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Abstract

Sufficient conditions are developed for the null controllability of neutral control systems with infinite delay when the values of the control lie in an m-dimensional unit cube. Conditions are placed on the perturbation function which guarantee that; if the uncontrolled system is uniformly asymptotically stable and the control system satisfies a full rank condition so that $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$, for every complex λ, where $K(\lambda)$ is an $n \times n$ polynomial matrix in λ constructed from the coefficient matrices of the control system and $\xi(\exp(-\lambda h))$ is the transpose of $[1, \exp(-\lambda h), \cdots, \exp(-(n-1)\lambda h)]$, then the control system is null controllable with constraint.

Keywords: Neutral systems, infinite delay, null controllability, rank condition, stable
Subject classification codes : 93Cxx, 93C23, 34k40, 93Bxx, 93B05

1. Introduction

Neutral functional differential systems have applications in many areas of study because of its importance as mathematical models for phenomena in both science and engineering; see Corduneanu (2002), Khartovskii and Pavlovskaya (2013) and references therein. The controllability of neutral functional differential systems, in particular null controllability of neutral functional differential systems have been studied, see for example Onwuatu (1984) and Underwood and Chukwu (1988). These studies have been extended to neutral functional differential systems with infinite delays in (Onwuatu, 1993; Balachandran and Leelamani, 2006; Umana, 2008, 2011; Dauer et al. 1998; Davies, 2006). In Dauer et al. (1998), null controllability of neutral functional differential systems having infinite delay has been studied using the Schauder fixed point theorem. In Umana (2008), the result was based on the uniform asymptotical stability of the uncontrolled system and the controllability of the linear control system. In Davies (2006), it was obtained by exploiting the stability of the free system and the well known rank criterion for properness.
The approach used in this paper is different. It is shown that when the controls of a neutral functional differential system with infinite delays are restrained to lie in the unit cube, the requirements for null controllability are that the system has full rank with the condition that $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$ and an additional condition that the trivial solution of the uncontrolled system be uniformly asymptotically stable. The results obtained by this method will extend that of (Underwood and Chukwu, 1988; Jacobs and Langenhop, 1976; Rivera Rodas and Langenhop, 1978) to neutral functional differential system having infinite delays by using the Schauder fixed point theorem.
The rest of the paper is organised as follows. Section 2 contains the mathematical notations, preliminaries and problem definition. In Section 3 stability theorems for the system are stated. Section 4 develops and proves the controllability theorems for the system; the main result of the paper is also developed and proved in this section. Finally, Section 5 contains numerical examples of the theoretical results prior to the conclusions.

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2. Basic notations, preliminaries and definitions

Suppose $h > 0$ is a given number, $\mathbb{E} = (-\infty, \infty)$, $\mathbb{E}^n$ is a real n – dimensional Euclidean space with norm $|\cdot|$. Let J be any interval in $\mathbb{E}$**,** the convention $\mathbb{W}_2^{(0)}(J, \mathbb{E}^n)$ will represents the Lebesgue space of square-integrable functions from J to $\mathbb{E}^n$, and $\mathbb{W}_2^{(1)}([-h, 0], \mathbb{E}^n)$ is the Sobolev space of all absolutely continuous functions $x: [-h, 0] \to \mathbb{E}^n$ whose derivatives are square integrable.
$\mathbb{C} = \mathbb{C}([-h, 0], \mathbb{E}^n)$ is the space of continuous function mapping the interval $[-h, 0]$ into $\mathbb{E}^n$ with the norm $\|\cdot\|$, where $\|\phi\| = \sup_{-h<s\le 0}|\phi(s)|$. Define the symbol $x_t \in \mathbb{C}$ by

$x_t(s) = x(t+s), -h \le s \le 0.$

This paper will consider neutral functional differential system with infinite delay of the form

$$\left.\begin{aligned} \frac{d}{dt}D(t)x_t = L(t, x_t, u) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta \\ x_\sigma = \phi, \ \ t \ge \sigma \end{aligned}\right\}, \tag{1}$$

and its perturbation

$$\frac{d}{dt}D(t)x_t = L(t, x_t, u) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta + f\big(t, x_t, u(t)\big) \tag{2}$$

through its linear base control system

$$\frac{d}{dt}D(t)x_t = L(t, x_t, u), \tag{3}$$

and its free system

$$\frac{d}{dt}D(t)x_t = L(t, x_t, 0) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta, \tag{4}$$

where,
$D(t)x_t = x(t) - A_0(t)x(t-h)$,
$L(t, x_t, u) = A_1(t)x(t) + A_2(t)x(t-h) + B(t)u(t)$, with the following assumptions:

(i) $A_0(t)$, $A_1(t)$ and $A_2(t)$ are continuous $n \times n$ matrices
(ii) $B(t)$, is a continuous $n \times m$ constant matrix
(iii) $A(\theta)$ is an $n \times n$ matrix whose elements are square integrable on $(-\infty, 0]$
(iv) $f: [\sigma, \infty) \times \mathbb{W}_2^{(1)} \times \mathbb{E}^n \to \mathbb{E}^n$ is a nonlinear continuous matrix function.

