Computation of equivalent poles placement for class of 2nd order discrete bilinear systems

Gadek, L. , Koszalka, L. and Burnham, K.

Original citation:
Gadek, L. , Koszalka, L. and Burnham, K. (2015) Computation of equivalent poles placement for class of 2nd order discrete bilinear systems. Journal of Physics: Conference Series, volume 659 : 012003

URL: http://dx.doi.org/10.1088/1742-6596/659/1/012003
DOI: 10.1088/1742-6596/659/1/012003

Publisher: IOP Science

Computation of equivalent poles placement for class of 2nd order discrete bilinear systems

2015 J. Phys.: Conf. Ser. 659 012003

(http://iopscience.iop.org/1742-6596/659/1/012003)

Computation of equivalent poles placement for class of 2nd order discrete bilinear systems

Lukasz Gadek [1], Leszek Koszalka [2], Keith Burnham [3]

Abstract. This paper introduces an adaptation of the classical linear control theory representation of zeros, poles and gain into a bilinear approach. The placement of poles at the complex plane is a complete description of plants dynamics; hence it is a convenient form from which calculation of various properties, e.g. rise time, settling time, is plausible. Such technique can be adjusted into the bilinear structure if poles of a quasi-linear representation (linear with respect to input) are concerned. The research outcomes with conclusion on the equivalent poles displacement and generalized rules for a 2nd order bilinear system equivalent poles input dependent loci. The proposed approach seems to be promising, as simplification of design and identification of a bilinear system increases transparency during modelling and control in practical applications and hence it may be followed by applicability of such structure in common industrial problems.

1. Introduction

Bilinear structure allows approximation of a non-linear (NL) plant into a decomposable form of linear model with NL term [1]. Initial industrial application utilizes mostly the property of bent steady state gain which improves modelling of water flow systems and industrial furnaces [2] where response is saturated gradually for high operating point (OP). By extending with the bilinear term, properties of the response become time variant with respect to current state (exceeding simplification to gain slide) as described in Section 2 and therefore a robust stability and behaviour prediction is required - pole-placement method, e.g. [3].

To achieve satisfactory performance in designing a bilinear plant controller ([4] and [3]), an efficient identification of the plant must be performed. The method on bilinear plant varying properties prediction is an extension allowing of more comprehensive understanding of the bilinear design. A similarity of equivalent poles movement with respect to OP and root locus of gain feedback system is observed in Section 3. Correlation of the classical root locus [5], pole-placement with output feedback [6] and equivalent (bilinear) poles locus will be a following stage of the research as highlighted in Section 4.1.

2. Bilinear structure

This chapter is aimed to introduce the bilinear structure in terms of mathematical representations (Section 2.1 and 2.2) and capabilities overview (Section 2.3). By extending

[1] Department of Systems and Computer Networks , Wroclaw University of Technology, Poland, e-mail: lukasz.gadek@pwr.edu.pl
[2] Department of Systems and Computer Networks , Wroclaw University of Technology, Poland
[3] Control Theory and Application Centre, Coventry University, United Kingdom

a linear model with a bi-linear term, the bilinear model is obtained. It can be illustrated with an example of auxiliary function $f(x,u)$ where x and u are time-dependant entities.

$$
\begin{aligned}
y = f(x,u) \qquad x \in \Re^n \qquad u \in \Re^m \\
y_{lin} = \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} u_j \\
y_{bil} = y_{lin} + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i u_j
\end{aligned} \tag{1}
$$

In (1) y_{lin} is an output of linear model which fulfils superposition rule. The following equation contains bilinear term $\sum_{i=1}^{n}\sum_{j=1}^{m} x_i u_j$ being attached to y_{lin}. A similar approach can be applied to the State Space form and other representations.

