Function Read_in_parameters(ByRef this_GBM_process As GBM_process) As Boolean

Read_in_parameters = False

On Error GoTo error_label

this_GBM_process.mu = Sheet1.Cells(15, 4).Value

this_GBM_process.sigma = Sheet1.Cells(16, 4).Value

this_GBM_process.rr = Sheet1.Cells(17, 4).Value

this_GBM_process.S_0 = Sheet1.Cells(18, 4).Value

this_GBM_process.T = Sheet1.Cells(16, 7).Value

this_GBM_process.N = Sheet1.Cells(15, 7).Value

 $this_GBM_process.d_t = this_GBM_process.T / this_GBM_process.N$ 'The

time step

this_GBM_process.sigma_sqrt_d_t = this_GBM_process.sigma * Sqr(this_GBM_process.d_t) If this_GBM_process.mu > 0 And _

this_GBM_process.sigma > 0 And _

this_GBM_process.S_0 > 0 And _

this_GBM_process.T > 0 And $_$

this_GBM_process.N > 0 _

Then

Read_in_parameters = True

Exit Function

End If

error_label:

Beep

MsgBox ("Data is invalid")

End Function

'* normal_poor()

'* generates a normal variate from 12 rnd

Function normal_poor() As Double

Dim running_total As Double

Dim i As Integer

 $running_total = 0$

For i = 1 To TWELVE

running_total = running_total + Rnd()

Next i

normal_poor = running_total - SIX

End Function

'* Next_S(Previous_S as double, z as double) as double

'* Generates the next value of the stock price S

Function Next_S(this_GBM_process As GBM_process, current_value As Double, z As Double) As Double

'Evolves state variable using an Euler discretisation lg ln S.

'z is ~N(0,1)

 $Next_S = current_value$

* Exp((this_GBM_process.rr - 0.5 * this_GBM_process.sigma ^ 2) _

* this_GBM_process.d_t + z * this_GBM_process.sigma_sqrt_d_t)

End Function

'* my_DoubleClick traps the DoubleClick event

Sub my_DoubleClick()

keep_on_going = False

End Sub

11. APPENDIX II - MATHEMATICAL DERIVATIONS -

CASE 1.0:

Starting with the master equation (chapter 4 eq., 19), when $n \ge 0$ and $m \ge 0$, we seek a time-independent solution of the form:

$$f(\chi, t) = \psi(\chi)\xi(t)$$
$$\frac{\partial f(\chi, t)}{\partial \chi} = \frac{\partial \psi(\chi)}{\partial x}\xi(t)$$
$$\frac{\partial^2 f(\chi, t)}{\partial \chi^2} = \frac{\partial^2 \psi(\chi)}{\partial \chi^2}\xi(t)$$
$$\frac{\partial f(\chi, t)}{\partial t} = \frac{\partial \xi(t)}{\partial t}\psi(\chi)$$

Where $\gamma(\chi, t) = \kappa \eta(\chi)$ and k is a constant.

Substituting back to the master equation, we obtain

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \kappa\eta(\chi)\right]\psi(\chi) = \left(i^n\right)^m \left[\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right]\psi(\chi)$$
(C1-1)

where $i = \sqrt{-1}$.

Two functions of different separable variables can only be equal for all values of their arguments if each is equal to the same constant, denoted here by α . We subsequently obtain

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \kappa\eta(\chi)\right]\psi(\chi) = \left(i^n\right)^m \left[\frac{1}{\xi(t)}\frac{\partial\xi(t)}{\partial t}\right]\psi(\chi) = \alpha\psi(\chi) \qquad (C1-2)$$

Which we express in a compact and generic form as

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \kappa\eta(\chi)\right]\psi(\chi) = \alpha\psi(\chi) \qquad (C1-3)$$

where α is given by the expression

$$\alpha = \left(i^{n}\right)^{m} \left[\frac{1}{\xi(t)} \frac{\partial \xi(t)}{\partial t}\right].$$
(C1-4)

CASE 3.0:

Under the latter case, equation (24) in chapter 4.0 can be written as

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \kappa\eta(\chi)\right]_{\Pi(\chi) = -r_{\rm f}\Pi(\chi)}$$
(C3-1)

