Dimer model on a triangular lattice

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Abstract

We analyze the partition function of the dimer model on an $\mathcal{M} \times \mathcal{N}$ triangular lattice wrapped on torus obtained by Fendley, Moessner and Sondhi [Phys. Rev. B **66**, 214513 (2002)]. From a finite-size analysis we have found that the dimer model on such a lattice can be described by conformal field theory having central charge $c = -2$. The shift exponent for the specific heat is found to depend on the parity of the number of lattice sites $\mathcal{N}$ along a given lattice axis: e.g., for odd $\mathcal{N}$ we obtain the shift exponent $\lambda = 1$, while for even $\mathcal{N}$ it is infinite ($\lambda = \infty$). In the former case, therefore, the finite-size specific-heat pseudocritical point is size dependent, while in the latter case, it coincides with the critical point of the thermodynamic limit.

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I. INTRODUCTION

In experiments and in numerical studies of critical phenomena, it is essential to take into account finite-size effects in order to extract correct infinite-volume predictions from the data. Therefore, in recent decades there have been many investigations of finite-size scaling, finite-size corrections, and boundary effects for critical model systems. In the quest to improve our understanding of realistic systems of finite extent, two-dimensional models play a crucial role in statistical mechanics as they have long served as a testing ground to explore the general ideas of finite-size scaling under controlled conditions. Very few of them have been solved exactly, the Ising model [1–9] and the dimer model [10–15] being the most prominent examples.

The dimer model is a two-particle system. The main difference between it and one-particle systems such as Ising, Heisenberg, or Potts models etc., is that occupation of a given lattice site ensures that at least one of its nearest-neighbor sites is also occupied. It is well known that, due to this non-locality, the critical behavior exhibited by the dimer model can depend on the lattice structure and shape (square, triangle, honeycomb, etc). Previous studies have shown that finite-size corrections in the free energy can exhibit a strong dependence upon the parity of the lattice, and this has provoked controversial conclusions about the value of the central charge from $c = -2$ to $c = 1$. One can expect that such unusual finite-size behavior should also hold for the dimer model on other lattices and here we investigate the triangular lattice in particular.

We are particularly interested in the finite-size scaling behavior of the specific-heat pseudocritical point. In finite systems the counterparts of the singularities which mark higher-order phase transitions in the thermodynamic limit are smooth peaks the shapes of which depend on the critical exponents. In particular, let $C(t, L)$ be the specific heat at a reduced temperature given by t for a system of linear extent characterized by L. In the infinite-volume limit, $C(t, \infty)$ diverges at the critical point $t = t_c = 0$. In finite volume, the analog to the divergence is a finite peak the shape of which is characterized by (i) its position t_{pseudo} (ii) its height $C(t_{\text{pseudo}}, L)$ and (iii) its value at the infinite-volume critical point $C(0, L)$. In particular, the position of the specific heat peak, t_{pseudo}, is a pseudocritical point which approaches $t_c = 0$ as $L^{-\lambda}$, where λ is called the shift exponent. In most models exhibiting higher-order phase transitions, the shift exponent coincides with the inverse of

the correlation-length critical exponent $1/\nu$, but this is not a direct conclusion of FSS theory is not always true.

For example, for the Ising model in two dimensions, Ferdinand and Fisher determined that behavior of the specific-heat pseudocritical point matches that of the correlation length with $\lambda = 1/\nu = 1$ [1]. However, Ising models defined on two-dimensional lattices with other topologies have shift exponents which differ from the inverse correlation length critical exponent (see Ref.[2] and references therein). This is despite the fact that the critical properties on such lattices are the same as for the torus in the thermodynamic limit. A question we wish to address here is the corresponding status of the shift exponent in the dimer model.

In contrast to spin models, the critical behavior of dimer models are strongly influenced by the structure of the underlying lattice. For example the square lattice dimer model is critical with algebraic decay of correlators [16, 17], while the dimer model on the anisotropic honeycomb lattice, which is equivalent to five-vertex model on the square lattice [18], exhibits a KDP-type singularity and the dimer model on the Fisher-type lattice exhibits Ising-type transitions [19]. Thus, it appears that the dimer model itself has not a single critical behavior, but several critical behaviors associated with different classes of universality.

It has been shown explicitly [20] that the free energy per site for the dimer model on the square lattice is insensitive to the precise form of the boundary conditions in the limit of a large lattice. This is in contrast to its finite-size counterpart, for which sensitivity to boundary conditions is notable feature, in particular to the parity of the number of lattice sites along a given lattice axis [11, 21]. Similar statements hold for the dimer model on the honeycomb and triangular lattices.