2.1. State space

State Space is the most popular in a context of bilinear modelling and can be found in recent publications, e.g. [7]. The formulation of a bilinear MIMO term from [8] is as presented in (2).

$$
\begin{aligned}
Fu \otimes x \\
F = [F_1 \; F_2 \; F_3 \; ... \; F_m] \\
F_i \in \Re^{nxn} \qquad u \in \Re^m \qquad x \in \Re^n
\end{aligned} \tag{2}
$$

In this paper only SISO systems are considered, hence (2) can be simplified into form of (3).

$$
\begin{aligned}
Fu \otimes x \rightarrow Nux \\
N \in \Re^{nxn} \qquad u \in \Re \qquad x \in \Re^n
\end{aligned} \tag{3}
$$

With (3) SISO State Space is established in (4).

$$
\begin{aligned}
x_{k+1} = Ax_k + Bu_k + Nu_k x_k \\
y_k = Cx_k \\
y_k, u_k \in \Re \\
x_k, B, C \in \Re^n \\
A, N \in \Re^{nxn}
\end{aligned} \tag{4}
$$

The root locus method introduced in this paper is based on the Transfer Function (TF) equivalent (presented in Section 2.2). The transmittance from state space is explicit if a canonical form [9] is utilized. Assuming observability and diagonal bilinear matrix, coefficient matrices from (5) are used.

$$
\begin{aligned}
A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & 0 & \end{bmatrix} \quad N = \begin{bmatrix} n_1 & 0 & 0 & \cdots & 0 \\ n_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_n & 0 & 0 & 0 & \end{bmatrix} \\
B = \begin{bmatrix} b_1 & b_2 & \cdots & b_m & 0 & \cdots \end{bmatrix}^T \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}
\end{aligned} \tag{5}
$$

Canonical form coefficients (5) are interchangeably used in difference equation (DE). Eq. 6 presents according Bilinear DE.

$$
y_k = -\sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=1}^{m} b_i u_{k-i} + \sum_{i=1}^{n} n_i u_{k-i} y_{k-i} \tag{6}
$$

Under strong assumption of constant input u the bilinear form may be approximated with a linear equivalent where $\tilde{a}_i = a_i - n_i u_{k-i}$. However such approximation can be used in the open loop control only, therefore it is impractical in the majority of industrial application.

2.2. Equivalent transfer function

Transfer Function (TF) can be achieved from DE by introducing z^i - time shift operator where i is quantity of samples shifted forward. In the case of bilinear model two types of the equivalent TF are computable:

- Assuming sluggish change of input (eg. slow PID or manual control) followed by u in denominator as in (7),
- For fast controllers or systems with high inertia where y assumed to be sufficiently constant be approximate as coefficient in the numerator.

Both forms are equivalents of TF as in each denominator or numerator contains time-varying variable while in classical approach these are static.

$$\frac{Y}{U} = \frac{b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} \cdots + a_n z^{-n} - n_1 z^{-1} u_{k-1} - \cdots - n_n z^{-n} u_{k-n}} \tag{7}$$

Natural step to recapture polynomial of static coefficients, is to assume u or y as approximately constant, e.g. if input change is negligible within last m samples then $U \approx u_{k-1} \approx \cdots \approx u_{k-m}$ (8). Such reasoning is a root of establishing the two types mentioned above.

$$\frac{Y}{U} = \frac{b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + (a_1 - n_1 u_{k-1}) z^{-1} \cdots + (a_n - n_n u_{k-n}) z^{-n}} \tag{8}$$

In this paper u in the denominator approach is used, although following the same procedure for both types leads to the identical conclusion. However, the latter approach requires more significant mathematical effort.

2.3. Properties

From equivalent TF (8) two main properties of bilinear system can be highlighted. Time shift operator z is a discrete substitute of Laplace's e^{st_s} where t_s is a sampling time of discrete system.

Therefore, knowing that for $t \to \inf$ Laplace operator is converging to zero, z assembles to 1. Input/output gain of a discrete system can be calculated as $\frac{\sum b}{\sum a}$. In case of bilinear system denominator coefficient is impacted by input. If input does not change for a certain number of samples after excitation (e.g. step change) then bilinear system gain can be represented as in Fig. 1 where ratio of steady state output to input (Y_{ss}/U_{ss}) is shaped depending on N.