Or, expressed in a more compact form as

$$-\frac{1}{2}\left[\frac{\partial^2\Pi(\chi)}{\partial\chi^2}\right] + q(\chi)\Pi(\chi) = 0 \qquad (C3-2)$$

where

$$q(\chi) = r_{f} + \kappa \eta(\chi) \qquad (C3-4)$$

Before any possible evaluation of chapter 4.0 equations (23) and (C3-1) can be done, a concrete expression for $q(\chi)$ needs to be established. We consider price of stock at time t to be S_t, where S_t is determined by the stochastic differential equation

$$dS_{t} = S_{t} (\mu dt + \sigma dW_{t})$$
 (C3-5)

with { W_t , $t \ge 0$ } being a standard Brownian motion and $\sigma > 0$, μ are constants; parameter σ is known as the volatility and μ the rate of return (Karatzas and Shreve, 1998). We consider next a stochastic differential equation for exponential Brownian motion. Consider the exponential Brownian motion $S_t = S_0 \exp \{\sigma W_t + \mu t\}$, where W_t is standard Brownian motion and S_0 is a constant (Karatzas and Shreve, 1998; Malliaris, 1982; Øksenda, (2000, Bru et al., 2002). We then apply Itô's Lemma with $X_t = W_t$, so that $Y_t \equiv 0$ and $Z_t \equiv 1$, and with $f(\chi, t) = S_0 \exp \{\sigma \chi + \mu t\}$ to obtain

$$dS_{t} = df(W_{t}, t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^{2} f}{\partial \chi^{2}}\right)dt + \frac{\partial f}{\partial \chi}dW_{t} = S_{t}\left[\left(\mu - \frac{1}{2}\sigma^{2}\right)dt + \sigma dW_{t}\right]$$
(C3-6)

Comparing the terms, allows us to extract the two expressions below:

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{t}} + \frac{1}{2}\frac{\partial^2 \mathbf{f}}{\partial \chi^2}\right) = \mathbf{s}_t \left(\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\sigma}^2\right) \qquad (C3-7)$$

and

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \sigma \mathbf{S}_{\mathbf{t}} \qquad (C3-8)$$

where we consider $S_t \equiv f$, and re-write (C3-7), using the same asset price function notation on both sides.

$$\frac{\partial \mathbf{S}}{\partial t} + \frac{1}{2} \frac{\partial^2 \mathbf{S}}{\partial \chi^2} = \mathbf{S}_t \left(\mu - \frac{1}{2} \sigma^2 \right) \qquad (C3-9)$$

which is then re-arranged and reset at a quasi-zero point in the time evolving path.

$$-\frac{1}{2}\frac{\partial^{2}S}{\partial\chi^{2}} + \left[\left(\mu - \frac{1}{2}\sigma^{2}\right) - \frac{\partial}{\partial t}\right]S_{t} = 0 \qquad (C3-10)$$

Comparing expressions (24 in chapter 4.0) and (C3-1), we obtain the identity of q to be independent of χ and of the form

$$q = \left[\left(\mu - \frac{1}{2} \sigma^2 \right) - \frac{\partial}{\partial t} \right] = \mu - \frac{1}{2} \sigma^2$$
 (C3-11)

At a quasi-zero dimension point in the asset price time-evolution path, operator $\partial/\partial t$ diminishes to zero, thus allowing us to simplify expression (C3-10).

$$\kappa \eta = \mu - \frac{1}{2}\sigma^2 \qquad (C3-12)$$

Subsequently, we establish the identity of q as

$$q = \mathbf{r}_{f} + \kappa \eta = \mathbf{r}_{f} + \left(\mu - \frac{1}{2}\sigma^{2}\right) \qquad (C3-13)$$

Finally, we can re-write equations (C3-10) and (C3-13) as

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \left(\mu - \frac{1}{2}\sigma^2\right)\right] \mathbf{s}(\chi) = 0 \qquad (C3-14)$$

and

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \left(r_{\rm f} + \left(\mu - \frac{1}{2}\sigma^2\right)\right)\right]\Pi(\chi) = 0 \qquad (C3-15)$$

CASE 4.0:

Consider the GSE with a simple harmonic $q(\chi) = k\eta(\chi)$ of the form:

$$\kappa \eta(\chi) = \frac{1}{2} \kappa \chi^2 \qquad (C4-1)$$

We substitute (C4-1) back to the GSE (master expression) to obtain

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\chi^2}\right) + \frac{1}{2}\kappa\chi^2\right]\psi(\chi) = \alpha\psi(\chi) \qquad (C4-2)$$

