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# Finite Size Scaling for $O(N)$ $\phi^4$ -Theory at the Upper Critical Dimension

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## Abstract

A finite size scaling theory for the partition function zeroes and thermodynamic functions of  $O(N)$   $\phi^4$ -theory in four dimensions is derived from renormalization group methods. The leading scaling behaviour is mean-field like with multiplicative logarithmic corrections which are linked to the triviality of the theory. These logarithmic corrections are independent of  $N$  for odd thermodynamic quantities and associated zeroes and are  $N$  dependent for the even ones. Thus a numerical study of finite size scaling in the Ising model serves as a non-perturbative test of triviality of  $\phi_4^4$ -theories for all  $N$ .

# 1 Introduction

The  $\phi^4$ -field theory, in  $d$  dimensions, with an  $O(N)$ -symmetry group is of fundamental interest in both high energy and statistical physics. Its strongly coupled limit is the non-linear  $\sigma$ -model, which for  $N = 0$  is the self-avoiding random-walk problem, and for  $N = 1, 2,$  and  $3$  is the Ising,  $XY$ , and Heisenberg model, respectively. The infinite component version is the spherical model [1].

Above the upper critical dimension,  $d_c = 4$ , the scaling behaviour of the  $O(N)$  theory simplifies and the critical exponents are exactly those given by mean field theory. It is rigorously known that the continuum limit is then trivial and described by free fields [2]. Perturbative arguments show that, for dimensionality  $d > d_c$ , the corrections to scaling are additive in nature and are governed by exponents which are independent of  $N$ . The  $N$ -dependency of the theory resides in the amplitudes of these corrections [3]. (Notwithstanding this, the precise nature of finite size effects in the  $O(N)$  theory above  $d_c$  has recently been the subject of some discussion and a better understanding is still required [4].)

The above situation contrasts with that of low dimensional models. Indeed, below the upper critical dimension the super-renormalizable  $\phi^4$ -theory is non-trivial [5]. There, the leading critical behaviour, and even the existence or otherwise of a phase transition, is strongly dependent on the value of  $N$ .

At the upper critical dimension,  $d_c = 4$ , leading scaling behaviour for  $\phi^4$ -theory is coincident with mean field theory; however it is modified by multiplicative logarithmic corrections. The universality class of a phase transition cannot, therefore, be determined through the leading critical exponents alone (since they are  $N$ -independent). The  $N$ -dependency resides in the logarithmic corrections and these can be used, in principle, to establish the universality class [6].

The logarithmic corrections at the upper critical dimension are intimately linked to the triviality of the theory there. In fact, it has been shown that if the leading scaling behaviour of the susceptibility differs from that of mean field theory only by multiplicative logarithmic corrections then the  $\phi_4^4$ -theory must be trivial [7]. These logarithmic corrections are the concern of this work.

The  $N = 4$  version of  $\phi^4$ -theory, in four dimensions, is of especially great interest to high energy physicists, as it forms the scalar sector of the standard model where it plays a central rôle in the generation of fermion and gauge boson mass via the Higgs mechanism. Triviality of the  $O(4)$   $\phi^4$ -theory in four dimensions is therefore an especially important issue. There is an abundance of analytical [1, 8, 9, 10, 11] and numerical [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] evidence in favour of triviality, mostly for the single-component theory. However, since a complete rigorous proof is still lacking, various ideas aimed at recovering a non-trivial continuum theory are also being explored [23].

In high energy physics, lattice gauge theories are also described by a mean field equation of state with logarithmic corrections. The issue of logarithmic triviality of these and related models in four dimensions was explored in [24]. Multiplicative logarithmic corrections in four dimensions also play an important role in non-equilibrium phase transitions (and in directed percolation in particular [25, 26]). Besides physics, such systems are of

direct interest in biology and chemistry [6, 27] as well as in the study of avalanches in sandpile models [28, 29].