Eigenvalues of the bilinear system are also impacted by bilinearity. Characteristic Equation (CE) from equivalent TF (8) has a form of:

$$z^n + (a_1 - n_1 u_{k-1}) z^{n-1} \cdots + (a_n - n_n u_{k-n}) = 0 \tag{9}$$

Where z satisfying (9) are poles of the system. Correlation between u and dislocation of poles is indisputable, its prediction and estimation is presented in Section 3.

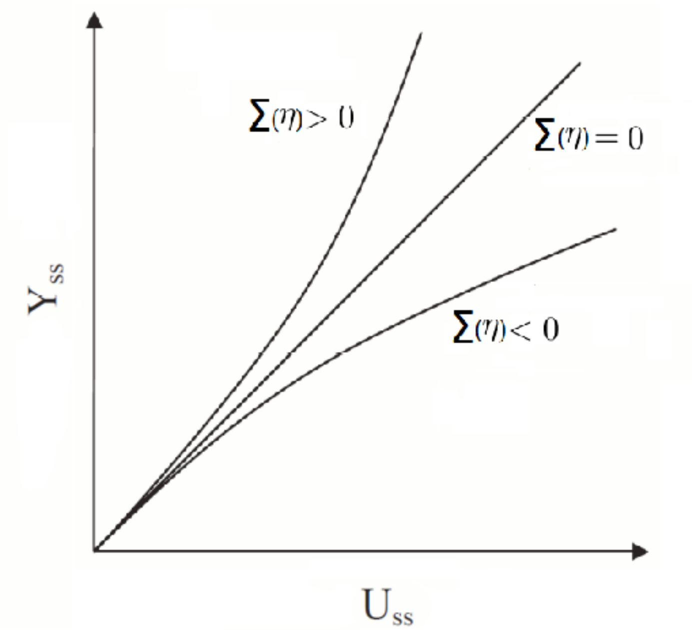

Figure 1. Steady State gain of bilinear continuous system accordingly to $\sum(n)$ - where n is first column of N from (5)

3. Equivalent poles loci

The aim of research is establishing a simple set of rules to predict perturbation of dynamical behaviour, i.e. movement of the equivalent poles, in a bilinear system. Derivation of LMI description of eigenvalues is performed in Section 3.1 while exemplary pole regions are plotted and commented in Section 3.2.

The reasoning is performed on diagonal bilinear and SISO system as described in Section 2.1. Due to simplification, initial derivation is utilizing approximated constant $U = u_{k-1\cdots n}$ which is valid if $u_{k-i} \approx u_{k-j} \forall i, j \in [0, m]$.

3.1. Analytical approach

In this section, solution of (9) with respect to U is derived. For simplicity 2nd order model with $n = 2$ is utilized. Pole computation (z) of a linear system is located in (10). Bilinear equivalent pole $\tilde{z}$ location is extended with n coefficient and U as presented in (11).

$$z = \frac{-a_1 \pm \sqrt{a_1^2 + 4a_2}}{2} \tag{10}$$

$$\tilde{z} = \frac{-a_1 + n_1 U \pm \sqrt{(a_1 - n_1 U)^2 - 4(a_2 - n_2 U)}}{2} \tag{11}$$

Initial step is calculation of eigenvalues with respect to imaginary plane, i.e. under condition $\Im(\tilde{z}) = 0$ - expanded in (12).

$$\Im(\tilde{z}) = 0 \iff 0 \leq (a_1 - n_1 U)^2 - 4(a_2 - n_2 U) \tag{12}$$
$$\Im(\tilde{z}) = 0 \iff 0 \leq n_1^2 U^2 - (2n_1 a_1 - 4n_2)U - 4a_2 + a_1^2$$

$$\Im(\tilde{z}) = 0 \iff U \in (-\infty, U^{out}] \vee U \in [U^{in}, \infty)$$

Basing on (12) break-out U^{out} and break-in U^{in} inputs can be calculated when (12) is equated to zero in. The result is presented in (13).