Further, we re-arrange (C4-2) in the compact form

$$\frac{\partial^2 \psi(\chi)}{\partial \chi^2} + (2\alpha - \kappa \chi^2) \psi(\chi) = 0 \qquad (C4-3)$$

We recall that random variable χ incorporated is a function $\chi : \Omega \to \mathbb{R}$ measurable with respect to \mathcal{F} ; that is all events ($\chi \leq c$) = ($\Omega : \chi(\Omega) \leq c$) $\in \mathcal{F}$ for all real numbers c $\in \mathbb{R}$, and the smallest σ -field with respect to which a random variable χ is measurable is the σ -field generated by χ , which we denote $\sigma(\chi)$ (Siminelakis, Paris, 2010; Shirayaev et al., 2006). We then use the differential equation and maximisation principles of Protter and Weinberger (1984) to obtain an expression for the optimum trading frequency

$$\upsilon = \frac{1}{2\pi} \sqrt{2\alpha - \kappa^2 \chi^2} \qquad (C4-5)$$

where the time that it takes to complete one cycle of one maximum and one minimum is given by

$$\tau = 2\pi\sqrt{2\alpha - \kappa^2 \chi^2} \qquad (C4-6)$$

Using the stochastic process that produces a harmonic pattern with the special case of k = 1, we obtain a compact expression.

$$\frac{\partial^2 \psi(z)}{\partial \chi^2} + \left(2\alpha - \chi^2\right) \psi(\chi) = 0 \qquad (C4-7)$$

We expect solutions of equation (C4-7) to show rapid decline at $\chi \rightarrow \pm \infty$. A simple inspection of the asymptotic form i.e. $\chi \gg 2\alpha$, shows $\psi \sim \exp(-\chi^2/2)$ is a solution in this region (Eugene and O'Donnell, 1997). Recall equation (29) from chapter 4.0, that a similar effect is achieved without the inspection of the asymptotic form, so therefore we can use this solution without loss of generality. This therefore suggests the presence of

general solutions of the form $F(\chi)exp(-\chi^2/2)$, where F is a polynomial (Eugene and O'Donnell, 1997). Substituting this form into equation (C4-7) yields an expression for the first and second order partial differentials. After differentiation with respect to χ , we obtain a first order partial derivative

$$\frac{\partial}{\partial \chi} \psi(\chi) = \frac{\partial}{\partial \chi} F(\chi) e^{\left(-\frac{1}{2}\chi^2\right)} - \chi F(\chi) e^{\left(-\frac{1}{2}\chi^2\right)}$$
(C4-8)

Furthermore, we also apply a second partial derivation to obtain

$$\frac{\partial^{2}}{\partial\chi^{2}}\psi(\chi) = \frac{\partial^{2}}{\partial\chi^{2}}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} - \chi\frac{\partial}{\partial\chi}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} - F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} - \chi\frac{\partial}{\partial\chi}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} + \chi^{2}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi^{2}}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} - 2\chi\frac{\partial}{\partial\chi}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} - F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} + \chi^{2}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi^{2}}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} - 2\chi\frac{\partial}{\partial\chi}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} - F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} + \chi^{2}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi^{2}}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi^{2}}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi}F(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi}F(\chi)e^{2}(\chi)e^{2}(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi}F(\chi)e^{2}(\chi)e^{2}(\chi)e^{\left(\frac{1}{2}\chi^{2}\right)} = \frac{\partial^{2}}{\partial\chi}F(\chi)e^{2}($$

(C4-9)

Then, we substitute (C4-8) back to (C4-9) to obtain

$$\frac{\partial^2}{\partial \chi^2} F(\chi) e^{\left(\frac{1}{2}\chi^2\right)} - 2\chi \frac{\partial}{\partial \chi} F(\chi) e^{\left(\frac{1}{2}\chi^2\right)} - F(\chi) e^{\left(\frac{1}{2}\chi^2\right)} + \chi^2 F(\chi) e^{\left(\frac{1}{2}\chi^2\right)} + 2\alpha F(\chi) e^{\left(\frac{1}{2}\chi^2\right)} - \chi^2 F(\chi) e^{\left(\frac{1}{2}\chi^2\right)} = 0$$

