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Scaling Relations for Logarithmic Corrections

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Multiplicative logarithmic corrections to scaling are frequently encountered in the critical behavior of certain statistical-mechanical systems. Here, a Lee-Yang zero approach is used to systematically analyze the exponents of such logarithms and to propose scaling relations between them. These proposed relations are then confronted with a variety of results from the literature.

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Conventional leading scaling behavior at a second-order phase transition is described by power laws in the reduced temperature t and field h. With $h = 0$, the correlation length, specific heat, and susceptibility behave as $\xi_\infty(t) \sim |t|^{-\nu}$, $C_\infty(t) \sim |t|^{-\alpha}$, and $\chi_\infty(t) \sim |t|^{-\gamma}$, while the magnetization in the broken phase has $m_\infty(t) \sim |t|^{\beta}$. Here the subscript indicates the extent of the system. At $t = 0$ the magnetization scales as $m_\infty(h) \sim h^{1/\delta}$ while the anomalous dimension η characterizes the correlation function at criticality. In the 1960s, it was shown that these six critical exponents are related via four scaling relations (see, e.g., Ref. [1] and references therein), which are now firmly established and fundamentally important in the theory of critical phenomena. With d representing the dimensionality of the system, the scaling relations are

$$\nu d = 2 - \alpha, \tag{1}$$

$$2\beta + \gamma = 2 - \alpha, \tag{2}$$

$$\beta(\delta - 1) = \gamma, \tag{3}$$

$$\nu(2 - \eta) = \gamma. \tag{4}$$

In the conventional scaling scenario, (2) and (3) can, in fact, be deduced from the Widom scaling hypothesis that the Helmholtz free energy is a homogeneous function [2]. Widom scaling and the remaining two laws can, in turn, be derived from the Kadanoff block-spin construction [3] and ultimately from Wilson's renormalization group (RG) [4]. The relation (1) can also be derived from the hyperscaling hypothesis, namely, that the free energy behaves near criticality as the inverse correlation volume: $f_\infty(t) \sim \xi_\infty^{-d}(t)$. Twice differentiating this relation recovers (1).

The scaling relations, (2) and (3), were both rederived using an alternative route by Abe [5] and Suzuki [6] exploiting the fact that the even and odd scaling fields can be linked by Lee-Yang zeros [7]. The locus of these zeros in the magnetic-field plane is controlled by the temperature. In the $t > 0$ (disordered) phase this locus terminates at the Yang-Lee edge [7], the distance of which from the critical point is denoted by $r_{\mathrm{YL}}(t)$. At a conventional second-order phase transition $r_{\mathrm{YL}}(t) \sim t^{\Delta}$ for $t > 0$, and the gap exponent Δ is related to the other exponents through [5,6]

$$\Delta = \frac{\delta\gamma}{\delta - 1} = \delta\beta = \beta + \gamma. \tag{5}$$

Logarithmic corrections are characteristic of a number of marginal scenarios (see, e.g., Ref. [8] and references therein). Hyperscaling fails at and above the upper critical dimension d_c and, while (1) holds there, it too fails above d_c, where mean-field behavior (which is independent of d) prevails. At d_c itself, multiplicative logarithmic corrections to scaling are manifest. Such corrections are found in marginal $d < d_c$ situations too [8,9]. The q-state Potts model in $d = 2$ dimensions possesses a first-order transition for $q > 4$ and a second-order one when $q < 4$. The $q = 4$ case is also characterized by a transition of second order, albeit with multiplicative logarithmic corrections to scaling. Also in two dimensions, the Ising model with uncorrelated, quenched random-site or random-bond disorder offers another example of such corrections at a demarcation point. According to the Harris criterion [10], when the critical exponent α of the specific heat for a pure system is positive, random quenched disorder is relevant (and exponents may change as disorder is added). If α is negative in the pure system, the critical behavior is not expected to be altered by such disorder. In the marginal case where $\alpha = 0$, no Harris prediction can be made, and logarithmic corrections to the pure scaling behavior may ensue. These are some examples of the rich and disparate variety of systems displaying such phenomena and which have been hitherto studied individually. Given the ubiquitous role that these logarithms play in such marginal cases, it is reasonable to ask if scaling relations for their exponents exist in analogy to (1)–(5) above.