It is assumed that f satisfy enough smoothness conditions to ensure that a solution of (2) exists through each (σ, ϕ), is unique, and depends continuously upon (σ, ϕ) and can be extended to the right as long as the trajectory remains in a bounded set $[\sigma, \infty) \times \mathbb{C}$. These conditions are given in Cruz and Hale, (1970).

Let $x(\sigma, \phi)$ be a solution of (3) with $u = 0$ and set $T(t, \sigma)\phi = x_t(\sigma, \phi)$, $\phi \in \mathbb{W}_2^{(1)}$. Then $T(t, \sigma)$ is a continuous linear operator from $\mathbb{W}_2^{(1)} \to \mathbb{W}_2^{(1)}$. There is an $n \times n$ matrix function $\mathbb{X}(t, s)$ which is defined on $0 \le t \le s \le t_1$, $t \in J = [\sigma, \infty)$, continuous in s from the right of bounded variation in s; $\mathbb{X}(t, s) = 0$, $t < s \le t_1$, such that $\mathbb{X}(t, s)$ satisfies

$$\frac{\partial D(t)\mathbb{X}(t, s)}{\partial s} = L(t, \mathbb{X}_t(\cdot, s), 0), \ \ t \ge s.$$

Now, define the $n \times n$ matrix function $\mathbb{X}_0$ as

$$\mathbb{X}_0(s) = \begin{cases} 0, & -h \le s < 0, \\ I, & s = 0, \end{cases}$$

where I is the identity matrix. Write $T(t,s)\mathbb{X}_0(s) = \mathbb{X}(t+\sigma,s) = \mathbb{X}_t(\cdot\,,s)(s)$, so that $T(t,s)I = \mathbb{X}(t,s)$.
A solution of (3) through (σ,ϕ) satisfies the equation

$$x_t(\sigma,\phi,u) = T(t,\sigma)\phi + \int_\sigma^t T(t,\sigma)\mathbb{X}_0 B(s)u(s)ds,$$

or

$$x_t(\sigma,\phi,u) = x_t(\sigma,\phi,0) + \int_\sigma^t \mathbb{X}(t,s)B(s)u(s)ds. \tag{5}$$

In a similar manner, any solution of system (2) will be given by

$$x_t(\sigma,\phi,u,f) = x_t(\sigma,\phi,0) + \int_\sigma^t \mathbb{X}(t,s)B(s)u(s)ds + \int_\sigma^t \mathbb{X}(t,s) \int_{-\infty}^0 A(\theta)x(t+\theta)d\theta ds$$
$$+ \int_\sigma^t \mathbb{X}(t,s)\, f\big(s,x_s,u(s)\big)ds \;, \tag{6}$$

Define the matrix functions Z by

$$Z(t,s) = \mathbb{X}(t,s)B(s), \tag{7}$$

for $t \geq s \geq \sigma$, it follows then from (6) that

$$x_t(\sigma,\phi,u,f) = x_t(\sigma,\phi,0) + \int_\sigma^t Z(t,s)u(s)ds + \int_\sigma^t \mathbb{X}(t,s) \int_{-\infty}^0 A(\theta)x(t+\theta)d\theta ds$$
$$+ \int_\sigma^t \mathbb{X}(t,s)\, f\big(s,x_s,u(s)\big)ds \quad . \tag{8}$$

The controls of interest will be functions $u{:}\,[\sigma,\infty) \to \mathbb{C}^m$ which are square integrable on finite intervals with values in $\mathbb{C}^m$, and such controls will be denoted by $\mathbb{U}$, where $\mathbb{C}^m = \{u{:}\, u \in E^m, |u_j| \leq 1,\ j = 1,2,\cdots,m\}$.

Definition 1. The system (3) is proper on $[\sigma,t_1]$ if $\eta^T Z(t_1,s) = 0$ almost everywhere $s \in [\sigma,t_1]$ implies $\eta = 0$ for $\eta \in \mathbb{E}^n$, where η^T is the transpose of η. If (3) is proper on each interval $[\sigma,t_1]$, then the system is said to be proper in $\mathbb{E}^n$

Definition 2. System (3) is said to be completely controllable on $[\sigma,t_1]$, if for each function $\phi \in \mathbb{W}_2^{(1)}$, $x_1 \in \mathbb{E}^n$ there is an admissible control $u \in \mathbb{L}_2([\sigma,t_1],\mathbb{E}^m)$ such that the solution $x(t,\sigma,\phi,u)$ of (3) satisfies $x_\sigma(\cdot\,,\sigma,\phi,u) = \phi$, $x_{t_1}(\cdot\,,\sigma,\phi,u) = x_1$. It is completely controllable on $[\sigma,t_1]$ with constraints, if the above holds with $u \in \mathbb{U}$.