Very recently, it has been shown [12] that the finite-size corrections of the dimer model on planar $\infty \times \mathcal{N}$ square lattices also depend crucially on the parity of $\mathcal{N}$ and the boundary conditions and such unusual finite-size behavior can be fully explained in the framework of the $c = -2$ logarithmic conformal field theory.

Our objective in this paper is to study the finite size properties of dimer model on the plane triangular lattice using the same techniques developed in papers [4, 11]. The paper is organized as follows. In Sec. II we introduce the dimer model on the triangular lattice with periodic boundary conditions. In Sec. III we discuss the finite-size corrections for an infinitely long cylinder of circumference $\mathcal{N}$ and find that the dimer model on the triangular
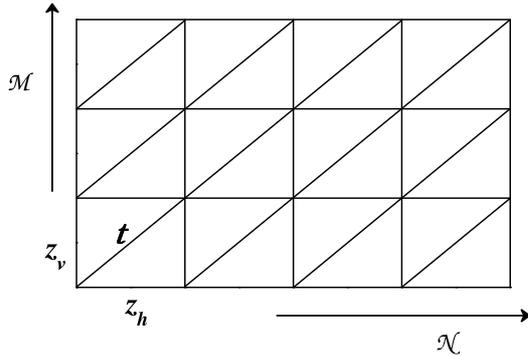
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Figure 1: The $\mathcal{M} \times \mathcal{N}$ triangular lattice with dimer weights z_h in the horizontal direction, z_v in the vertical direction, and t in the diagonal direction.

lattice can be described by conformal field theory with central charge $c = -2$. In Sec. IV we investigate the properties of the specific heat near the critical point and find that the specific-heat shift exponent λ depends on the parity of the number of lattice sites along the lattice axis $\mathcal{N}$. For odd $\mathcal{N}$ we obtain for the shift exponent $\lambda = 1$, while for even $\mathcal{N}$ we find that the shift exponent λ is infinity ($\lambda = \infty$). Our results are summarized and discussed in Sec. V.

II. PARTITION FUNCTION

In the present work, we consider the dimer model on $\mathcal{M} \times \mathcal{N}$ triangular lattice under periodic boundary conditions. The partition function is given by

$$Z_{\mathcal{M},\mathcal{N}}(z_h, z_v, t) = \sum z_h^{n_h} z_v^{n_v} t^{n_t}, \tag{1}$$

where summation is taken over all dimer covering configurations, where z_h, z_v and t are, respectively, dimer weights in the horizontal, vertical and diagonal directions, and where n_h, n_v and n_t are, respectively, the number of horizontal, vertical and diagonal dimers (Fig.1). The dimer model on the triangular lattice undergoes phase transition at the point $t = t_c = 0$ (and likewise for z_h and z_v), where the partition function is singular. Thus the general triangular lattice model is critical in the square lattice limit. The dimer weight t plays a role similar to the reduced temperature in the Ising model. In what follows, we will set $z_h = z_v = 1$.

An explicit expression for the partition function of the dimer model on an $\mathcal{M} \times \mathcal{N}$

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triangular lattice wrapped on torus has been obtained by Fendley, Moessner and Sondhi [22] and can be written as [15]

$$Z_{\mathcal{M},\mathcal{N}}(t) = \frac{1}{2} \left[G_{0,0}(t,\mathcal{M},\mathcal{N}) + G_{0,1/2}(t,\mathcal{M},\mathcal{N}) + G_{1/2,0}(t,\mathcal{M},\mathcal{N}) + G_{1/2,1/2}(t,\mathcal{M},\mathcal{N}) \right],$$

(2)

where

$$\begin{aligned}
G_{\alpha,\beta}^2(t,\mathcal{M},\mathcal{N}) = \prod_{m=0}^{\mathcal{M}/2-1} \prod_{n=0}^{\mathcal{N}-1} & 4 \left[\sin^2 \frac{2\pi(n+\alpha)}{\mathcal{N}} \right. \\
& \left. + \sin^2 \frac{2\pi(m+\beta)}{\mathcal{M}} + t^2 \cos^2 \left(\frac{2\pi(n+\alpha)}{\mathcal{N}} + \frac{2\pi(m+\beta)}{\mathcal{M}} \right) \right],
\end{aligned}$$

(3)

for even $\mathcal{M}$. The notation $\alpha = 0$ corresponds to periodic boundary conditions for the underlying free fermion in the $\mathcal{N}$-direction while $\alpha = \frac{1}{2}$ represents anti-periodic boundary conditions. The boundary conditions in the $\mathcal{M}$-direction are similarly controlled by the parameter β.