(C4-10)

which, after simplification, can be written

$$\frac{d^2F}{d\chi^2} - 2\chi \frac{dF}{d\chi} + (2\alpha - 1)F = 0 \qquad (C4-11)$$

There might also exist non-polynomial i.e. infinite series, solutions of equation (C4-11), but they all have divergent behaviour at large χ (Birkoff and Rota, 1962). Suppose the leading term of F is χ^n . This contributes

$$n(n-1)\chi^{n-2} - 2n\chi^n + (2\alpha - 1)\chi^n$$
 (C4-12)

to the left of equation (C4-11). The coefficient of χ^n must vanish to comply with equation (C4-11) and as lower-order terms in the polynomial F only contribute to χ^{n-1} or lower powers, we demand from equation (C4-12) that

$$\alpha = n + \frac{1}{2}$$
 where $n = 0, 1, 2, 3, \dots$ (C4-13)

The probability density ground state is obtained at n = 0, in which case F is a constant, so that $u \propto \exp(-\chi^2/2)$, and with the application of the normalization condition, we obtain a Gaussian function (Malliaris, 1982; Øksendal, 2000, Bru et al., 2002). Subsequently, we establish the expectation value for χ at the ground state to be

$$E_{0}(\chi) = \int_{\Omega} \chi(\Omega) dP_{0} = \int_{-\infty}^{+\infty} \chi u_{0}^{2} d\chi = 0 \qquad (C4-14)$$

To construct the $n \ge 1$ probability density functions, one must substitute a full polynomial for F in equation (C4-11) and equate the coefficients of all powers (not just χ^n) to zero. One obtains the following set of functions for F (to within an overall constant) 1, 2χ , $4\chi^2 - 2$, $8\chi^3 - 12\chi$, for n =0, 1, 2, 3,.... These functions are well known to 394 mathematicians as Hermite polynomials, denoted $H_n(\chi)$, and are discussed in great detail in Walter (1977), and (Szego, 1939). Equation (106) is a Hermite equation. The $H_n(\chi)$ may all be generated by successive operations as follows:

$$H_n(\chi) = (-1)^n e^{\chi^2} \frac{d^n}{d\chi^n} e^{-\chi^2}$$
 (C4-15)

The normalization factors may be evaluated using the properties of Hermite polynomials (Szego, 1939). One obtains the ψ function

$$\psi_{n}(\chi) = \left(\frac{\alpha}{\sqrt{\pi}}2^{n}n!\right)^{\frac{1}{2}}H_{n}(\alpha\chi)e^{-\frac{1}{2}(\alpha\chi)^{2}} \quad (C4-16)$$

and subsequently in full alignment with the work of Schwartz (1967), the probability density function is:

$$u_n(\chi) = |\psi_n(\chi)|^2 = \left(\frac{\alpha}{\sqrt{\pi}} 2^n n!\right) |H_n(\alpha \chi)|^2 e^{-(\alpha \chi)^2}$$
 (C4-17)

CASE 5.0:

Starting with GSE⁸⁵, we seek a solution of the form:

$$f(x,t) = \psi(x,t)\xi(t)$$
 (C5-1)

Next, we take the partial derivatives with respect to χ and time, following the rules of differentiation (Birkoff and Rota, 1962)

⁸⁵ Generalised Schrodinger Equation or referred equivalently as the Master expression.
$$\frac{\partial f(\chi,t)}{\partial \chi} = \xi(t) \frac{\partial \psi(\chi,t)}{\partial \chi}$$
(C5-2)

$$\frac{\partial^2 f(\chi, t)}{\partial \chi^2} = \xi(t) \frac{\partial^2 \psi(\chi, t)}{\partial \chi^2}$$
(C5-3)

$$\frac{\partial f(\chi, t)}{\partial t} = \xi(t) \frac{\partial \psi(\chi, t)}{\partial t} + \psi(\chi, t) \frac{\partial \xi(t)}{\partial t}$$
(C5-4)

Substituting these expressions back into the master formula, we obtain

$$\left[-\frac{1}{2}\frac{\partial^{2}}{\partial\chi^{2}}+\gamma(\chi,t)\right]\xi(t)\psi(\chi,t)=\left(i^{n}\right)^{m}\left[\xi(t)\frac{\partial \psi(\chi,t)}{\partial t}+\psi(\chi,t)\frac{\partial \xi(t)}{\partial t}\right]$$
(C5-5)