Definition 3. The system (2) is null-controllable on $[\sigma,t_1]$ if for each

$\phi \in \mathbb{W}_2^{(1)}([-h\,,0],\ \mathbb{E}^n)$, there exists a $u \in \mathbb{L}_2([\sigma,t_1],\mathbb{E}^m)$ such that the solution of (2) satisfies $x_\sigma(\cdot\,,\sigma,\phi,u,f) = \phi$, $x_{t_1}(\cdot\,,\sigma,\phi,u,f) = 0$. The system (2) is null-controllable with constraints if the above holds with control $u \in \mathbb{U}$

Definition 4. The domain $\mathfrak{U}$ of null-controllability of (3) with constraints is the set of all initial functions $\phi \in \mathbb{W}_2^{(1)}$ for which the solution $x(\sigma,\phi,u)$ of (3) with $x_\sigma(\cdot\,,\sigma,\phi,u) = \phi$, $x_{t_1}(\cdot\,,\sigma,\phi,u) = 0$ at some $t_1, u \in \mathbb{U}$

Definition 5. The reachable set of (3) is a subset of $\mathbb{E}^m$ given by

$$P(\sigma,t)=\left\{\int_{\sigma}^{t} Z(t,s)u(s)ds:\ u\in\mathbb{L}_2([\sigma,t],\mathbb{E}^m)\right\}.$$

If the controls are in $\mathbb{L}_2([\sigma,t],\mathbb{C}^m)$, we define the constraint reachable set by

$$R(\sigma,t)=\left\{\int_{\sigma}^{t} Z(t,s)u(s)ds:\ u\in\mathbb{L}_2([\sigma,t],\mathbb{C}^m)\right\}.$$

Note that $P(\sigma,t)$ is a subset of $\mathbb{E}^m$ which is symmetric about zero.

Definition 6. The controllability matrix of (3) will be given by

$$W(\sigma,t)=\int_{\sigma}^{t} Z(t,s)Z^T(t,s)ds\ ,$$

where Z^T is the transpose of Z.

3. STABILITY RESULTS

Here, some definitions, lemmas and theorem which are fundamental to the development of the stability results for the system (4) are given. Consider the systems (3) defined by

$$\frac{d}{dt}D(t)x_t=L(t\ ,x_t,0)\ , \tag{9}$$

Definition 7. The solution $x=0$ of (9) is called stable at σ if $\sigma\geq 0$ and there exists a $b=b(\sigma)>0$ such that if $\|\phi\|\leq b$, then the solution $x(\sigma,\phi)$ exists for $t\geq\sigma$.

Also, for each $\varepsilon>0$ there exists a $\delta=\delta(\sigma,\varepsilon)>0$ such that if $\|\phi\|\leq\delta$, then the solution $x(\sigma,\phi)$ of (9) satisfies $\|x_t(\sigma,\phi)\|\leq\varepsilon$ for $t\geq\sigma$.

The trivial solution of (9) is called stable if it is stable for each $\sigma\geq 0$. It is called uniformly stable if it is stable and δ does not depend on σ. It is uniformly asymptotically stable if it is uniformly stable for every $\eta>0$ and every $\sigma>0$ there exists $T(\eta)$ independent of σ and $H_0>0$ independent of η,σ such that $\|\phi\|\leq H_0$ implies $\|x_t(\sigma,\phi)\|\leq\eta$, for all

$t\geq\sigma+T(\eta)$.

Definition 8. The solution $x=0$ of (9) is uniformly asymptotically stable if and only if there exists constant $c>0,\ k>0$ such that $\|x_t(\sigma,\phi)\|\leq k\ exp[-c(t-\sigma)]\|\phi\|$, for all $t\geq\sigma$.

Definition 9. Let $D(t,\phi)=D\phi$, and consider the homogeneous difference equation

$$\begin{cases} D(t)x_t=0,\ \ t\geq\sigma \\ x_\sigma=\phi,\qquad D\phi=0 \end{cases} \tag{10}$$

$D(t)$ is uniformly stable if there are constants $\alpha,\beta>0$ such that for $\sigma\in J,\ \phi\in\mathbb{W}_2^{(1)}$ the solution of (10) satisfies $\|x_t(\sigma,\phi)\|\leq\beta\ exp[-\alpha(t-\sigma)]\|\phi\|$, for all $t\geq\sigma$.

The next two lemmas are due to Cruz and Hale (1970), they are very important for the analysis and development of the properties for operator $D(\cdot)$, and for the overall stability result in this section.

Lemma 1. Let A_0 be an $n \times n$ constant matrix. The operator $D\phi = \phi(0) - A_0\phi(-h)$ is uniformly stable if all the roots of the equation $\det[I - A_0 r^{-h}] = 0$ have moduli less than 1. This holds if $\|A_0\| < 1$.

Lemma 2. $D\phi$ is uniformly stable if there are constants $\alpha, \beta > 0$ such that for any $\phi \in \mathbb{W}_2^{(1)}$, $\sigma \in [\tau, \infty)$, the solution $x(\sigma, \phi)$ of (10) satisfies $\|x_t(\sigma, \phi)\| \leq \beta \|\phi\| e^{\alpha(t-\sigma)},\ t \geq \sigma$.

The next theorem is developed following Theorem 1 of Sinha (1985) and Corollary 2 of Hale (1974) for functional differential equations with infinite delay; see also Corollary 3.8 of Davies (2006) and references therein for neutral functional differential systems with infinite delays.