Since the total number of sites must be even if the lattice is to be completely covered by dimers, we will consider two cases, namely even-even (ee) case when $\mathcal{M} = 2M$ and $\mathcal{N} = 2N$, and even-odd (eo) case when $\mathcal{M} = 2M$ and $\mathcal{N} = 2N+1$. Note that due to the symmetry of the lattice the odd-even case (oe) ($\mathcal{M} = 2M+1$, $\mathcal{N} = 2N$) can be obtained from even-odd case by simple transformation $\xi \to 1/\xi$, where $\xi = \mathcal{M}/\mathcal{N}$ is a aspect ratio.

a). Dimers on $2M \times 2N$ lattices

In the even-even case where $(\mathcal{M},\mathcal{N}) = (2M,2N)$, the second product in (3) may be compactly written as $\prod_{n=0}^{2N-1} F(n+\alpha, m+\beta)$ where the function $F(x,y)$ is given by

$$F(x,y) = 4 \left[\sin^2 \frac{\pi x}{N} + \sin^2 \frac{\pi y}{M} + t^2 \cos^2 \left(\frac{\pi x}{N} + \frac{\pi y}{M} \right) \right].$$

(4)

Splitting this product into two parts,

$$\prod_{n=0}^{2N-1} F(n+\alpha, y) = \prod_{n=0}^{N-1} F(n+\alpha, y) \times \prod_{l=N}^{2N-1} F(l+\alpha, y),$$

(5)

shifting the index in the second part from l to $n = l - N$, and noting the translation symmetry $F(N+x) = F(x)$, this may be expressed as

$$\prod_{n=0}^{2N-1} F(n+\alpha, y) = \left(\prod_{n=0}^{N-1} F(n+\alpha, y) \right)^2.$$

(6)

Defining the partition function with twisted boundary conditions

$$Z_{\alpha,\beta}^2(t, M, N) = \prod_{m=0}^{M-1} \prod_{n=0}^{N-1} 4 \left[\sin^2 \frac{\pi(n+\alpha)}{N} \right.$$
$$\left. + \sin^2 \frac{\pi(m+\beta)}{M} + t^2 \cos^2 \left(\frac{\pi(n+\alpha)}{N} + \frac{\pi(m+\beta)}{M} \right) \right], \qquad (7)$$

one has

$$G_{\alpha,\beta}(t, 2M, 2N) = Z_{\alpha,\beta}^2(t, M, N). \qquad (8)$$

The even-even partition function given by Eq.(2) and Eq.(3) can now be written in the form [15]

$$Z_{2M,2N}(t) = \frac{1}{2} \left[Z_{0,0}^2(t, M, N) + Z_{0,1/2}^2(t, M, N) + Z_{1/2,0}^2(t, M, N) + Z_{1/2,1/2}^2(t, M, N) \right]. \qquad (9)$$

Note that Eq.(7) at $t = 0$ coincides with the corresponding expressions for the square lattice for which a general theory about its asymptotic expansion has been given in Ref.[4].

Note also that for all $(\alpha, \beta) \neq (0, 0)$, the partition function $Z_{\alpha,\beta}(t, M, N)$ is even with respect to its argument t. Hence, near the critical point $(t = 0)$ we have

$$Z_{\alpha,\beta}(t, M, N) = Z_{\alpha,\beta}(0, M, N) + \frac{t^2}{2} Z''_{\alpha,\beta}(0, M, N) + ... \qquad \text{for} \quad (\alpha, \beta) \neq (0, 0). \quad (10)$$

The only exception is the point where both α and β are equal to zero. This case has to be treated separately since at this point $(t = 0)$ the partition function $Z_{0,0}(t, M, N)$ vanishes. As a result, we have

$$Z_{0,0}(t, M, N) = t Z'_{0,0}(0, M, N) + \frac{t^3}{6} Z'''_{0,0}(0, M, N) + ... \qquad \text{for} \quad (\alpha, \beta) = (0, 0). \quad (11)$$

At the critical point $t = 0$ we have

$$Z_{0,0}(0, M, N) = 0 \qquad (12)$$

$$Z_{\alpha,\beta}(0, M, N) = \prod_{n=0}^{N-1} \left| 2 \sinh \left[M\omega \left(\frac{\pi(n+\alpha)}{N} \right) + i\pi\beta \right] \right| \qquad \text{for} \quad (\alpha, \beta) \neq (0, 0), \quad (13)$$

where $\omega(k) = \text{arcsinh}(\sin k)$. In the derivation of the Eq.(13) we have used the identity [23]

$$\prod_{m=0}^{M-1} 4 \left[\sinh^2 \omega + \sin^2 \left(\frac{\pi(m+\beta)}{M} \right) \right] = 4 \left| \sinh(M\omega + i\pi\beta) \right|^2. \qquad (14)$$