The renormalization group yields predictions on how criticality is approached in the infinite-volume four-dimensional case [8, 9, 14]. With the reduced temperature,  $t$ , denoting the distance from criticality, the perturbative formula for the susceptibility is

$$\chi_{\infty}(t) \propto |t|^{-1}(-\ln |t|)^{\frac{N+2}{N+8}} \left\{ 1 + O\left(\frac{\ln(-\ln |t|)}{\ln |t|}\right) \right\} \quad , \quad (1.1)$$

while that of the singular part of the specific heat is

$$c_{\infty}(t) = \begin{cases} (-\ln |t|)^{\frac{4-N}{N+8}} \left\{ 1 + O\left(\frac{\ln(-\ln |t|)}{\ln |t|}\right) \right\} & \text{if } N \neq 4 \\ \ln(-\ln |t|) \left\{ 1 + O\left(\frac{1}{\ln |t|}\right) \right\} & \text{if } N = 4 \end{cases} \quad , \quad (1.2)$$

and that of the correlation length is

$$\xi_{\infty}(t) = |t|^{-\frac{1}{2}}(-\ln |t|)^{\frac{N+2}{2(N+8)}} \left\{ 1 + O\left(\frac{\ln(-\ln |t|)}{\ln |t|}\right) \right\} \quad . \quad (1.3)$$

(In (1.1), and henceforth in this paper,  $\chi$  refers to the longitudinal susceptibility in the case where  $N \geq 2$  and  $t < 0$ .)

In each of these formulae, the leading power-law scaling behaviour coincides with the predictions of mean field theory. One notes that the multiplicative logarithmic corrections are  $N$ -dependent in each case. Equations (1.1) and (1.3) have been rigorously proven for the weakly coupled version of the single-component theory [10]. No proof exists for the strongly coupled or general- $N$  theories.

Finite size scaling [FSS] is an important tool used in the determination of universality classes and most modern Monte Carlo studies exploit it to determine critical exponents [30]. In particular, recent analyses of systems at the upper critical dimension have adopted a finite-sized approach [21, 29, 31, 32, 33, 34, 35, 36].

FSS was formulated as a hypothesis by Fisher and co-workers [30] and proved for  $d < d_c = 4$  on the basis of the renormalization group by Brézin [37]. Let  $P_L(t)$  be a thermodynamic quantity measured on a system of linear extent  $L$  at a distance  $t$  from the bulk critical point. The traditional FSS hypothesis states that if  $P_{\infty}$  exhibits an algebraic singularity (at  $t = 0$  in the infinite system), then its finite size behaviour (in this region) is given by

$$\frac{P_L(t)}{P_{\infty}(t)} = \mathcal{F}_P\left(\frac{L}{\xi_{\infty}(t)}\right) \quad , \quad (1.4)$$

where  $\xi_{\infty}(t)$  is the correlation length of the infinite-size system. Here,  $\mathcal{F}_P$  is an a priori unknown function of its argument which is called the scaling variable.

Brézin's theoretical justification of (1.4) relies on two assumptions over and above the usual assumptions of renormalization group theory. These are that (i) the system length,  $L$ , is not renormalized in the flow equations and that (ii) the infra-red fixed point is not zero. This latter assumption fails in and above four dimensions.

At the upper critical dimension, then, the FSS formulae have had to be calculated directly from the (perturbative) renormalization group equations [37]. The  $N = 0$  case introduces significant simplifications which were exploited in [12] to test the renormalization group predictions for logarithmic corrections numerically. (However precise agreement in this case is obscured by large subleading corrections [20].)

Early attempts to verify the perturbative predictions numerically, in the Ising case, were hampered by limited computational resources [13]. Logarithmic triviality of the Ising model in four dimensions was eventually established using sophisticated techniques in [16]. At present, no direct calculations have been carried out for finite  $N > 1$ . Given the recent enormous advances in computer hardware and algorithms, direct computational detection of logarithms in high dimensional  $O(N)$  systems is now feasible and it is reasonable to ask what results might be expected.