Theorem 1. In system (4), assume there is a $v > 0$, and a constant M such that $|A(\theta)| \leq M exp(v\theta) \leq M$ for $\theta \in (-\infty, 0]$ and if $B(\lambda) = \{\lambda \in \mathbb{C}: Re\, \lambda \geq\ 0\ ,\ \det\ \triangle\ (\lambda) = 0\} = \emptyset$, where

$$\triangle\ (\lambda) = \lambda\big(I - A_0 exp(-\lambda h)\big) - A_1 - A_2 \exp(-\lambda h) - \int_{-\infty}^{0} \exp(\lambda\theta)\, A(\theta) d\theta,$$

Then the solution of (4) is uniformly asymptotically stable if $|\mathbb{X}(t, s)| \leq k\ exp\big(-\alpha(t - \sigma)\big),\ t \geq s \geq \sigma, k > 0,\ \alpha > 0$

4. Controllability Results

This section develops and proves necessary and sufficient controllability conditions for systems (3) by exploiting the method in (Jacobs and Langenhop, 1976; Rivera Rodas and Langenhop, 1978). Some relevant controllability results which are relevant to this study are also given.

Lemma 3. The system (3) is completely controllable if and only if $W(\sigma, t_1)$ is non-singular

Proof. The proof can be observed from Proposition 3.1 of Dauer and Gahl (1977). □

Proposition 1. The system (3) is controllable on $[\sigma, t_1]$ if and only if $0 \in \operatorname{int} R(\sigma, t)$

Proof. This is Lemma 3.2 of Davies (2006). □

Proposition 2. The system (3) is completely controllable on $[\sigma, t_1]$ if and only if

$0 \in R(\sigma, t)$.

Proof. This is Theorem 3.3 of Davies (2006). □

Proposition 3. The following are equivalent.

(i) $W(\sigma, t_1)$ is non-singular
(ii) system (3) is completely controllable on $[\sigma, t_1],\ t_1 > \sigma$
(iii) system (3) is proper on $[\sigma, t_1],\ t_1 > \sigma$

Proof. This is Proposition 3 of Onwuatu (1993). □

Lemma 4. The system (3) is completely controllable on $[\sigma, t_1]$ if and only if it is controllable on $[\sigma, t_1]$.

Proof. The proof follows immediately from Proposition 4.1 and 4.2. □

Let $x: [\alpha, \beta] \to \mathbb{E}^p$, p a positive integer, be absolutely continuous and define the differential operator for neutral systems D by $(Dx)(t) = \dot{x}(t) = dx(t)/dt$, almost everywhere on $[\alpha, \beta]$. Higher powers of the operator D are defined inductively by $D^{k+1} = DD^k$, with domain equal to all $x: [\alpha, \beta] \to \mathbb{E}^p$, such that $D^k x$ is absolutely continuous on $[\alpha, \beta]$. Note that, by D^0 the identity $(D^0 x)(t) = x(t)$, $t \in [\alpha, \beta]$.

Define $\mathbb{W}_{2,0}^{\upsilon}(\tau, \mathbb{E}^{\mu})$, μ a positive integer and υ a nonnegative integer to be the collection of all $x: (-\infty, \tau] \to \mathbb{E}^{\mu}$ such that $x(t) = 0$ for $t \le 0$ and the restriction $x|_{[0,\tau]}$ is in $\mathbb{W}_2^{\upsilon}([0, \tau], \mathbb{E}^{\mu})$, and adopt the convention $\mathbb{W}_2^0([0, \tau], \mathbb{E}^{\mu}) = \mathbb{L}_2([0, \tau], \mathbb{E}^{\mu})$. For

$f \in \mathbb{W}_{2,0}^{\upsilon}(\tau, \mathbb{E}^{\mu})$ define the shift operator S by

$$(Sf)(t) = f(t - h), \quad t \le \tau . \tag{11}$$

Define S^0 to be the identity operator on $\mathbb{W}_{2,0}^{\upsilon}(\tau, \mathbb{E}^{\mu})$ and take $S^{k+1} = SS^k$, $k = 0, 1, 2, \cdots$ by inductively using (11).

Observe from the definition of the differential operator D, and the shift operator S that for $\upsilon \ge 1$, if the function space $\mathbb{W}_{2,0}^{\upsilon}(\tau, \mathbb{E}^{\mu})$ is taken as a common domain for the operators S and D, then S and D commute in this setting and each commutes with multiplication by a scalar (element in $\mathbb{E}$).

For any $n \times n$ matrix A and $n \times m$ matrix B, one can define $n \times \upsilon m$ matrix by

$$P_{\upsilon}[A, B] = [B, \cdots, A^{\upsilon - 1} B] ,$$

for integers $\upsilon \ge 1$.