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Taking the derivative of Eq.(7) with respect to variable t and then considering limit $t \to 0$ we obtain

$$Z'_{0,0}(0, M, N) = 2M \prod_{n=1}^{N-1} 2 \left| \sinh \left(M\omega \left(\frac{\pi n}{N} \right) \right) \right|, \tag{15}$$

$$Z'_{\alpha,\beta}(0, M, N) = 0 \quad \text{for} \quad (\alpha, \beta) \neq (0, 0). \tag{16}$$

b). Dimers on $2M \times (2N + 1)$ lattices

In Ref.[15], it has been shown that in the even-odd case, the partition function given by Eq.(2) and Eq.(3) can be written as

$$Z_{2M,2N+1}(t) = Z_{0,0}(t, M, 2N + 1) + Z_{0,1/2}(t, M, 2N + 1). \tag{17}$$

Note that Eqs. (9) and (17) in the case $t = 0$ coincide with the corresponding expressions for the square lattice (see Ref.[11]).

Thus we can see that partition function for the dimer model on triangular lattice under periodic boundary conditions can be expressed in terms of the only one subject, namely, $Z_{\alpha,\beta}(t, M, N)$ with $(\alpha, \beta) = (0, 0), (0, 1/2), (1/2, 0)$ and $(1/2, 1/2)$.

III.   DIMER ON THE INFINITELY LONG STRIP

Conformal invariance of the model in the continuum scaling limit would dictate that the asymptotic finite-size scaling behavior of the critical free energy of an infinitely long two-dimensional strip of finite width $\mathcal{N}$ has the form

$$f = f_{\text{bulk}} + \frac{2 f_{\text{surf}}}{\mathcal{N}} + \frac{A}{\mathcal{N}^2} + ..., \tag{18}$$

where f_{bulk} is the bulk free energy, f_{surf} is a surface free energy and A is constant. Unlike the free energy densities f_{bulk} and f_{surf}, the constant A is universal. The value of A is related to the central charge c and the highest conformal weight Δ of the underlying conformal theory, and depends on the boundary conditions in the transversal direction. These two dependencies combine into a function of the effective central charge $c_{eff} = c - 24\Delta$ [24–26],

$$A = -\frac{\pi}{24} c_{eff} = \pi \left(\Delta - \frac{c}{24} \right) \quad \text{on a strip,} \tag{19}$$

$$A = -\frac{\pi}{6} c_{eff} = 4\pi \left(\Delta - \frac{c}{24} \right) \quad \text{on a cylinder.} \tag{20}$$

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Let us now consider the dimer model on the infinitely long strip of width $\mathcal{N}$ under periodic boundary conditions.

Considering the logarithm of the partition function given by Eq.(13), we note that it can be transformed as

$$\ln Z_{\alpha,\beta}(0, M, N) = M \sum_{n=0}^{N-1} \omega\left(\frac{\pi(n+\alpha)}{N}\right) + \sum_{n=0}^{N-1} \ln\left|1 - e^{-2\left[M\omega\left(\frac{\pi(n+\alpha)}{N}\right) - i\pi\beta\right]}\right|. \qquad (21)$$

The second sum here vanishes in the formal limit $M \to \infty$. The asymptotic expansion of the first sum can be found with the help of the Euler-Maclaurin summation formula

$$M \sum_{n=0}^{N-1} \omega\left(\frac{\pi(n+\alpha)}{N}\right) = \frac{S}{\pi} \int_0^\pi \omega(x)\,\mathrm{d}x - \pi\lambda_0\rho\,\mathrm{B}_2^\alpha - 2\pi\rho \sum_{p=1}^\infty \left(\frac{\pi^2\rho}{S}\right)^p \frac{\lambda_{2p}}{(2p)!} \frac{\mathrm{B}_{2p+2}^\alpha}{2p+2}, \qquad (22)$$

where $\int_0^\pi \omega(x)\,\mathrm{d}x = 2\gamma$, $\gamma = 0.915965...$ is Catalan's constant and B_p^α are so-called Bernoulli polynomials and $S = MN$. We have also used the symmetry property, $\omega(k) = \omega(\pi - k)$, of the lattice dispersion relation $\omega(k)$ and its Taylor expansion

$$\omega(k) = \sum_{p=0}^\infty \frac{\lambda_{2p}}{(2p)!} k^{2p+1}, \qquad (23)$$

where $\lambda_0 = 1$, $\lambda_2 = -2/3$, $\lambda_4 = 4$, etc.