$$U^{out}, U^{in} = \frac{2n_1a_1 - 4n_2 \pm \sqrt{(-2n_1a_1 + 4n_2)^2 - 4(-4a_2 + a_1^2)n_1^2}}{2n_1^2} \tag{13}$$

Replacing U in (11) with (13) results with calculation of the break-out and -in points in the complex plane in (14).

$$\begin{aligned} \tilde{z}^{out}, \tilde{z}^{in} &= \frac{-a_1 + n_1 \left[\frac{2n_1a_1 - 4n_2 \pm \sqrt{(-2n_1a_1+4n_2)^2 - 4(-4a_2+a_1^2)n_1^2}}{2n_1^2}\right]}{2} \\ &= -\frac{n_2 \pm \sqrt{n_2^2 - n_1n_2a_1 + n_1^2a_2}}{n_1} \end{aligned} \tag{14}$$

From (12) it can be seen that the square root of (11) is a quadratic function. Therefore, $\Re(\tilde{z}) = \frac{|\tilde{z}^{out} - \tilde{z}^{in}|}{2}$ denotes a point on real axis at which extrema of pole imaginary part is achieved. Hence $U = -\frac{n_2}{n_1}$ is inserted into the square root in (11) resulting with (15).

$$\max|\Im(\tilde{z})| = \frac{\pm\sqrt{n_2^2 + n_1n_2a_1 - n_1^2a_2}}{n_1} i \tag{15}$$

Basing on extrema points of equivalent poles loci form (14) and (15) it can be deduced that the locus trajectory is a circle with centre in ($x = -\frac{n_2}{n_1}$, $y = 0$) (where x is real and y is imaginary axis) and range $r = \Im(\frac{\sqrt{n_2^2 - n_1n_2a_1 + n_1^2a_2}}{n_1})$ if $U \in [U^{out}, U^{in}]$. The circle equation in (16) for 2nd order bilinear system is formed.

$$\begin{aligned} (\Re(\tilde{z}) - x)^2 + (\Im(\tilde{z}) - y)^2 &= r^2 \\ (\frac{-a_1 + n_1U}{2} + \frac{n_2}{n_1})^2 + \frac{(a_1 - n_1U)^2 - 4(a_2 - n_2U)}{4} &= \frac{n_2^2 - n_1n_2a_1 + n_1^2a_2}{n_1^2} \end{aligned} \tag{16}$$

The equality is satisfied for all $U \in [U^{out}, U^{in}]$. Exemplary results obtained with numerical simulation are presented in Section 3.2.

Second order loci is always based on circle in range of $U \in [U^{out}, U^{in}]$ for the assumption of constant U. If U is changing within known range then the equivalent pole position must be amened accordingly to rate of change of input (17). However, according to LMI approach [10] it does not include internal instabilities which may result in discrepancies between prediction and actual state.

$$u_1 = Uu_2 = U + \Delta \tag{17}$$

Replacing U in (11) with (17) leads to amendment in the following calculations. Resulting locus formula is a circle based accordingly to (16) amended into (18).

$$\begin{aligned} (\Re_\Delta(\tilde{z}) - x_\Delta)^2 + (\Im_\Delta(\tilde{z}) - y_\Delta)^2 &= r_\Delta^2 \\ (\frac{-a_1 + n_1U}{2} + \frac{n_2}{n_1})^2 + \frac{(a_1 - n_1U)^2 - 4a_2 + 4n_2(U + \Delta)}{4} &= \\ &= \frac{n_2^2 - n_1n_2a_1 + n_1^2a_2 - n_1^2n_2\Delta}{n_1^2} \end{aligned} \tag{18}$$

Range of the circle is impacted with Δ while centre position is not. Moreover, it can be observed that the occurrence of Δ has impact exclusively on the vertical position of the pole; regardless to the rate of change, the position is not changed respect to the real axis. This allows to represent equivalent poles as a region in which (19) holds if maximum Δ is definable.

$$r - |n_2\Delta| \leq \left[\Re(\tilde{z}) + \frac{n_2}{n_1}\right]^2 + \Im_\Delta(\tilde{z})^2 \leq r + |n_2\Delta| \tag{19}$$

Where in (19) properties r, $\Re(\tilde{z})$ and centre of gravity are identical to (16).