Consider the neutral system with delay of the form

$$(d/dt)\big(x - A_0 x(t - h)\big) = A_1 x(t) + A_2 x(t - h) + Bu(t), \tag{12}$$

where A_i , $i = 0, 1, 2$ are $n \times n$ constant matrices, B is chosen to be $n \times 1$ constant real matrix. The solution $x(\cdot, 0, u)$ of (12) is the restriction to $[-h, 0]$ of the solution $x \in \mathbb{W}_{2,0}^{(1)}(\tau, \mathbb{E}^n)$ of the equation

$$(ID - A_0 SD - A_1 - A_2 S)x = Bu .$$

Now define the matrix $Q(D, S)$ by the equation

$$Q(D, S) = ID - A_0 SD - A_1 - A_2 S,$$

and let

$$P(D, S) = \operatorname{adj} Q(D, S) ,$$

where "adj" denotes the transposed matrix of cofactors. Some basic relationship exists between these two operators which by Jacobs and Langenhop (1976) can be expressed as

$$P(D, S) = \sum_{i=0}^{n-1} P_i(D) S^i = \sum_{i=0}^{n-1} \hat{P}_i(S) D^i, \tag{13}$$

where the $n \times n$ matrix polynomials $P_i(D)$, $\hat{P}_i(S)$ are at most of degree $n - 1$ in their argument. Using the polynomial $P_i(D)$ in (13) define a unique matrix operator by

$$K(D) = [P_0(D)B, P_1(D)B, \cdots, P_{n-1}(D)B] ,$$

Now, the operator $K(D)$ can be written in the form of a polynomial to get

$$K(D) = \sum_{i=0}^{n-1} K_i D^{n-1-i} \quad , \tag{14}$$

where, the K_i , $i = 0, 1, \cdots, n - 1$ are $n \times n$ constant real matrices , and let $\xi(\exp(-\lambda h))$ be the transpose of $[1, \exp(-\lambda h), \cdots, \exp(-(n - 1)\lambda h)]$ for all complex numbers λ , $h > 0$.

Theorem 2. Let $\tau > nh$, then for (12) to be controllable on $[0, \tau]$ it is necessary and sufficient for rank $P_n[A_0, B] = n$ and $K(\lambda)\xi(\exp(-\lambda h)) \ne 0$ for every complex λ.

Proof. This is Theorem 3.4 of Rivera Rodas and Langenhop, (1978). □

Corollary 1. Let $\tau > nh$, $[0, \tau]$ and assume that system (12) satisfies the following

(i) rank $P_n[A_0, B] = n$;
(ii) $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$, for every complex λ.

Then, system (12) is completely controllable on $[0, \tau]$.

Proof. If condition (i) and (ii) holds, then by Theorem 2, the system (12) is controllable on $[0, \tau]$. This by Lemma 4 implies that system (12) is completely controllable on $[0, \tau]$. Conversely, if system (12) is completely controllable on $[0, \tau]$, then it is controllable by Lemma 4, and by Theorem 2; rank $P_n[A_0, B] = n$ and $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$ for every complex λ, and the proof is complete. □

Theorem 3. In system (1), assume the following

(i) A_0, A_1, A_2 are $n \times n$ constant matrices, B is $n \times 1$ constant real matrix
(ii) for $\tau > nh$, rank $P_n[A_0, B] = n$;
(iii) $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$, for every complex λ,
(iv) $sup\{\mathrm{Re}(\lambda), \det \Delta(\lambda) = 0\} < 0$,

$$\Delta(\lambda) = \lambda(I - A_0 \exp(-\lambda h)) - A_1 - A_2 \exp(-\lambda h) - \int_{-\infty}^{0} \exp(\lambda\theta)\, A(\theta) d\theta$$

(v) $D\phi = \phi(0) - A_0\phi(-h)$ is uniformly stable.

Then, system (1) is null controllable with constraints on $(0, \sigma)$, $\sigma > \tau$.
Proof. Because of (i), (ii) and (iii) system (1) is controllable on $[0, \tau]$ by Theorem 2. Hence, $0 \in \mathrm{Int}\, R(0, \tau)$ by Proposition 1. By condition (iv), and (v) the system (1) with $u = 0$ satisfies $x_t(\cdot, \phi, 0) \to 0$ as $t \to \infty$. Hence, at some $t_1 > 0$, $x_{t_1}(0, \phi, 0) \in \mathrm{Int}\, R(0, \tau)$ and hence $0 \in \mathrm{Int}\, \mathfrak{U}$, the domain of null controllability of (1). Suppose for the contrary that $0 \notin \mathrm{Int}\, \mathfrak{U}$. Since $x = 0$ is a solution of (1) with $u = 0$, then $0 \in \mathfrak{U}$. This implies that, there exists a sequence $\{\phi_i\} \subseteq \mathbb{W}_2^{(1)}$ such that $\phi_i \to 0$ as $i \to \infty$ and $\phi_i \notin \mathfrak{U}$, for any i, therefore $\phi_i \neq 0$. It follows from the variation of constant formula (5) that;
$x_{t_i}(0, \phi_i, u) = x_t(0, \phi_i, 0) + \int_\sigma^{t_1} Z(t_1, s)u(s)ds$. Let $z_i = x_{t_1}(0, \phi_i, 0)$. Then, since $\phi_i \notin \mathfrak{U}$, $x_{t_1}(0, \phi_i, u) \neq 0$, for any i, and so $z_i \notin R(0, t_1)$, for any $t_1 > 0$ and $z_i \neq 0$. But $z_i \to 0$ as $i \to \infty$, and $0 \notin \mathrm{Int}\, R(0, t_1)$ which is a contradiction. Therefore, $0 \in \mathrm{Int}\, \mathfrak{U}$, and hence there exists a ball S_2 around 0 which is contained in $\mathfrak{U}$. Again, by (iv) there exists some $t_2 < \infty$, $x_{t_2}(\cdot, \phi, 0) \in S_2$. Therefore, using t_2 as initial point and $x_{t_2}(\cdot, \phi, 0) \equiv \psi$ as initial function, there exists $u \in \mathbb{U}$ and $t_3 > t_2$ such that, the solution $x(t_2, x_{t_1}(\cdot, \phi, 0), u)$ of (1) satisfies $x_{t_3}(\cdot, t_1, x_{t_1}, u) = 0$, and the proof is complete. □

4.1. Main result

The main result for the neutral control system with infinite delay will now be developed and proved in this section.