Thus one can easily write down all the terms of the exact asymptotic expansion for the $F_{\alpha,\beta}(N) = -\lim_{M\to\infty} \frac{1}{M} \ln Z_{\alpha,\beta}(M, N)$

$$F_{\alpha,\beta}(N) = -\lim_{M\to\infty} \frac{1}{M} \ln Z_{\alpha,\beta}(M, N) = -\frac{2\gamma}{\pi} N + 2 \sum_{p=0}^\infty \left(\frac{\pi}{N}\right)^{2p+1} \frac{\lambda_{2p}}{(2p)!} \frac{B_{2p+2}^\alpha}{2p+2}. \qquad (24)$$

From $F_{\alpha,\beta}(N)$, we can obtain the asymptotic expansion of the free energy per bond of an infinitely long cylinder of circumference $\mathcal{N}$. Since the expression for the partition function is different for even $\mathcal{N}$ and odd $\mathcal{N}$, we will consider these two cases separately. For even $\mathcal{N}$ ($\mathcal{N} = 2N$), we have

$$f = -\lim_{M\to\infty} \frac{1}{4MN} \ln Z_{2M,2N}(0) = -\lim_{M\to\infty} \frac{1}{2MN} \ln Z_{1/2,0}(M, N) = \frac{1}{2N} F_{1/2,0}(N), \qquad (25)$$

and for odd $\mathcal{N}$

$$f = -\lim_{M\to\infty} \frac{1}{2M(2N+1)} \ln Z_{2M,2N+1}(0) = -\lim_{M\to\infty} \frac{1}{2M(2N+1)} \ln Z_{0,1/2}(M, 2N+1)$$

$$= \frac{1}{2(2N+1)} F_{0,1/2}(2N+1). \qquad (26)$$

From Eq.(25) using Eq.(24) one can easily obtain that for even $\mathcal{N}$ the asymptotic expansion of the free energy is given by

$$f = f_{\text{bulk}} + \frac{1}{\pi} \sum_{p=0}^{\infty} \left(\frac{2\pi}{\mathcal{N}} \right)^{2p+2} \frac{\lambda_{2p}}{(2p)!} \frac{B_{2p+2}^{1/2}}{2p+2}$$

$$= f_{\text{bulk}} - \frac{\pi}{6} \frac{1}{\mathcal{N}^2} + \dots \quad (\text{for } \mathcal{N} = 2N), \tag{27}$$

while for odd $\mathcal{N}$ from Eqs.(26) and (24) one can obtain

$$f = f_{\text{bulk}} + \frac{1}{\pi} \sum_{p=0}^{\infty} \left(\frac{\pi}{\mathcal{N}} \right)^{2p+2} \frac{\lambda_{2p}}{(2p)!} \frac{B_{2p+2}}{2p+2}$$

$$= f_{\text{bulk}} + \frac{\pi}{12} \frac{1}{\mathcal{N}^2} - \dots \quad (\text{for } \mathcal{N} = 2N+1). \tag{28}$$

The bulk free energy $f_{\text{bulk}} = -\frac{\gamma}{\pi}$ is the same for $\mathcal{N}$ even and odd cases. Thus we find that the finite-size corrections in a crucial way depend on the parity of $\mathcal{N}$. In particular it means that due to the certain non-local features present in the dimer model, a change of parity of $\mathcal{N}$ induces a change of boundary condition. The similar situation also happen in the dimer model on the square lattice, see [12], where a detailed analysis of boundary conditions and parity dependence effects has been carried out in this context.

Since the effective central charge merely determines some combination of c and Δ, one cannot obtain the values of both without some assumption about one of them. This assumption can be a posteriori justified if the conformal description obtained from it is fully consistent. Surprisingly, there are two consistent values of c that can be used to describe the dimer model, namely $c = -2$ and $c = 1$. For example for the dimer model on an infinitely long cylinder of even circumference $\mathcal{N}$ one can obtained from Eqs.(18), (20) and (27) that the central charge c and the highest conformal weight Δ can take values $c = 1$ and $\Delta = 0$ or $c = -2$ and $\Delta = -1/8$. For the dimer model on an infinitely long cylinder of odd circumference $\mathcal{N}$ one can obtained from Eqs.(18), (20) and (28) that the central charge c and the highest conformal weight Δ can take values $c = 1$ and $\Delta = 1/16$. It turns out in this case that another consistent conformal description exists, with $c = -2$ and $\Delta = 0$ [12]. In particular, it has been shown (for more details see [12]) that although the dimer model is originally defined on a cylinder with odd circumference $\mathcal{N}$, it shows the finite-size corrections expected on a strip and must really be viewed as a model on a strip.

Thus from the finite size analyzes we can see that two conformal field theories with the central charges $c = 1$ and $c = -2$ can be used to described the dimer model on the triangular

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lattice. But since the general triangular lattice model is critical in the square lattice limit and the dimer model on the square lattice belongs to $c = -2$ universality class [12], we come to the conclusion that the dimer model on the triangular lattice can also be described by conformal field theory having central charge $c = -2$.