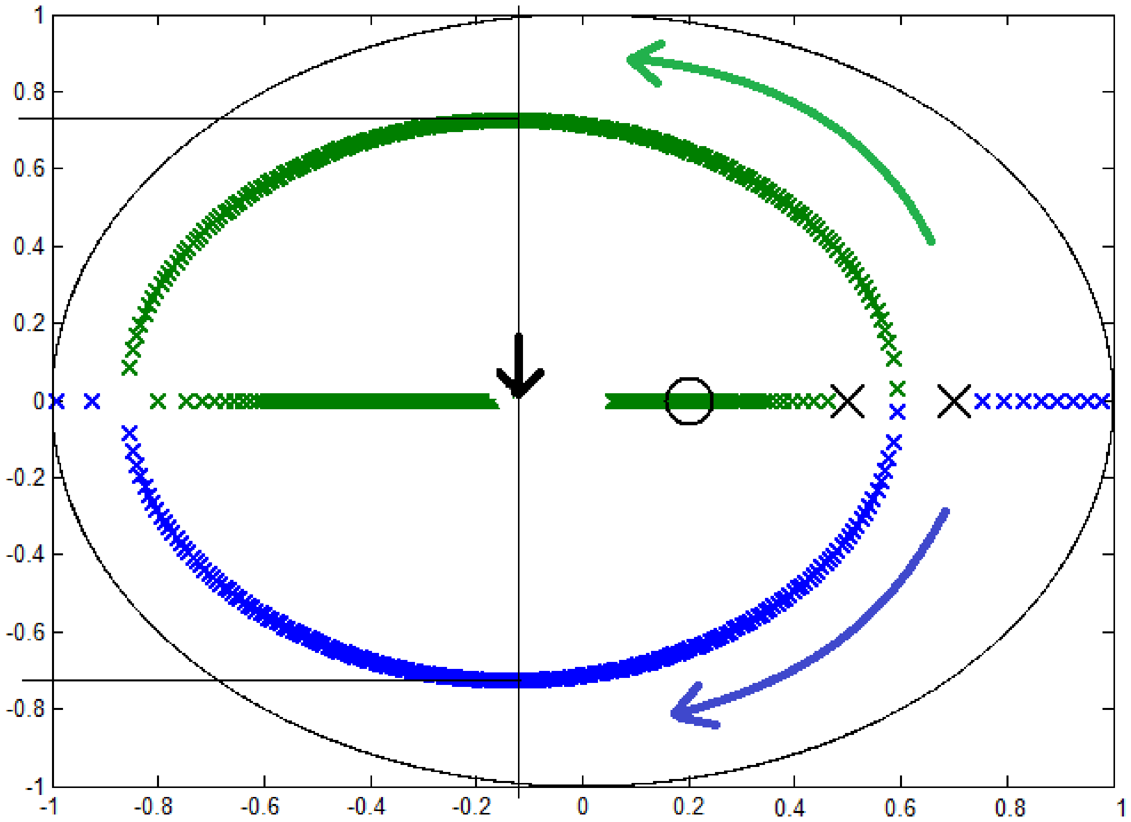

Figure 2. Equivalent poles loci with respect to increasing step input (arrow direction shows location for incrementing u)

3.2. Simulation

Method described in Section 3.1 has been validated numerically using MATLAB environment. Exemplary result of the locus for second order system with SS coefficients $A = \begin{bmatrix} 1.2 & 1 \\ -0.35 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -0.2 \end{bmatrix}^T$ and $N = \begin{bmatrix} 0.015 & 0 \\ 0.002 & 0 \end{bmatrix}$ is presented in Fig. 2. According to the (16) the trace of the poles is within circle with range of $r = \Im(\frac{\sqrt{0.002^2+0.002\cdot 0.015\cdot 1.2+0.015^2\cdot 0.35}}{0.015}) = 0.72$ and centre in $x = -\frac{0.002}{0.015} = -0.13, y = 0$.