The purpose of this paper is twofold. Firstly, some results concerning the FSS of the  $O(N)$  model are presented, paying particular attention to the multiplicative logarithmic corrections. Wherever possible, these results are compared with ones already available in the literature (specifically for the Ising and spherical models) and in each case complete agreement is found. It turns out that the leading logarithmic corrections to the finite size dependency are independent of  $N$  in the odd sector. This means that numerical confirmation of their existence in the Ising model also serves as a non-perturbative test for all  $N$  [16]. Secondly, all of the renormalization group finite size results can be recovered from a modified version of the FSS hypothesis, extending its validity to the upper critical dimension.

The layout of the remainder of this paper is as follows. For completeness, renormalization group techniques are recalled in Sec.2 and applied to the finite-sized  $O(N)$  four-dimensional theory. The substantive results concerning the detailed  $N$ -dependence of the logarithmic corrections to FSS for the partition function zeroes and thermodynamic functions are presented in Sec.3. The general applicability of the modified FSS hypothesis, valid at and below the upper critical dimension, is verified in Sec.4 and conclusions are drawn in Sec.5.

## 2 Finite Size Renormalisation Group Analysis

The starting point is the action for the  $N$ -component  $\phi^4$ -theory in  $d$ -dimensional space, which is

$$S = \int d^d x \left[ \frac{1}{2}(\nabla\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g_0}{4!}(\nabla\phi^2)^2 + \mathbf{H}(x)\nabla\phi(x) \right] , \quad (2.1)$$

where  $\phi$  is an  $N$ -dimensional vector,  $m_0$  the bare bosonic mass in the quantum field theory,  $g_0$  the bare quartic self-coupling and  $\mathbf{H}$  an external field. Writing  $m_0^2 = m_{0c}^2 + t(x)$ , where  $m_{0c}$  is the critical bare mass for which the renormalized theory is massless,  $t(x)$  becomes a source for quadratic composite fields. Its inclusion facilitates the derivation of energy-energy correlation functions (i.e., the specific heat). If  $t$  is allowed to become independent of  $x$ , it is a measure of the distance from criticality [9]. In the statistical physics theory, the squared mass corresponds to the Boltzmann factor and  $t$  is called the

reduced temperature.

The generating functional for the connected Green's functions for the  $N$ -component theory is

$$\exp W[t, \mathbf{H}] \propto \int \prod_x \prod_\alpha d\phi_\alpha(x) \exp(-S) \quad , \quad (2.2)$$

where the constant of proportionality is such that  $W[0, \mathbf{0}] = 0$ , and  $\alpha = 1, \dots, N$  label the field components. The function conjugate to  $H_\alpha$  is

$$M_\alpha(t, \mathbf{H}, x) = \frac{\delta W[t, \mathbf{H}]}{\delta H_\alpha(x)} \quad (2.3)$$

and facilitates the following expression for the generating functional for one-particle irreducible vertex functions:

$$\Gamma[t, \mathbf{M}] + W[t, \mathbf{H}] = \int dx \mathbf{H}(x) \mathbf{M}(t, \mathbf{H}, x) \quad . \quad (2.4)$$

This gives

$$H_\alpha(x) = \frac{\delta \Gamma[t, \mathbf{M}]}{\delta M_\alpha} \quad . \quad (2.5)$$

The four-dimensional version of the theory possesses certain subtleties not present in lower dimensions. The infra-red fixed point of the Callan-Symanzik function moves to the origin as the dimensionality becomes four (i.e., the perturbative approach predicts that the theory is trivial) and becomes a double zero. This latter phenomenon is responsible for the occurrence of multiplicative logarithmic corrections. There is also an inhomogeneous term in the renormalization group equations. The graph responsible for this term is not divergent in less than four dimensions where singular behaviour comes from the homogeneous term. In four dimensions the inhomogeneous term can and does contribute to the leading scaling behaviour.