Here three such relations, together with one for the Yang-Lee edge, are derived through the medium of partition function zeros and confronted with the literature. From the outset we mention that the scaling relations proposed herein do not all apply to the very special circumstance of the Ising model in two dimensions and its bond-disordered counterpart. These require special treatment beyond the general considerations presented here. For a scaling theory appropriate to the former, see Refs. [8,11] and references therein. With this in mind, we address the situation with the following scaling behavior:

$$\xi_\infty(t) \sim |t|^{-\nu}|\ln|t||^{\hat{\nu}}, \tag{6}$$

$$C_\infty(t) \sim |t|^{-\alpha}|\ln|t||^{\hat{\alpha}}, \tag{7}$$

$$\chi_\infty(t) \sim |t|^{-\gamma}|\ln|t||^{\hat{\gamma}}, \tag{8}$$

$$m_\infty(t) \sim |t|^{\beta}|\ln|t||^{\hat{\beta}} \quad \text{for } t<0, \tag{9}$$

$$r_{\rm YL}(t) \sim t^{\Delta}|\ln|t||^{\hat{\Delta}} \quad \text{for } t>0, \tag{10}$$

while at $t=0$,

$$m_\infty(h) \sim |h|^{1/\delta}|\ln|h||^{\hat{\delta}}. \tag{11}$$

In the thermodynamic limit the free energy may be written as

$$f_\infty(t,h) = 2\,{\rm Re}\int_{r_{\rm YL}(t)}^{R} \ln[h-h(r,t)]g_\infty(r,t)dr, \tag{12}$$

where R is a cutoff, $g_\infty(r,t)$ is the density of zeros, with locus $h(r,t)=r\exp[i\phi(r,t)]$. If the Lee-Yang circle theorem holds the locus is given by $\phi=\pi/2$, $R=\pi$ [7]. While the validity of the Lee-Yang circle theorem is not assumed in what follows (it does not hold for the Potts model, for example), it is assumed that the small-t critical behavior is dominated by the zeros closest to the critical point, and that the locus of these zeros can be approximated by $\phi(r,t)=\phi$, a constant.

The magnetic susceptibility is the second field derivative of the free energy, and, at $h=0$ [substituting $r=xr_{\rm YL}(t)$] is

$$\chi_\infty(t) = -\frac{2\cos(2\phi)}{r_{\rm YL}(t)}\int_1^{R/r_{\rm YL}(t)} \frac{g_\infty(xr_{\rm YL},t)}{x^2}dx. \tag{13}$$

Expanding (13) about $r_{\rm YL}(t)/R=0$ gives

$$g_\infty(r,t) = \chi_\infty(t)r_{\rm YL}(t)\Phi\left(\frac{r}{r_{\rm YL}(t)}\right), \tag{14}$$

up to additive corrections in $r_{\rm YL}(t)/R$ and where Φ is an undetermined function of its argument. The ratio $r_{\rm YL}(t)/R$ is sufficiently small near criticality so that these additive corrections may be dropped. Similar considerations yield for the magnetization

$$m_\infty(t,h) = \chi_\infty(t)r_{\rm YL}(t)\Psi_\phi\left(\frac{h}{r_{\rm YL}(t)}\right), \tag{15}$$

in which

$$\Psi_\phi\left(\frac{h}{r_{\rm YL}(t)}\right) = 2\,{\rm Re}\int_1^\infty \frac{\Phi(x)}{h/r_{\rm YL}(t)-xe^{i\phi}}dx. \tag{16}$$