IV.   SPECIFIC HEAT NEAR THE CRITICAL POINT

Let us now consider the behavior of the specific heat near the critical point. The specific heat $C(t, \mathcal{M}, \mathcal{N})$ of the dimer model on $\mathcal{M} \times \mathcal{N}$ triangular lattice is defined as

$$C(t, \mathcal{M}, \mathcal{N}) \; = \; -\frac{\partial^2}{\partial t^2} f(t, \mathcal{M}, \mathcal{N}), \tag{29}$$

where $f(t, \mathcal{M}, \mathcal{N})$ is free energy of the system

$$f(t, \mathcal{M}, \mathcal{N}) \; = \; -\frac{1}{\mathcal{S}} \ln Z_{\mathcal{M}, \mathcal{N}}(t), \tag{30}$$

and where $\mathcal{S} = \mathcal{M}\mathcal{N}$ is the lattice area.

The pseudocritical point t_{pseudo} is the value of the temperature at which the specific heat has its maximum for finite $\mathcal{M} \times \mathcal{N}$ lattice. One can determine this quantity as the point where the derivative of $C(t, \mathcal{M}, \mathcal{N})$ vanishes. The pseudocritical point approaches the critical point $t_c = 0$ as $L \to \infty$ in a manner dictated by the shift exponent λ,

$$|t_{\text{pseudo}} - t_c| \sim L^{-\lambda}. \tag{31}$$

where $L = \sqrt{\mathcal{S}}$ is the characteristic size of the system. The coincidence of λ with $1/\nu$, where ν is the correlation lengths exponent, is common to most models, but it is not a direct consequence of finite-size scaling and is not always true.

Since the expression for the partition function is different for even $\mathcal{N}$ and odd $\mathcal{N}$, we consider these two cases separately.

Let us start with case of odd $\mathcal{N}$ ($\mathcal{N} = 2N + 1$). Expanding the expression (29) about the critical point $t = 0$ with the help of Eqs.(10), (11) and (17) yields

$$C(t, \mathcal{M}, \mathcal{N}) = C(0, \mathcal{M}, \mathcal{N}) + t \, C^{(1)}(0, \mathcal{M}, \mathcal{N}) + \frac{t^2}{2} C^{(2)}(0, \mathcal{M}, \mathcal{N}) + O(t^3), \tag{32}$$

where $C(0, \mathcal{M}, \mathcal{N})$ is the critical specific heat, and $C^{(n)}(0, \mathcal{M}, \mathcal{N}) \equiv \dfrac{d^n}{dt^n} C(t, \mathcal{M}, \mathcal{N})|_{t=0}$. We

have

$$S\,C(0,\mathcal{M},\mathcal{N}) = \frac{Z_{0,1/2}^{(2)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})} - \left(\frac{Z_{0,0}^{(1)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})}\right)^2, \tag{33}$$

$$S\,C^{(1)}(0,\mathcal{M},\mathcal{N}) = \frac{Z_{0,0}^{(3)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})} - 3\frac{Z_{0,0}^{(1)}(0,\mathcal{M}/2,\mathcal{N})Z_{0,1/2}^{(2)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}^2(0,\mathcal{M}/2,\mathcal{N})}$$

$$+\, 2\left(\frac{Z_{0,0}^{(1)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})}\right)^3, \tag{34}$$

$$S\,C^{(2)}(0,\mathcal{M},\mathcal{N}) = \frac{Z_{0,1/2}^{(4)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})} + 12\left(\frac{Z_{0,0}^{(1)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})}\right)^2 \frac{Z_{0,1/2}^{(2)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})}$$

$$-\, 3\left(\frac{Z_{0,1/2}^{(2)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})}\right)^2 - 4\frac{Z_{0,0}^{(1)}(0,\mathcal{M}/2,\mathcal{N})Z_{0,0}^{(3)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}^2(0,\mathcal{M}/2,\mathcal{N})}$$

$$-\, 6\left(\frac{Z_{0,0}^{(1)}(0,\mathcal{M}/2,\mathcal{N})}{Z_{0,1/2}(0,\mathcal{M}/2,\mathcal{N})}\right)^4. \tag{35}$$

From Eq.(32), the first derivative of the specific heat on a finite lattice near the infinite volume critical point can be found, and it seen to vanish when

$$t_{\text{pseudo}} = -\frac{C^{(1)}(0,\mathcal{M},\mathcal{N})}{C^{(2)}(0,\mathcal{M},\mathcal{N})}. \tag{36}$$