Theorem 4. Assume for system (2) that

(i) the constraint set $\mathbb{U}$ is an arbitrary compact set of $\mathbb{E}^n$

(ii) $D\phi = \phi(0) - A_0\phi(-h)$ is uniformly stable

(iii) the system (4) is uniformly asymptotically stable; so that the solution of (4) satisfies $\|x_t(\sigma,\phi,0)\| \le k\, exp\big(-\alpha(t-\sigma)\big)\|\phi\|$, $\alpha > 0$, $k > 0$.

(iv) the system (3) is completely controllable

(v) The continuous function f satisfies $\big|f\big(t,x(\cdot),u(\cdot)\big)\big| \le \exp(-bt)\,\pi\big(x(\cdot),u(\cdot)\big)$, for all $\big(t,x(\cdot),u(\cdot)\big) \in [\sigma,\infty) \times \mathbb{W}_2^{(0)} \times L_2$, where $\int_\sigma^\infty \pi\big(x(\cdot),u(\cdot)\big)ds \le M < \infty$, and $b - \alpha \ge 0$.

Then, the system (2) is null controllable.

Proof. By (iv), W^{-1} exists for each $t_1 > \sigma$. Assuming the pair of functions x, u forms a pair to the integral equations

$$u(t) = -Z(t_1,s)^T W^{-1}(\sigma,t_1)\left[x_{t_1}(\sigma,\phi,0) + \int_\sigma^{t_1} \mathbb{X}(t_1,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta ds + \int_\sigma^{t_1} \mathbb{X}(t_1,s)\, f\big(s,x(\cdot),u(\cdot)\big)ds\right], \tag{15}$$

for some suitably chosen $t_1 \ge t \ge \sigma$, $u(t) = v(t)$, $t \in [\sigma-h,\sigma]$

$$x(t) = x_t(\sigma,\phi,0) + \int_\sigma^t Z(t,s)u(s)ds + \int_\sigma^t \mathbb{X}(t,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta ds + \int_\sigma^t \mathbb{X}(t,s)\, f\big(s,x(\cdot),u(\cdot)\big)ds\,, \tag{16}$$

$x(t) = \phi(t)$, $t \in [\sigma-h,\sigma]$.

Then u is square integrable on $[\sigma-h,t_1]$ and x is a solution of (2) corresponding to u with initial state $x(\sigma) = \phi$. Also, $x(t_1) = 0$. It is necessary to show now that $u\colon [\sigma,t_1] \to \mathbb{U}$ is in the arbitrary compact constraint subset of $\mathbb{E}^m$, that is $|u(t)| \le a_1$, for some constant $a_1 > 0$. By (ii) and (iii), and the continuity of B in compact intervals, it follows that

$$|Z(t_1,s)^T W^{-1}(\sigma,t_1)| \le d_1,$$

$$\left|x_{t_1}(\sigma,\phi,0) + \int_\sigma^{t_1} \mathbb{X}(t_1,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta ds\right| \le d_2 \exp\big(-\alpha(t_1-\sigma)\big)\,,$$

for some $d_1 > 0$, $d_2 > 0$. Hence,

$$|u(t)| \le d_1\left[d_2 \exp\big(-\alpha(t_1-\sigma)\big) + \int_\sigma^{t_1} k\, exp\big(-\alpha(t-s)\big)\exp(-bs)\,\pi\big(x(\cdot),u(\cdot)\big)ds\right]$$

and therefore,

$$|u(t)| \le d_1 d_2 \exp\big(-\alpha(t_1-\sigma)\big) + d_1 kMexp\big(-\alpha(t_1)\big), \tag{17}$$

since $b - \alpha \ge 0$ and $s \ge \sigma \ge 0$. Hence, t_1 from (17) can be chosen sufficiently large that $|u(t)| \le a_1$, $t \in [\sigma,t_1]$, showing that u is admissible control. It remains to prove the existence of a pair of the integral equations (15) and (16). Let $\mathbb{W}_2^{(0)}$ represent the Banach space of all functions $(x,u)\colon [\sigma-h,t_1] \times [\sigma-h,t_1] \to \mathbb{E}^n \times \mathbb{E}^m$, where $x \in \mathbb{W}_2^{(0)}([\sigma-h,t_1],\mathbb{E}^n)$; $u \in \mathbb{L}_2([\sigma-h,t_1],\mathbb{E}^m)$ with the norm defined by

$\|(x,u)\| = \|x\|_2 + \|u\|_2$, where $\|x\|_2 = \left\{\int_{\sigma-h}^{t_1} |x(s)|ds\right\}^{1/2}$; $\|u\|_2 = \left\{\int_{\sigma-h}^{t_1} |u(s)|ds\right\}^{1/2}$.