The extension from the single component  $\phi_4^4$ -theory to the  $N$ -component version is straightforward [9, 13, 16]. The critical theory is firstly renormalized. The bulk renormalization constants are sufficient to renormalize the finite size theory and this renormalization is performed at an arbitrary non-zero mass parameter,  $\mu$ , in order to control infra-red divergences. One then applies a Taylor expansion about  $t = 0$  and  $\mathbf{M} = \mathbf{0}$  to the vertex functions to find the renormalization group equations for the massive theory. So even though the symmetry might be (explicitly or spontaneously) broken, the vertex functions in the critical region are evaluated with  $\mathbf{M} = \mathbf{0}$ . The question of whether there exists spontaneous breaking of symmetry (in the infinite-volume case) is answered by appealing to (2.5) with  $\mathbf{H} = \mathbf{0}$  and searching for solutions for which  $\mathbf{M} \neq \mathbf{0}$  [38].

Following [9, 13, 16] and using dimensional analysis, the solution for the renormalized free energy is given by

$$f_R(t, \mathbf{M}, g_R, \mu, L) = L^{-4} f_R(L^2 t(\lambda), L \mathbf{M}(\lambda), g_R(\lambda), L \mu(\lambda), 1) + \Pi(\lambda; t) \quad , \quad (2.6)$$

where

$$\Pi(\lambda; t) = -\frac{1}{2!} \int_1^\lambda \frac{d\lambda'}{\lambda'} t(\lambda')^2 \Upsilon(g_R(\lambda')) \quad . \quad (2.7)$$

Here,  $\mu(\lambda) = \lambda\mu$  is a rescaling of the arbitrary mass  $\mu$ . The functions  $g_R(\lambda)$ ,  $\mathbf{M}(\lambda)$ ,  $t(\lambda)$  and  $\Upsilon(g_R(\lambda))$  respond to this rescaling through the flow equations. To leading order in perturbation theory the flow equations for the  $O(N)$  theory are [9, 13]

$$\frac{dg_R(\lambda)}{d \ln \lambda} = \frac{N+8}{6} g_R(\lambda)^2 \{1 + O(g_R(\lambda))\} \quad , \quad (2.8)$$

$$\frac{d \ln t(\lambda)}{d \ln \lambda} = \frac{N+2}{6} g_R(\lambda) \{1 + O(g_R(\lambda)^2)\} \quad , \quad (2.9)$$

$$\frac{d \ln M_\alpha(\lambda)}{d \ln \lambda} = -\frac{N+2}{144} g_R(\lambda)^2 \{1 + O(g_R(\lambda))\} \quad , \quad (2.10)$$

$$\Upsilon(g_R(\lambda)) = \frac{N}{2} \{1 + O(g_R(\lambda))\} \quad . \quad (2.11)$$

For  $\lambda \ll 1$  the solutions to these equations are

$$g_R(\lambda) = a_2 (-\ln \lambda)^{-1} \quad , \quad (2.12)$$

$$t(\lambda) = a_1 t (-\ln \lambda)^{-\frac{N+2}{N+8}} \quad , \quad (2.13)$$

$$M_\alpha(\lambda) = b_1 M_\alpha \quad , \quad (2.14)$$

$$\Pi(\lambda; t) \propto \begin{cases} a_3 t^2 (-\ln \lambda)^{\frac{4-N}{N+8}} & \text{for } N \neq 4 \\ b_2 t^2 \ln(-\ln \lambda) & \text{for } N = 4 \end{cases} \quad , \quad (2.15)$$

where the  $a_j$  have the form

$$a \left\{ 1 + O\left(\frac{\ln |\ln \lambda|}{\ln \lambda}\right) \right\} \quad , \quad (2.16)$$

and the  $b_j$  are

$$b \left\{ 1 + O\left(\frac{1}{\ln \lambda}\right) \right\} \quad . \quad (2.17)$$

The  $a_j$  and  $b_j$  are dependent on  $N$  through the prefactors of (2.16) and (2.17) and there is no other  $N$  dependency in (2.12) or (2.14). This is due to the fact that, in perturbation theory in four dimensions, (2.8) and (2.10) begin with quadratic terms in the running coupling. Only the logarithmic term coupled to the reduced temperature  $t$  in (2.13) has an  $N$ -dependent exponent and this is due to the term linear in  $g_R(\lambda)$  in (2.9).