To find the exponent λ one need the finite-size corrections to $C^{(1)}(0,\mathcal{M},\mathcal{N})$ and $C^{(2)}(0,\mathcal{M},\mathcal{N})$. The exact asymptotic expansions of $C^{(1)}(0,\mathcal{M},\mathcal{N})$ and $C^{(2)}(0,\mathcal{M},\mathcal{N})$ can be found along the same lines as in Ref.[27] and leading finite size behavior is

$$C^{(1)}(0,\mathcal{M},\mathcal{N}) \sim \sqrt{S}, \tag{37}$$

$$C^{(2)}(0,\mathcal{M},\mathcal{N}) \sim S. \tag{38}$$

Expressions Eq.(37) and (38) now gives the FSS of the pseudocritical point to be

$$t_{\text{pseudo}} \sim L^{-1}, \tag{39}$$

where $L = \sqrt{S}$ is the characteristic size of the system. Thus from Eqs. (31) and (39) we find that the shift exponent is $\lambda = 1$ for odd $\mathcal{N}$. In Fig. 2(a) and 2(b) we plot the t dependence of the specific heat for 16×17 lattice. We can see from Fig. 2(b) that the position of the specific heat peak is shifted from zero.

For even $\mathcal{N}$ ($\mathcal{N} = 2N$), one can see from Eqs.(9), (10), (11), (29) and (30) that the partition function $Z_{\mathcal{M},\mathcal{N}}(t)$ and the specific heat $C(t,\mathcal{M},\mathcal{N})$ are an even function with

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respect to its argument t

$$C(t, \mathcal{M}, \mathcal{N}) = C(0, \mathcal{M}, \mathcal{N}) + \frac{t^2}{2} C^{(2)}(0, \mathcal{M}, \mathcal{N}) + \frac{t^4}{4!} C^{(4)}(0, \mathcal{M}, \mathcal{N}) + O(t^6). \qquad (40)$$

The partition function given by Eq. (1) is a polynomial function of its argument t. In the case of even $\mathcal{M}$ and $\mathcal{N}$ the partition function is an even function with respect of its argument t and hence the number of diagonal bonds n_t also should be even. There is also a simple geometrical explanation why, in the case of even-even lattices, the number of diagonal dimers should be even. Let us consider an even-even $(2\mathcal{M} \times 2\mathcal{N})$ lattice with periodic boundary conditions in the vertical and horizontal directions. Such a lattice can be divided into two sublattice A and B, as shown in Fig. 3. Each sublattice consists of $2\mathcal{M}\mathcal{N}$ sites in a such way that every horizontal or vertical edge connects a site in sublattice A to one in sublattice B, while diagonal edges connect two sites within sublattice A or two sites within sublattice B. Note that such a division into A and B sublattices is impossible for even-odd $(2\mathcal{M} \times 2\mathcal{N} + 1)$ lattices with periodic boundary conditions, since in that case, one can always find a horizontal or vertical edge which connects two sites in the same sublattice A or B. Thus, in the even-even case, each horizontal or vertical dimer occupies one site from sublattice A and another site from sublattice B, while diagonal dimer occupies two sites from sublattice A or B. If one diagonal dimer occupies two sites from sublattice A one should have another diagonal dimer which occupies two sites of sublattice B in order to insure that remaining sites can be occupied by horizontal and vertical dimers. Thus, in the case of even M and N, only even number of diagonal bonds are allowed. Similar geometrical considerations for the dimer model on $(\mathcal{M} \times \mathcal{N})$ lattice with free boundary conditions lead to the conclusion that for both even-even and even-odd lattices the number of diagonal bonds should be even.

Thus the first derivative of $C(t, \mathcal{M}, \mathcal{N})$ vanishes exactly at