Letting $h\to 0$ in (15), and comparing to (9), recovers the scaling relation (5) and yields

$$\hat{\Delta} = \hat{\beta}-\hat{\gamma}. \tag{17}$$

Furthermore, fixing the argument of the function Ψ_ϕ in (15) gives $t\sim h^{1/\Delta}|\ln|h||^{-\hat{\Delta}/\Delta}$ from (10), so that (15) may be written

$$m_\infty(t,h) \sim h^{1-\gamma/\Delta}|\ln h|^{\hat{\gamma}+\gamma\hat{\Delta}/\Delta}\Psi_\phi\left(\frac{h}{r_{\rm YL}(t)}\right). \tag{18}$$

Now taking $t\to 0$ and comparing with (11) recovers the known leading behavior for the edge (5), together with the correction relation $\hat{\Delta}=\delta(\hat{\delta}-\hat{\gamma})/(\delta-1)$. The former recovers (3), while the latter, with (17), gives

$$\hat{\beta}(\delta-1) = \delta\hat{\delta}-\hat{\gamma}. \tag{19}$$

It is convenient at this point to define the cumulative distribution function of zeros as

$$G_\infty(r,t) = \int_{r_{\rm YL}(t)}^{r} g_\infty(s,t)ds = \chi_\infty(t)r_{\rm YL}^2(t)I\left(\frac{r}{r_{\rm YL}(t)}\right), \tag{20}$$

in which $I(y)=\int_1^y\Phi(z)dz$. Integrating (12) by parts then gives the singular part of the free energy at $h=0$, $f_\infty(t)=2\int_{r_{\rm YL}(t)}^R[G_\infty(r,t)/r]dr$. Again substituting $r=xr_{\rm YL}(t)$, differentiating twice with respect to reduced temperature and comparing the resulting expression for the specific heat with (7) yields $\alpha=2+\gamma-2\Delta$ and $\hat{\alpha}=\hat{\gamma}+2\hat{\Delta}$. From (3) and (5), the first of these is the scaling law (2). From (17), the second can be conveniently expressed as another relation between the correction exponents, namely

$$\hat{\alpha} = 2\hat{\beta}-\hat{\gamma}. \tag{21}$$

Using these scaling relations, and fixing the ratio $r/r_{\rm YL}(t)$ in (20) and then taking the $t\to 0$ limit, gives the critical cumulative distribution function to be

$$G_\infty(r,0) \sim r^{(2-\alpha)/\Delta}|\ln r|^{\hat{\alpha}-(2-\alpha)\hat{\Delta}/\Delta}. \tag{22}$$

Consider now a system of finite extent L, and let $h_j(L)=r_j(L)\exp(i\phi_j)$ be the jth zero there. The finite-size scaling (FSS) of first zero is expressible as

$$\frac{r_1(L)}{r_{\rm YL}(t)} = \mathcal{F}\left(\frac{\xi_L(0)}{\xi_\infty(t)}\right), \tag{23}$$

in which $\xi_L(0)$ is the correlation length of the finite-size system at $t=0$. On dimensional grounds, we may assume this quantity takes the generic form

$$\xi_L(0) \sim L(\ln L)^{\hat{q}}, \tag{24}$$

having allowed for multiplicative logarithmic corrections. Recently, additional insights into the origin of FSS were given in Ref. [12]. For a finite system, the cumulative density of zeros is simply the fractional number of zeros up to a given point, and we write

$$G_L(r_j(L)) = \frac{2j-1}{2L^d}. \tag{25}$$

For large enough L, and at $t = 0$, this must coincide with the expression (22). In particular, it allows the scaling behavior of the lowest zero at thermodynamic criticality to be expressed as

$$r_1(L) \sim L^{-d\Delta/(2-\alpha)}(\ln L)^{\hat{\Delta}-\Delta\hat{\alpha}/(2-\alpha)}. \tag{26}$$

Inserting (6), (10), (24), and (26) into (23) recovers (1) and yields a new scaling relation for logarithmic corrections, namely

$$\hat{q} = \hat{\nu} + \frac{\nu\hat{\alpha}}{2-\alpha}. \tag{27}$$

Hyperscaling corresponds to $\hat{q} = 0$. Relations (19) and (21) but not (27) can be derived starting with a suitably modified phenomonological Widom ansatz [13,14].