Now, choosing  $\lambda = L^{-1}$  and applying perturbation theory to the homogeneous term of (2.6), one finds

$$f_R(t, \mathbf{M}, g_R, 1, L) = a'_1 t M^2 (\ln L)^{-\frac{N+2}{N+8}} + a'_2 M^4 (\ln L)^{-1} + \Pi(L^{-1}; t) \quad , \quad (2.18)$$

where  $M = |\mathbf{M}|$  and  $a'_1$  and  $a'_2$  are of the form (2.16) above, with  $\lambda$  replaced by  $L^{-1}$ .

Applying (2.5) to this yields, for the ( $x$ -independent) external field,

$$H_\alpha(t, \mathbf{M}, g_R, 1, L) = 2a'_1 t M_\alpha (\ln L)^{-\frac{N+2}{N+8}} + 4a'_2 M^2 M_\alpha (\ln L)^{-1} \quad . \quad (2.19)$$

Note, again, that there is no  $N$  dependence in the logarithm in the second term on the right hand side of (2.18) and consequently in the logarithmic part of the last term of (2.19).

This crucial fact will ultimately lead to the  $N$  independence of the multiplicative logarithmic corrections to the FSS of the Lee-Yang zeroes and associated odd thermodynamic functions.

Indeed,  $N$  independence in the odd sector can already be seen in the infinite-volume magnetization. There, the expressions for  $f_R$  and  $H = |\mathbf{H}|$  are similar to those of (2.18) and (2.19), except that  $\ln L$  is replaced by  $\ln M$  [9]. The strategy there is to set  $H = 0$  and to eliminate the spontaneous magnetization in favour of  $t$ . (Thus the thermodynamic scaling behaviour (scaling with  $t$ ) in (1.1) to (1.3) exhibit  $N$ -dependent logarithmic corrections.) If, however,  $t$  is set to zero in the infinite-volume counterpart of (2.19), one finds, up to leading logarithms (using  $H_\alpha = H\delta_{\alpha,1}$  to isolate the first component as the longitudinal one),

$$M(H) \propto H^{\frac{1}{3}}(-\ln H)^{\frac{1}{3}} \quad , \quad (2.20)$$

independent of  $N$  [9].

From the Legendre transformation (2.4), and from (2.18) and (2.19), the free energy per unit volume in the presence of an external field is

$$W_L(t, \mathbf{H}) = a'_1 t M^2 (\ln L)^{-\frac{N+2}{N+8}} + 3a'_2 M^4 (\ln L)^{-1} - \Pi(L^{-1}; t) \quad . \quad (2.21)$$

From this expression the FSS relations can be derived. From (2.19) and (2.21), if  $t = 0$ ,

$$W_L(0, \mathbf{H}) \propto H^{\frac{4}{3}} (\ln L)^{\frac{1}{3}} \left\{ 1 + O\left(\frac{\ln(\ln L)}{\ln L}\right) \right\} \quad . \quad (2.22)$$

On the other hand, when the external magnetic field vanishes, (2.19) gives  $M = 0$  (corresponding to the symmetric phase) or (up to additive correction terms)

$$M^2 \propto (-t) (\ln L)^{\frac{6}{N+8}} \left\{ 1 + O\left(\frac{\ln(\ln L)}{\ln L}\right) \right\} \quad , \quad (2.23)$$

which corresponds to the  $t < 0$  phase. In either case, (2.21) gives

$$W_L(t, \mathbf{0}) \propto \begin{cases} t^2 (\ln L)^{\frac{4-N}{N+8}} \left\{ 1 + O\left(\frac{\ln(\ln L)}{\ln L}\right) \right\} & \text{if } n \neq 4 \\ t^2 \ln(\ln L) \left\{ 1 + O\left(\frac{1}{\ln L}\right) \right\} & \text{if } n = 4 \end{cases} \quad . \quad (2.24)$$

This functional form for the free energy holds regardless of phase.