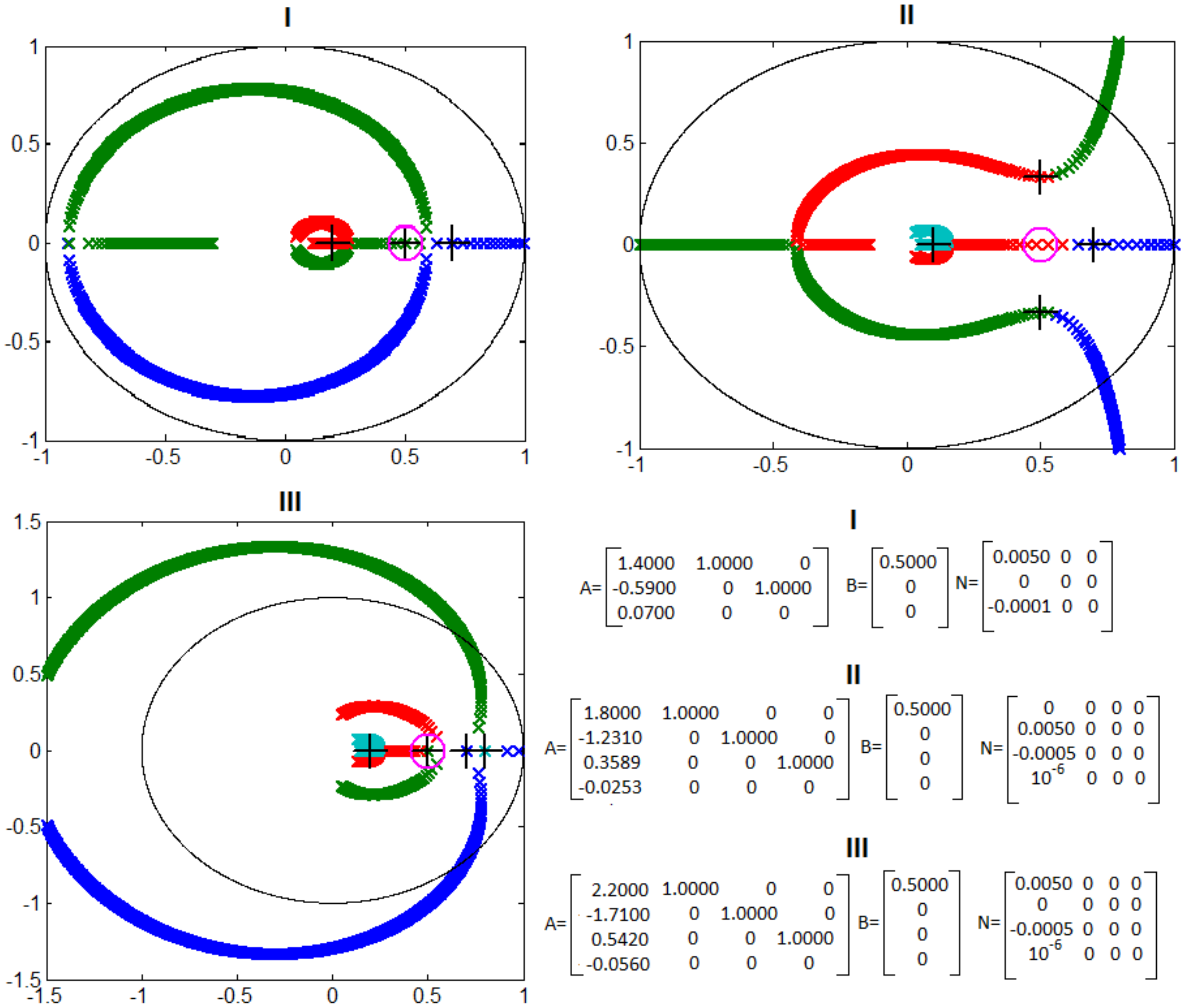

Figure 3. Exemplary Loci of higher order systems

For systems with higher orders the circular movement applies on corresponding basis. However, depending on number of poles and bilinear coefficient both: centres and trajectories might be marginally perturbed as presented in Figure 3 where:

- Plant I has two loci trajectories with $r_1 = 0.7842, r_2 = 0.08407$ and respective centres in $(-0.159, 0)$ and $(0.159, 0)$,
- Plant II poles starting in complex plane are moving perpendicular to the real axis for descending u (a loci defined as $r = \infty$?),
- In plant III around break away point trajectory is perturbed.