To summarize thus far, the three standard scaling laws (1)–(3) have been recovered and three analogous relations for the logarithmic corrections (19), (21), and (27) presented. Furthermore, the standard formula (5) for the edge has been recovered and its logarithmic-correction counterpart is given in (17). While the standard scaling laws for the leading critical exponents are well established, it is now necessary to confront the scaling relations for corrections with results from the literature, and a variety of models with logarithmic corrections are examined on a case-by-case basis.

The leading critical exponents for the 4-state Potts model in $d = 2$ dimensions were established in Ref. [15] as $\alpha = 2/3$, $\beta = 1/12$, $\gamma = 7/6$, $\delta = 15$, and $\nu = 2/3$, and their correction counterparts are [16,17] $\hat{\alpha} = -1$, $\hat{\beta} = -1/8$, $\hat{\gamma} = 3/4$, $\hat{\delta} = -1/15$, and $\hat{\nu} = 1/2$. FSS of the thermodynamic functions are given in Refs. [17,18], from which $\hat{q} = 0$. The standard scaling laws, of course, hold and one notes that the correction relations (19), (21), and (27) hold too, while (5) and (17) give $\Delta = 5/4$ and $\hat{\Delta} = -7/8$ for the edge. This latter prediction remains to be verified numerically.

The upper critical dimension for $O(N)$ symmetric ϕ_d^4 theories is $d = d_c = 4$, where hyperscaling fails and the leading critical exponents take on their mean-field values, $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$, $\nu = 1/2$, $\Delta = 3/2$. The RG predictions for the corrections are [14,19–21] $\hat{\alpha} = (4-N)/(N+8)$, $\hat{\beta} = 3/(N+8)$, $\hat{\gamma} = (N+2)/(N+8)$, $\hat{\delta} = 1/3$, $\hat{\nu} = (N+2)/2(N+8)$, $\hat{\Delta} = (1-N)/(N+8)$, $\hat{q} = 1/4$, and all of the correction relations (17), (19), (21), and (27) hold.

The universality class of $O(N)$ spin models can be adjusted by introducing long-range interactions decaying as $x^{-(d+\sigma)}$ (x being distance along the lattice), for which $d_c = 2\sigma$. The critical exponents for the N-component system were calculated in Ref. [22] and are $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$, $\nu = 1/\sigma$, and obey the leading scaling relations. The Privman-Fisher form for the free energy was calculated in Ref. [23], from which the RG predictions for the critical exponents could be verified and the logarithmic corrections observed. The logarithmic exponents are $\hat{\alpha} = (4-N)/(N+8)$, $\hat{\beta} = 3/(N+8)$, $\hat{\gamma} = (N+2)/(N+8)$, $\hat{\delta} = 1/3$, $\hat{\nu} = (N+2)/\sigma(N+8)$. One observes that (19) and (21) are obeyed, and (27) holds too if $\hat{q} = 1/2\sigma$. This recovers the known value $\hat{q} = 1/4$ for $O(N)\phi_4^4$ theory [19] when $\sigma = 2$, and leads to agreement with FSS in the long-range Ising case in two dimensions when $\sigma = 1$ [24]. Furthermore, (5) and (17) yield $\Delta = 3/2$ and $\hat{\Delta} = (1-N)/(N+8)$ for the edge, which are again identical to the $O(N)\phi_4^4$ values. The expression (26) then gives that the first Lee-Yang zero of such a system should scale as $r_1(L) \sim L^{-3\sigma/2}(\ln L)^{-1/4}$. This prediction for the Lee-Yang zeros of long-range systems remains to be verified.