### 3 Partition Function Zeroes and Thermodynamic Functions

The zeroes of the partition function present an analytically powerful and numerically precise approach to the study of critical phenomena, which is complimentary to the more traditional functional approach. Zeroes in the complex external-field plane are referred to as Lee-Yang zeroes [39] while their complex-temperature counterparts are Fisher zeroes [40]. In particular, the FSS behaviour of these zeroes can be used to extract ratios of



critical exponents. The precise scaling formulae were given for  $d < d_c$  in [41] and for the single-component four-dimensional  $\phi^4$ -theory in [16]. The  $O(N)$ -generalization of the latter can now be derived from the expressions (2.22) and (2.24) for the free energy.

From the expression (2.22), for the finite-size free energy at  $t = 0$ , the partition function must take the form

$$Z_L(0, H) = Q \left( H^{\frac{4}{3}} L^4 (\ln L)^{\frac{1}{3}} \left\{ 1 + O \left( \frac{\ln(\ln L)}{\ln L} \right) \right\} \right) , \quad (3.1)$$

for some unknown function  $Q$ . At a complex Lee-Yang zero,  $H_j(L)$ , this vanishes. Following [41], and inverting (3.1), the FSS formula for the Lee-Yang zeroes is found to be

$$H_j(L) \sim L^{-3} (\ln L)^{-\frac{1}{4}} \left\{ 1 + O \left( \frac{\ln(\ln L)}{\ln L} \right) \right\} . \quad (3.2)$$

To reiterate, there is no  $N$  dependence in the power of the logarithmic corrections here for the following reason. The expression (3.2) is a direct consequence of (2.22), which itself comes from the free energy in (2.21). There, only the logarithmic terms coupled to the reduced temperature,  $t$ , are  $N$ -dependent, for the reasons explained in Sec. 2. The  $N$  independency of the remaining terms is due to the fact that the perturbative expressions (2.8) and (2.10) begin with quadratic terms in the running coupling. In the derivation of the FSS behaviour of the Lee-Yang zeroes, the reduced temperature,  $t$ , has been set to zero and the associated  $N$ -dependent logarithms of (2.21) are therefore absent. Consequently, there will also be no  $N$  dependency in the multiplicative logarithmic corrections to derivable thermodynamic functions such as the magnetic susceptibility.

Writing the partition function as a product over its Lee-Yang zeroes,

$$Z_L(t, H) \propto \prod_j (H - H_j(L)) , \quad (3.3)$$

the (longitudinal) magnetic susceptibility is directly derived as the second derivative of the free energy with respect to  $H$ ,

$$\chi_L(t, H) \propto \frac{1}{L^4} \sum_j \frac{1}{(H - H_j(L))^2} . \quad (3.4)$$

The zero field susceptibility is then

$$\chi_L(t) \propto \frac{1}{L^4} \sum_j \frac{1}{H_j(L)^2} . \quad (3.5)$$

It is reasonable (and usual [16, 42]) to assume that the scaling of the susceptibility is dominated by the behaviour of the zeroes closest to the real axis. Then, (3.2) and (3.5) give the FSS formula

$$\chi_L(0) \propto L^2 (\ln L)^{\frac{1}{2}} \left\{ 1 + O \left( \frac{\ln(\ln L)}{\ln L} \right) \right\} . \quad (3.6)$$

This is independent of  $N$ , as anticipated.

Similar reasoning can be used to determine the FSS behaviour of the Fisher zeroes using (2.24). Setting the corresponding partition function to zero and solving for  $t$  gives the following FSS formula for the Fisher zeroes in four dimensions;

$$t_j(L) \sim \begin{cases} L^{-2}(\ln L)^{\frac{N-4}{2(N+8)}} \left\{ 1 + O\left(\frac{\ln(\ln L)}{\ln L}\right) \right\} & \text{if } n \neq 4 \\ L^{-2}(\ln(\ln L))^{-\frac{1}{2}} \left\{ 1 + O\left(\frac{1}{\ln L}\right) \right\} & \text{if } n = 4 \end{cases} . \quad (3.7)$$