Define the operator $T\colon \mathbb{W}_2^{(0)} \to \mathbb{W}_2^{(0)}$ by $T(x,u) = (y,w)$, where

$$w(t) = -Z(t_1,s)^T W^{-1}(\sigma,t_1)\left[x_{t_1}(\sigma,\phi,0) + \int_{\sigma}^{t_1} \mathbb{X}(t_1,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta ds \right.$$
$$\left. + \int_{\sigma}^{t_1} \mathbb{X}(t_1,s)\, f\big(s,x_s,u(s)\big)ds\right], \qquad (18)$$

for $t \in J = [\sigma, t_1]$ and $v(t) = w(t)$ for $t \in [\sigma - h, \sigma]$.

$$y(t) = x_t(\sigma,\phi,0) + \int_{\sigma}^{t} Z(t,s)u(s)ds + \int_{\sigma}^{t} \mathbb{X}(t,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta ds$$
$$+ \int_{\sigma}^{t} \mathbb{X}(t,s)\, f\big(s,x_s,u(s)\big)ds, \qquad (19)$$

for $t \in J$ and $y(t) = \phi(t)$ for $t \in [\sigma - h, \sigma]$.

It is clear from (17) that $|v(t)| \le a_1$ for $t \in J$ and also $v : [\sigma - h, \sigma] \to \mathbb{U}$, so that $|v(t)| \le k$. Hence, $\|v\|_2 \le a_1(t_1 + h - \sigma)^{1/2} = b_0$. Again,

$$|y(t)| \le d_2 \exp\big(-\alpha(t_1 - \sigma)\big) + d_3 \int_{\sigma}^{t} |v(s)|ds + kMexp\big(-\alpha(t_1)\big),$$

where $d_3 = sup|Z(t,s)|$. Since $\alpha > 0,\ t \ge \sigma \ge 0$, it follows that $|y(t)| \le d_2 + d_3 a_1(t - \sigma) + kM = b_1,\ t \in J$ and $|y(t)| \le \sup|\phi(t)| = \delta,\ t \in [\sigma - \tau, \sigma]$. Hence, if $\lambda = \max[b_1, \delta]$, then $\|y\|_2 \le \lambda(t_1 + h - \sigma)^{1/2} = b_2 < \infty$. Let $r = \max[b_1, b_2]$. Then letting $Q(r) = \left\{(x,u) \in \mathbb{W}_2^{(0)}: \|x\|_2 \le r,\ \|u\|_2 \le r\right\}$, it follows that $T: Q(r) \to Q(r)$. Now, since $Q(r)$ is closed, bounded and convex, by Riesz theorem (see Kantorovich and Akilov, 1982), it is relatively compact under the transformation T. Hence, the Schauder's fixed point theorem implies that T has a fixed point. Hence, system (2) is null controllable.

5. Example

Consider the neutral control system

$$(d/dt)\big(x(t) - A_0x(t-h)\big) = A_1x(t) + A_2x(t-h) + C_0 \int_{-\infty}^{0} \exp(v\theta)x(t+\theta)d\theta$$
$$+ Bu(t), \qquad (20)$$

and its perturbation

$$(d/dt)\big(x(t) - A_0x(t-h)\big) = A_1x(t) + A_2x(t-h) + C_0 \int_{-\infty}^{0} \exp(v\theta)x(t+\theta)d\theta$$
$$+ Bu(t) + f\big(t,x(t),x(t-h),u(t)\big). \qquad (21)$$

Its linear control base system is given by

$$(d/dt)\big(x(t) - A_0x(t-h)\big) = A_1x(t) + A_2x(t-h) + Bu(t), \qquad (22)$$

and its free system

$$(d/dt)\big(x(t) - A_0x(t-h)\big) = A_1x(t) + A_2x(t-h) + C_0 \int_{-\infty}^{0} \exp(v\theta)x(t+\theta)d\theta, \ (23)$$

where

$$A_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1/4 \end{pmatrix},$$

$$f\big(t, x(t), x(t-h), u(t)\big) = \begin{pmatrix} 0 \\ \exp(-\alpha t)\sin\big(x(t) + x(t-h)\big) \cdot \cos u(t) \end{pmatrix}, \alpha > 0,$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now, use Lemma 1 to check that, the operator D is uniformly stable as follows: The condition $\det[I - A_0 r^{-h}] = 0$ of Lemma 1 gives,

$$\begin{pmatrix} 1 & -\frac{1}{2} r^{-h} \\ -\frac{1}{2} r^{-h} & 1 \end{pmatrix} = 0,$$

which implies $1 - (1/4) r^{-2h} = 0$, and $r = (1/2)^{1/h}$. Hence, the operator D is uniformly stable if $h > 0$.

Next, observe by Theorem 1 that the characteristic root of (23) is

$$(4 - \exp(-2\lambda h))\lambda^2 + (12 - 2\exp(-\lambda h) - \exp(-2\lambda h))\lambda + 4$$
$$+ (\lambda + 1) \int_{-\infty}^{0} \exp(\lambda + v)\theta d\theta = 0, \tag{24}$$

and all the roots of (23) have negative real part. Hence by Theorem 1, system (23) is uniformly asymptotically stable.