Spin glasses, percolation, the Yang-Lee edge problem and lattice animals are all related to ϕ^3 field theory. For each of these $d_c = 6$ except for the lattice animal problem which has $d_c = 8$ [25]. Ruiz-Lorenzo gave a compact description of the scaling of the correlation length, susceptibility and specific heat for these models as [26]

$$\alpha = -1, \qquad \gamma = 1, \qquad \nu = \tfrac{1}{2}, \tag{28}$$

$$\hat{\alpha} = \frac{2(2b-3a)}{4b-a}, \quad \hat{\gamma} = \frac{2a}{4b-a}, \quad \hat{\nu} = \frac{5a}{6(4b-a)}. \tag{29}$$

The values of (a, b) are $(-4m, 1-3m)$ for the m-component spin glass, $(-1, -2)$ for percolation, and $(-1, -1)$ for Yang-Lee singularities (which in d dimensions is closely related to the lattice animal problem in $d+2$ dimensions). The mean-field values of the critical exponents for spin glasses and percolation were calculated in Refs. [27,28], respectively, as $\beta = 1$, $\delta = 2$, and, together with (28), obey the usual scaling relations (1)–(3). The correction exponents (29) satisfy the scaling relations (19) and (21) provided that $\hat{\beta} = 2(b-a)/(4b-a)$ and $\hat{\delta} = b/(4b-a)$. In the percolation case these give $\hat{\beta} = \hat{\delta} = 2/7$, values which are in agreement with explicit calculations [28]. Also, (5) and (17) now yield $\Delta = 1$ and $\hat{\Delta} = 2(b-2a)/(4b-a)$ for these models, while (27) gives $\hat{q} = 1/6$ in each case. Ruiz-Lorenzo's prediction for this quantity is $\hat{q} = 1/3$ [26] while Ref. [29] contains an implicit assumption that $\hat{q} = 0$.

The strong universality hypothesis predicts that the quenched, disordered Ising model in $d = 2$ dimensions has the same leading critical exponents as in the pure case with logarithmic corrections to scaling [30]. In particular, Shalaev and later Shankar and Ludwig (SSL) gave [31] $\alpha = 0$, $\beta = 1/8$, $\gamma = 7/4$, $\delta = 15$, $\nu = 1$, $\hat{\alpha} = 0$, $\hat{\gamma} = 7/8$, $\hat{\nu} = 1/2$, with the specific heat predicted to be double-logarithmically divergent [30], and a more recent RG calculation gave [32] $\hat{\beta} = -1/16$ and $\hat{\delta} = 0$.

Among the SSL values, that for $\hat{\gamma}$ of the random-bond version has been the most clearly confirmed [33]. While the majority of published opinion favors the double-logarithmically divergent specific heat (see Refs. [9,34,35] and references therein), there have been persistent claims in the literature that the specific heat, in fact, remains finite at criticality in the site-diluted model [36,37]. Compatibility between the proposed scaling relations and the values $\hat{\beta} = -1/16$, $\hat{\gamma} = 7/8$, $\hat{\delta} = 0$, and $\hat{\nu} = 1/2$ is established if $\hat{\alpha} = -1$ in this case. This value indeed leads to a finite specific heat in the random-site version and would neatly explain the persistent claims to that effect in the literature [36,37] while still being consistent with the strong universality hypothesis. These values are also consistent, via (27), with SSL's $\hat{\nu} = 1/2$ provided $\hat{q} = 0$, a value actually claimed on the basis of numerical evidence in Ref. [34] (see also Ref. [37]). However, any value of $\hat{\nu}$ and $\hat{q}$, differing by $1/2$, cannot be ruled out on the basis of (27).

Finally, the fact that the logarithmic divergence in the specific heat for the pure Ising model in $d = 2$ (see Refs. [8,11]) does not directly fit into the scaling scheme proposed here is to do with special features of that model, which are shared by the random-bond version [31,32]. These special features are the vanishing of the specific-heat exponent α coupled with the property of self-duality and give rise to an extra logarithmic factor beyond those discussed herein. The apparent incompatibility in these special cases with the scaling relations proposed herein is perhaps a reason why they have gone unnoticed as such before.

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