Further investigation on the higher order systems is proposed in Section 4.1

Fig. 4 represents a region of the equivalent poles against locus of U approximated as a constant. The system is described by following: $A = \begin{bmatrix} 1.2 & 1 \\ -0.35 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -0.2 \end{bmatrix}^T$, $N = \begin{bmatrix} 0.010 & 0 \\ 0.002 & 0 \end{bmatrix}$ and $|\Delta| < 100$. Hence, basing on (18), locus range perturbation is

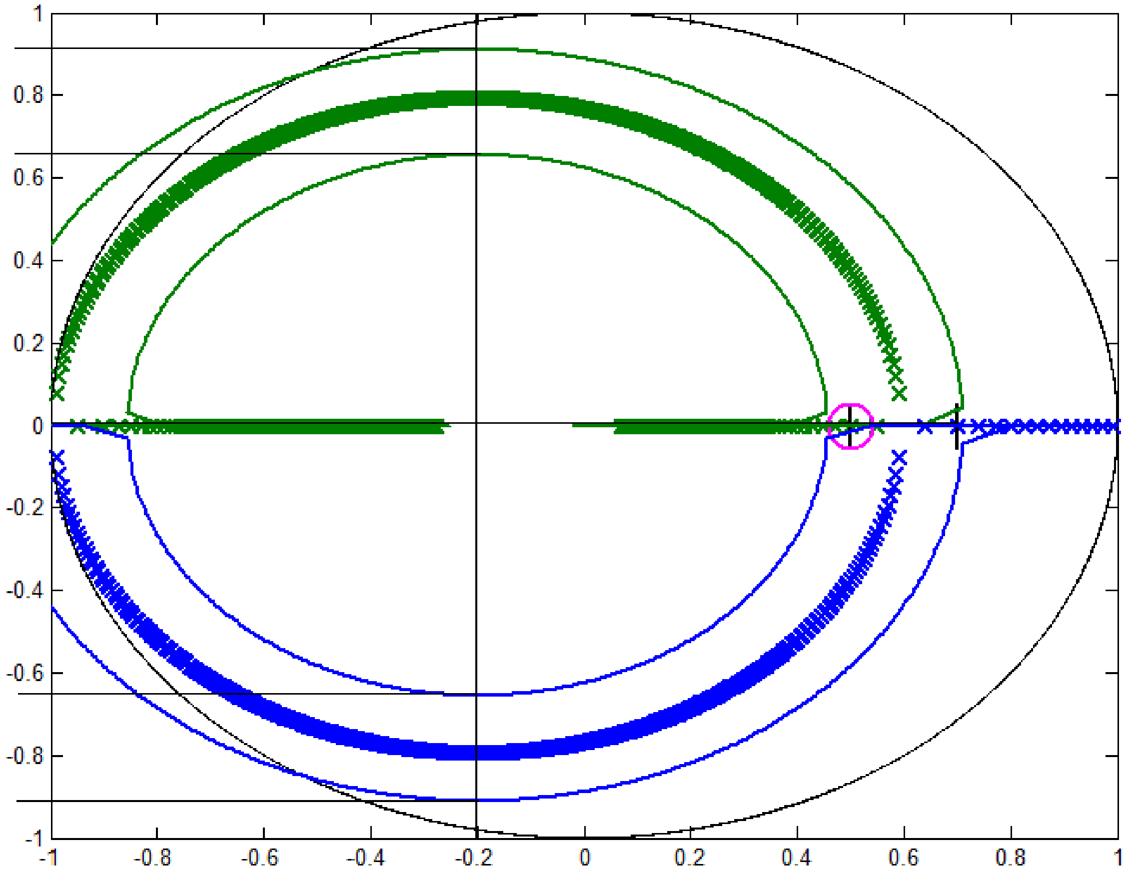

Figure 4. Equivalent poles loci as inequality region where $\Delta|u_{k-1..m}| < 100$

$r \in [0.65, 0.91]$ around the gravity centre in $(x = -0.2, y = 0)$.