Next, we divide both sides by ξ to obtain:

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2} + \gamma(\chi, t)\right]\psi(\chi, t) = \left(i^n\right)^m \left[\frac{\partial \psi(\chi, t)}{\partial t} + \psi(\chi, t)\left(\frac{1}{\xi(t)}\frac{\partial \xi(t)}{\partial t}\right)\right] \quad (C5-6)$$

We set $\alpha = (i^n)^m [\frac{1}{\xi(t)} \frac{\partial \xi(t)}{\partial t}]$ in exactly the same manner as in all scenarios in the previous

chapter, and re-arrange the equation to obtain

$$-\frac{1}{2}\frac{\partial^2\psi(\chi,t)}{\partial\chi^2} + \left[\gamma(\chi,t) - \alpha\right]\psi(\chi,t) = \left(i^n\right)^m \left[\frac{\partial\psi(\chi,t)}{\partial t}\right] \qquad (C5-7)$$

Thus obtaining the desired expression as set out in this theorem.

CASE 7:

Starting with the GSE, we seek a general solution of the form

$$f(\chi,t) = \psi(\chi,t)\xi(\chi,t) \qquad (C7-1)$$

We assume that both functions are dependent on χ and t. Further on we take the partial derivatives of (C7-1). First order partial derivative of $f(\chi, t)$ with respect to χ :

$$\frac{\partial f(\chi, t)}{\partial \chi} = \xi \frac{\partial \psi}{\partial \chi} + \psi \frac{\partial \xi}{\partial \chi} \qquad (C7-2)$$

Second order partial derivative of $f(\chi, t)$ with respect to χ :

$$\frac{\partial^2 f(\chi, t)}{\partial \chi^2} = \xi \frac{\partial^2 \psi}{\partial \chi^2} + 2 \frac{\partial \xi}{\partial \chi} \frac{\partial \psi}{\partial \chi} + \psi \frac{\partial^2 \xi}{\partial \chi^2} \qquad (C7-3)$$

First order partial derivative of $f(\chi, t)$ with respect to t:

$$\frac{\partial f(\chi, t)}{\partial t} = \xi \frac{\partial \psi}{\partial t} + \psi \frac{\partial \xi}{\partial t}$$
(C7-4)

Which are then substituted back in the master expression to obtain

$$-\frac{1}{2}\left[\xi\frac{\partial^2\psi}{\partial\chi^2} + 2\frac{\partial\xi}{\partial\chi}\frac{\partial\psi}{\partial\chi} + \psi\frac{\partial^2\xi}{\partial\chi^2}\right] + \gamma(\chi)\xi\psi = (i^n)^m\left[\xi\frac{\partial}{\partial t} + \frac{\partial\xi}{\partial t}\right]\psi \qquad (C7-5)$$

We divide each term by ξ to obtain

$$-\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial \chi^2} + 2\frac{1}{\xi}\frac{\partial \xi}{\partial t}\frac{\partial t}{\partial \chi}\frac{\partial \psi}{\partial \chi} + \psi\frac{1}{\xi}\frac{\partial^2 \xi}{\partial t^2}\frac{\partial t^2}{\partial \chi^2}\right] + \gamma(\chi)\psi = (i^n)^m \left[\frac{\partial}{\partial t} + \frac{1}{\xi}\frac{\partial \xi}{\partial t}\right]\psi \quad (C7-6)$$

Then set

$$\frac{1}{\xi}\frac{\partial \xi}{\partial t} = \alpha \Longrightarrow \xi = e^{\alpha t}; \ \frac{\partial^2 \xi}{\partial t^2} = \alpha^2 \xi, \quad (C7-7)$$

and substitute back in equation (C7-6) to obtain

$$-\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial \chi^2} + 2\alpha \frac{\partial t}{\partial \chi} \frac{\partial \psi}{\partial \chi} + \psi \alpha^2 \frac{\partial t^2}{\partial \chi^2}\right] + \gamma(\chi)\psi = (i^n)^m \left[\frac{\partial}{\partial t} + \alpha\right]\psi \qquad (C7-8)$$