$$t_{\text{pseudo}} = 0. \qquad (41)$$

In Fig. 4(a) and 4(b) we plot the t dependence of the specific heat for a 16×16 lattice. We can see from Fig. 3(b) that the position of the specific heat peak t_{pseudo} is equal exactly to zero. Therefore the maximum of the specific heat (the pseudocritical point t_{pseudo}) always occurs at vanishing reduced temperature for any finite $M \times 2N$ lattice and coincides with the critical point t_c at the thermodynamic limit. From Eqs. (31) and (41) we find that the shift exponent is $\lambda = \infty$ for even $\mathcal{N}$.
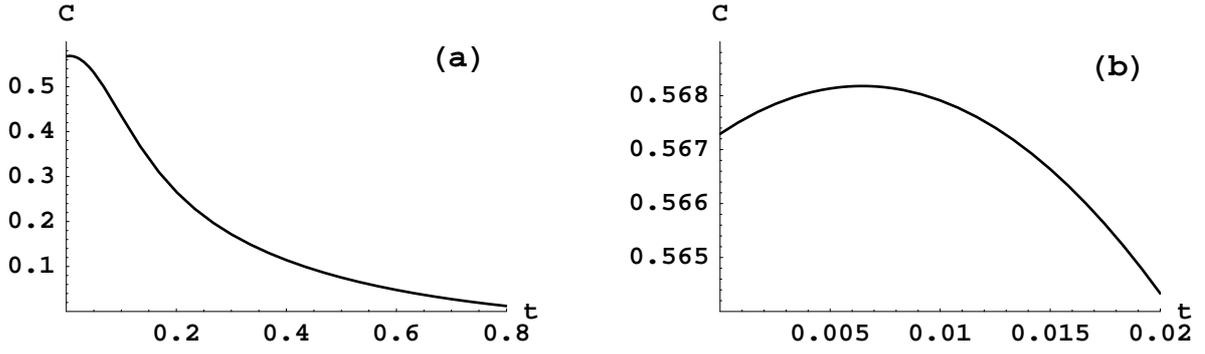
Figure 2: (a) The behavior of the specific heat on $\mathcal{M} \times \mathcal{N}$ lattices with even $\mathcal{M} = 16$ and odd $\mathcal{N} = 17$. (b) The behavior of the specific heat on the same lattice for small t.
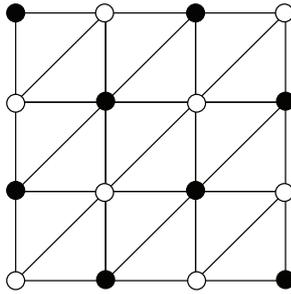

Figure 3: Division of lattice on two sublattice A (black circles) and B (white circles).

Thus we have found that the shift exponent λ for the specific heat depend in a crucial way on the parity of the number of lattice sites along the lattice axis $\mathcal{N}$. For odd $\mathcal{N}$ we obtain for the shift exponent $\lambda = 1$, while for even $\mathcal{N}$ we have found that the shift exponent λ is infinity ($\lambda = \infty$).

V. CONCLUSION

We analyze the partition function of the dimer model on $\mathcal{M} \times \mathcal{N}$ triangular lattice wrapped on torus obtained by Fendley, Moessner and Sondhi [22]. From a finite-size analysis we have
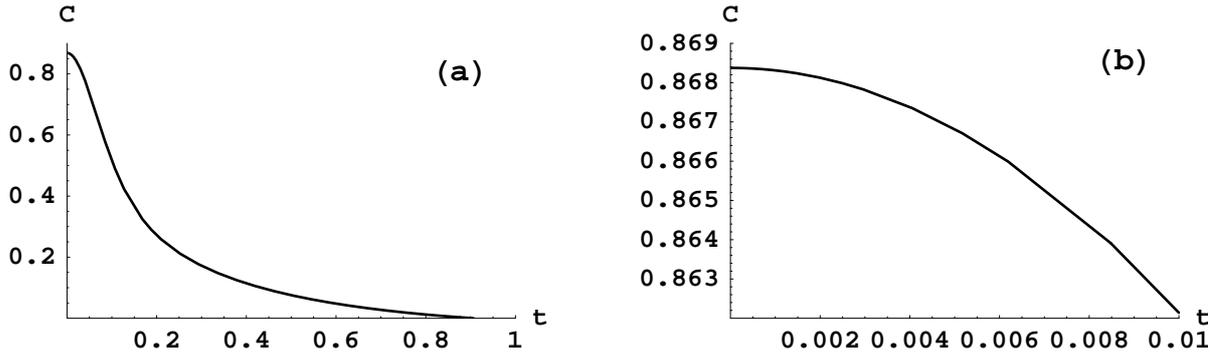
Figure 4: (a) The behavior of the specific heat on $M \times N$ lattices with even $M = 16$ and $N = 16$. (b) The behavior of the specific heat on the same lattice for small t.

found that the dimer model on the triangular lattice can be described by conformal field theory having central charge $c = -2$. Thus we have shown that the dimer model on the triangular lattice belongs to the same universality class as the dimer model on the square lattice, while the dimer model on the honeycomb lattice belongs to another $c = 1$ universality class. In addition, we have found that the shift exponent λ depends in a crucial way on the parity of the number of lattice sites along the lattice axis $\mathcal{N}$: for odd $\mathcal{N}$ we obtain $\lambda = 1$, while for even $\mathcal{N}$ we have found that $\lambda = \infty$. In the former case, therefore, the finite-size specific-heat pseudocritical point is size dependent, while in the latter case it coincides with the critical point of the thermodynamic limit. This adds to the catalog of anomalous circumstances where the shift exponent is not coincident with the correlation-length critical exponent. The present circumstance manifests the additional feature that the shift-exponent is boundary-condition dependent.

VI. ACKNOWLEDGMENT

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