From the expression (3.7), the FSS of the even thermodynamic functions are easily derived. In the absence of an external ordering field, the partition function may be written

$$Z_L(t, 0) \propto \prod_j (t - t_j(L)) \quad . \quad (3.8)$$

The specific heat, given by the second derivative of the free energy with respect to  $t$ , is

$$C_L(t) = -\frac{1}{L^4} \sum_j \frac{1}{(t - t_j)^2} \quad . \quad (3.9)$$

At the bulk critical point,  $t = 0$ , this becomes

$$C_L(0) = -\frac{1}{L^4} \sum_j \frac{1}{t_j(L)^2} \quad , \quad (3.10)$$

which, together with (3.7) and the reasonable assumption that the first few zeroes dominate scaling [16, 42], yields

$$C_l(0) \sim \begin{cases} (\ln L)^{\frac{4-N}{N+8}} \left\{ 1 + O\left(\frac{\ln(\ln L)}{\ln L}\right) \right\} & \text{if } n \neq 4 \\ \ln(\ln L) \left\{ 1 + O\left(\frac{1}{\ln L}\right) \right\} & \text{if } n = 4 \end{cases} \quad . \quad (3.11)$$

In particular, the  $N = 1$  result recovers the theoretical results of [13] and [15] for the singular part of the Ising specific heat and the  $N \rightarrow \infty$  limit recovers the result of [43] for the spherical model in four dimensions. The result (3.11) is also in agreement with a *conjecture* made in [43], except in the case of  $N = 4$ . In that case, log-log corrections are dominant in (3.11). Furthermore, the  $N$ -independent result (3.6) is coincident with the Ising result of [15] as well as with results for the spherical model (given by  $N \rightarrow \infty$ ) explicitly obtained in [43] and separately in [44].

The real part of the first Fisher zero may be considered a pseudocritical point. For general  $N \neq 4$ , (3.7) gives that this pseudocritical point approaches the critical one as  $L^{-2}(\ln L)^{(N-4)/2(N+8)}$ . In the  $N = 1$  case, this is coincident with the specific-heat pseudocritical-point scaling,  $L^{-2}(\ln L)^{-1/6}$ , derived from direct renormalization group considerations of the Ising model in [13] and with the equivalent susceptibility peak position derived in [15]. For  $N = 4$  the pseudocritical point scaling changes to  $L^{-2}(\ln(\ln L))^{-1/2}$ , from (3.7).

The  $N$  independence of the leading multiplicative logarithmic corrections in the odd sector is fortuitous. That is the sector to which the rigorous results of [7] concerning

the triviality of the  $O(N)$ - $\phi_4^4$  theory apply. These results state that if the leading scaling behaviour of the susceptibility differs from that of mean field theory only by multiplicative logarithmic corrections then the theory must be trivial. The perturbative arguments above demonstrate that the logarithmic corrections in the susceptibility are intimately linked to those of the Lee-Yang zeroes and that these are, in fact, independent of  $N$ . Thus independent numerical confirmation of their existence is strong evidence for the triviality of the theory for all  $N$ . Such non-perturbative evidence was provided in [16, 19, 22] (see also [12, 13, 15, 18, 20]).

On the other hand, any future numerical attempts to detect the  $N$  dependence of the leading logarithms must focus on the even sector. To our knowledge, the only non-perturbative attempts that have so far been made to verify logarithmic corrections in this sector are confined to  $N = 1$  [13, 15, 16, 18, 22].

## 4 A Modified Finite Size Scaling Hypothesis

The traditional FSS hypothesis for a thermodynamic function,  $P$ , is given in (1.4). The hypothesis is based on the physical assumption that the only relevant length scales involved are the actual length,  $L$ , of the finite-sized system and the correlation length,  $\xi_\infty(t)$ , of the bulk system. As a hypothesis, (1.4) is a non-perturbative statement, but can be backed up by renormalization group arguments [37].

It has long been known that the standard FSS hypothesis, (1.4), fails for  $d \geq d_c$  [45]. A modified hypothesis was proposed in [16], which correctly recovers the perturbative renormalization group predictions for FSS in the case of the single-component  $\phi_4^4$ -theory as well as recovering the traditional form of the hypothesis below four dimensions. In fact this modified hypothesis is successful for all  $N$  in four dimensions, as demonstrated in the sequel (see also [21, 32]).