Finally, check that (22) is controllable as follows:

$$\operatorname{rank}[B, A_0 B] = \operatorname{rank} \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} = 2,$$

$$P_0(\lambda) = \operatorname{adj}(I\lambda - A_1)$$
$$= \operatorname{adj} \begin{pmatrix} \lambda + 1 & -1 \\ -1 & \lambda + 2 \end{pmatrix} = \begin{pmatrix} \lambda + 2 & 1 \\ 1 & \lambda + 1 \end{pmatrix},$$

$$P_1(\lambda) = \operatorname{adj}(-A_0\lambda - A_2)$$
$$= \operatorname{adj}\left[\begin{pmatrix} 0 & -\lambda/2 \\ -\lambda/2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \end{pmatrix}\right] = \begin{pmatrix} 1/2 & (\lambda+1)/2 \\ \lambda/2 & 0 \end{pmatrix},$$

$$K(\lambda) = [P_0(\lambda)B, P_1(\lambda)B] = \begin{pmatrix} 1 & (\lambda+1)/2 \\ 1 + \lambda & 0 \end{pmatrix},$$

$$K(\lambda)\xi(\exp(-\lambda h)) = \begin{pmatrix} 1 & (\lambda+1)/2 \\ 1 + \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \exp(-\lambda h) \end{pmatrix}.$$

Observe that for all complex λ, $h > 0$,

$$K(\lambda)\xi(\exp(-\lambda h)) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore system (22) is controllable on $(0, \tau), \tau > 2h$.

Moreover,

$$\big|f\big(t, x(t), x(t-h), u(t)\big)\big| = \big|\exp(-\alpha t)\sin\big(x(t) + x(t-h)\big) \cdot \cos u(t)\big|$$
$$\leq \exp(-\alpha t) \cdot 1 \tag{25}$$

Hence, all the conditions of Theorem 4 are satisfied and system (21) is null controllable.

5.1 Simulation studies

The stability and controllability of the open loop system (21) can be illustrated using Simulink and MATLAB based simulation studies. The simulation model parameters are as given (21) with the default parameter setting and a square wave input where α and v are chosen to be 2 and 1 respectively with $h = 0.25s$. Fig. 5.1 depicts the stability and controllability of the states when simulation is done with the linear control base system i.e. (22), and when the simulation is done with the perturbation function (see (21)). The amplitude of simulation is observed to be slightly higher with the perturbation function and faster response when the simulation is done without the perturbation function. The settling times for the systems without the perturbation function are also observed to be faster; this is as expected and depends on the assumptions placed on the perturbation function (see (25)). The simulation showed that, the system (23) is stable and the overall control system (21) is controllable.

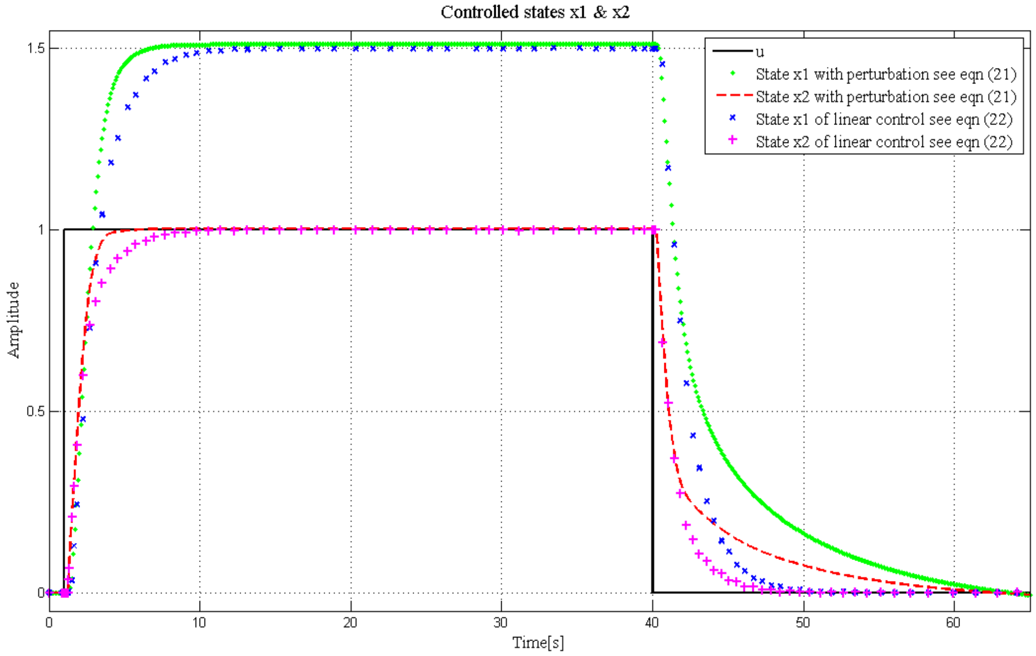

Figure 5.1. Simulation of control input and system states with perturbation function and linear control base system

Conclusion

Null controllability for neutral functional differential system with infinite delay have been developed and proved. The results were obtained with respect to the stability of the free system and the linear control base system having full rank with the condition that $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$, with the assumption that the perturbation function f satisfies the smoothness and growth condition placed on it.

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