Under the assumption of no explicit probability density identity presence (n=0), we obtain $(i^{\circ})^{m} = 1$, and then re-arrange equation (C7-8) to obtain

$$-\frac{1}{2}\left[\frac{\partial\chi^{2}}{\partialt^{2}}\frac{\partial^{2}\psi}{\partial\chi^{2}}+2\alpha\frac{\partial\chi}{\partialt}\frac{\partial\psi}{\partial\chi}+\psi\alpha^{2}\right]+\gamma(\chi)\psi=\frac{\partial\psi}{\partialt}+\alpha\psi \qquad (C7-9)$$

which is then further simplified into

$$-\frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} + 2\alpha \frac{\partial \psi}{\partial t} + \psi \alpha^2 + \gamma(\chi)\psi = \frac{\partial \psi}{\partial t} + \alpha \psi$$
(C7-10)

We re-arrange it

$$\left[\frac{\partial^2 \psi}{\partial t^2} + 2\alpha \frac{\partial \psi}{\partial t} + \psi \alpha^2\right] + 2 \frac{\partial \psi}{\partial t} + 2\alpha \psi = 2\gamma(\chi)\psi \qquad (C7-11)$$

to further obtain

$$\left[\frac{\partial^2 \psi}{\partial t^2} + 2(\alpha + 1)\frac{\partial \psi}{\partial t} + (\alpha^2 + 2\alpha)\psi\right] = 2\gamma(\chi)\psi \qquad (C7-12)$$

Which clearly has resulted in a separation of variables χ and t. We add one unit of ψ on both sides to reduce the expression to a more compact form

$$\left[\frac{\partial^2 \psi}{\partial t^2} + 2(\alpha + 1)\frac{\partial \psi}{\partial t} + (\alpha^2 + 2\alpha + 1)\psi\right] = \left[2\gamma(\chi) + 1\right]\psi \qquad (C7-13)$$

or

$$\left[\frac{\partial^2 \psi}{\partial t^2} + 2(1+\alpha)\frac{\partial \psi}{\partial t} + (1+\alpha)^2 \psi\right] = [2\gamma(\chi) + 1]\psi \qquad (C7-14)$$

Separation of variables is persistent and the left hand side can be equal to the right hand side if both are equal to a constant. We may contemplate a special case by setting it to 0, which implies that $\gamma(\chi) = -1/2$. Thus we obtain a PDE representation that is entirely independent of stochastic process variable χ .

$$\left[\frac{\partial^2 \psi}{\partial t^2} + 2(1+\alpha)\frac{\partial \psi}{\partial t} + (1+\alpha)^2 \psi\right] = 0$$
 (C7-15)

We further re-arrange and write it in a compact form as

$$\left[\frac{\partial}{\partial t} + (1+\alpha)\right]^2 \psi = 0 \qquad (C7-16)$$

In all of the examples we have considered, the true identity of α has been either the interest rate, risk-free rate of return, or hypothesized rate of return. We consider a financial option-like identity for ψ with the $\alpha = r_f$, the rate of return of a riskless asset (Black, 1973; Jarrow and Turnbull, 1998; Hull, 2014; Ho and Lee, 2015).

$$\left[\frac{\partial}{\partial t} + (1 + r_{f})\right]^{2} \psi = [1 + 2\gamma]\psi \qquad (C7-17)$$

If we apply the same logic as that leading to expression (80) in CASE 5.0 of chapter 5.0, where we consider a portfolio of one option with a legal right to claim h shares, where h is hedge and given by $h = \partial \psi / \partial s$ (Kennedy, 2010). Alternatively, we could consider a portfolio made up of a risk free investment and a certain number of shares (Bodie et al., 2009). The right-hand side of (104)

can be written

$$\psi + 2\gamma\psi = \psi + hs = \psi + s\frac{\partial\psi}{\partial s} = \Pi$$
 (C7-18)

Where Π represents the value of such portfolio. Expression (C7-18) allows us to establish the identity of cut-off price potential as an operator $\gamma = s(1/2) (\partial/\partial s)$, which is similar to that of CASE 5.0.

We re-write equation (C7-18) to obtain

$$-\left(\frac{\partial}{\partial t} + (1+r)\right)^2 \psi = \left(1 + s\frac{1}{2}\frac{\partial}{\partial s}\right)\psi = 0 \qquad (C7-19)$$

or

$$-\left(\frac{\partial}{\partial t} + (1+r)\right)^2 \psi + \frac{1}{2}s\left(\frac{\partial}{\partial s}\right)\psi + \psi = 0 \qquad (C7-20)$$