In the modified FSS hypothesis, the actual length of the finite-sized system is replaced by its correlation length,

$$\frac{P_L(t)}{P_\infty(t)} = \mathcal{F}_P \left( \frac{\xi_L(0)}{\xi_\infty(t)} \right) \quad . \quad (4.1)$$

The finite size behaviour of correlation length for four dimensions was calculated up to leading logarithms in [37] as

$$\xi_L(0) \propto L (\ln L)^{\frac{1}{4}} \quad , \quad (4.2)$$

independent of  $N$ . A change in system size necessitates a corresponding change in the temperature to keep the scaling variable,  $x = \xi_L(0)/\xi_\infty(t)$ , fixed. This change is

$$|t| \propto x^2 L^{-2} (\ln L)^{\frac{N-4}{2(N+8)}} \quad . \quad (4.3)$$

Inserting (4.3) into (1.1) and (1.2) recovers the perturbative renormalization group predictions (3.6) and (3.11) for the critical susceptibility and specific heat, respectively. The modified hypothesis, (4.1), also reduces to the traditional one, (1.4), for  $d < d_c$  as  $\xi_L(0) \propto L$  there.

Recently, Aktekin argued that the singular part of the free energy of the Ising model at the upper critical dimension obeys the Privman-Fisher-type ansatz [21, 46]

$$f_L(t, H) = L^{-4} \mathcal{F} \left( tL^2 \ln^{1/6} L, HL^3 \ln^{1/4} L \right) \quad . \quad (4.4)$$

From this, the behaviour of the zeroes again follows. Setting  $t = 0$ , the partition function takes the leading form of (3.1) and hence recovers the leading scaling for the Lee-Yang zeroes (3.2). Likewise setting  $h = 0$  in (4.4) recovers the correct form, (3.7), for the Fisher zeroes when  $N = 1$ . A simple extension of Aktekin's formula, (4.4), can now be proposed, which recovers the FSS formulae in the general- $N$  case:

$$f_L(t, h) = L^{-4} \mathcal{F} \left( tL^2 (\ln L)^{\frac{4-N}{2(N+8)}}, HL^3 (\ln L)^{\frac{1}{4}} \right) \quad . \quad (4.5)$$

As well as recovering (3.2) in the odd sector, this recovers the correct form, (3.7), for the zeroes in the even sector, except in the  $N = 4$  case. For the latter, the Aktekin's formula should be further modified, to read

$$f_L(t, h) = L^{-4} \mathcal{F} \left( tL^2 (\ln (\ln L))^{\frac{1}{2}}, HL^3 (\ln L)^{\frac{1}{4}} \right) \quad . \quad (4.6)$$

The functional form (4.5) also coincides with the large- $L$  behaviour of the free energy for  $N \neq 4$  spin models with long-range interactions [31] (see also [26, 47]).

## 5 Conclusions

The issue of multiplicative logarithmic corrections to mean field scaling behaviour is one of importance in both high energy and statistical physics. No rigorous proof of their presence in  $O(N)$   $\phi_4^4$ -theories exists, however there is strong analytical and numerical evidence that this is, in fact, the case.

Long before such logarithms were verified nonperturbatively in the Ising case [16], explicit calculations had been performed to see what subtleties simulators may expect in order to achieve a proper finite size extrapolation there [13]. At present, there still exists no direct non-perturbative verification of the existence and nature of these logarithms in the general  $N$  case in four dimensions. Here we have seen that any future numerical attempts to detect the  $N$  dependence of the leading logarithms will only come from the even sector. On the other hand, the fortuitous result that the logarithms in the odd sector are independent of  $N$  means that a nonperturbative test of their behaviour in the Ising case may be interpreted as a test of triviality for all  $N$  [16].

The results derived here are in agreement with previous results in the literature, which pertain specifically to the Ising ( $N = 1$  case) and spherical ( $N \rightarrow \infty$  case) models at the upper critical dimension.

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