Method derived in Section 3.1 has been validated for second order systems in numerical tests presented in this section (Fig. 2 and 4). The resemblance of equivalent poles and the gain feedback root loci can be observed due to the similar rotational behaviour and poles convergence at inf. It applies when close loop gain is mapped as U. The question to be asked is if this similarity is fully convertible.

4. Conclusion

Method derived for a second order bilinear system in Section 3.1 has been validated and presented graphically in Section 3.2. Moreover, similar vein pattern for third and higher order systems can be observed. The method is visually similar to the root locus of closed loop gain control [5]. Hence, parts of the method may be applied interchangeably between bilinear and linear structures.

Due to the more accurate dynamics prediction, a calculation of stable bilinear plant range of operation can be performed. It may be done utilising a correspondent method in the gain feedback loop locus. The critical (boundary) operating region of the bilinear model will be defined as a curve with respect to U and Δ.

Improved predictability is followed by enhanced identification capabilities - when observing trace of both: gain and dynamical properties (e.g. rise time and overshoot) engineer would have additional resource for assessing plant's structure type and the parameter estimation.

4.1. Future works

The intuitive continuation of the research is expanding of the reasoning for higher order systems. Although obtaining analytical solution will be less feasible due to mathematical complexity, results of simulation may be used to back calculate mathematical principle.

Another solution would be to compare equivalent poles with the classical P-gain feedback loop root locus as a correspondence between these methods has been observed (eg. in Figure 3). The potential of integration of the output feedback stabilization method [6] and bilinear model stability problem should be investigated.

Prediction of the pole location or the region of displacement may be useful for developing model based controllers. This approach contrary to the existing methods:

- frequent update of the linearised model around current OP,
- linearisation of the actual plant with a compensator [4]

has potential of detecting the internal instabilities and therefore ensuring robustness.

References

[1] Tan N and Atherton D P 2002 Stability margin computation for nonlinear systems: a parametric approach *15th IFAC Triennial world congr.* vol 15 ed L Basanez and J A de la Puente (Barcelona: IFAC) p 257

[2] Martineau S, Burnham K J, Haas O, Andrews G and Heeley A 2004 Four-term bi-linear PID controller applied to an industrial furnace *Cont. Eng. Pract.* **12** 457-64

[3] Taylor C J, Chotai A and Burnham K J 2011 Controllable forms for stabilising pole assignment design of generalised bilinear systems *Electr. Lett.* **47** 437-9

[4] Gadek L, Koszalka L and Burnham K J 2015 Compensation of n-th order nilinearity applied with model based controller *Progress in systems Engineering, Proc. of 23rd Int. Conf. on Systems Engineering* vol 330, ed H Selvaraj, D Zydek and G Chmaj (Las Vegas: Springer) pp 49-54

[5] Eydgahi A M 2001 Complementary root locus revisited *IEEE Trans. on Edu.* **44** 137-43

[6] Konigorski U 2012 Pole placement by parametric output feedback *Sys. & Cont. Lett.* **61** 292-7

[7] Lee C H and Juang J N 2012 Nonlinear system identificationa continuous-time bilinear state space approach *J. of the Astr. Sc.* **59** 398-420

[8] Verdult V and Verhaegen M 2000 Bilinear state space systems for nonlinear dynamical modelling *Th. in Biosc* **119** 1-9

[9] Reuter H 1994 State space identification of bilinear canonical forms Int. Conf. on Control vol 1 Coventry, IET pp 833-6

[10] Chilali M, Gahinet P and Apkarin P 1999 Robust Pole Placement in LMI Regions *IEE Trans. on Aut. Cont.* **44** 2257-70