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# Entropic equation of state and scaling functions near the critical point in scale-free networks

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We analyze the entropic equation of state for a many-particle interacting system in a scale-free network. The analysis is performed in terms of scaling functions which are of fundamental interest in the theory of critical phenomena and have previously been theoretically and experimentally explored in the context of various magnetic, fluid, and superconducting systems in two and three dimensions. Here, we obtain general scaling functions for the entropy, the constant-field heat capacity, and the isothermal magnetocaloric coefficient near the critical point in scale-free networks, where the node-degree distribution exponent  $\lambda$  appears to be a global variable and plays a crucial role, similar to the dimensionality  $d$  for systems on lattices. This extends the principle of universality to systems on scale-free networks and allows quantification of the impact of fluctuations in the network structure on critical behavior.

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## I. INTRODUCTION

Phase transitions and critical behavior in complex networks currently attract much attention [1] because of their unusual features and broad array of applications, ranging from socio- [2] to nanophysics [3]. It is by now well established that the critical behavior of a many-particle interacting system located on the nodes of a general network may crucially differ from that of a system located on the sites of a  $d$ -dimensional regular lattice. Of particular interest are the so-called scale-free networks, for which the probability to find a node of degree  $k$  (i.e. with  $k$  nearest neighbors) vanishes for large  $k$  as a power law

$$P(k) \sim k^{-\lambda}. \quad (1)$$

The questions we address in this paper concern two fundamental principles of critical phenomena: universality and scaling [4]. Both of these questions have to be reconsidered when a system resides on a network. Usually, the universality of critical phenomena is understood as stating that the thermodynamic properties near the critical point  $T_c$  are governed by a small number of global features, such as dimensionality, symmetry, and the type of interaction. In turn, the scaling hypothesis states that the singular part of a thermodynamic potential near  $T_c$  has the form of a generalized homogeneous function. To

be specific, for the Helmholtz potential of a magnetic system the latter can be written as [5]

$$F(\tau, M) = \tau^{2-\alpha} f_{\pm}(M/\tau^{\beta}), \quad (2)$$

where  $M$  is the magnetization,  $\tau = |T - T_c|/T_c$ ,  $\alpha$ ,  $\beta$  are the universal exponents, and the sign  $\pm$  corresponds to  $T > T_c$  or  $T < T_c$  respectively. The essence of relation (2) is that the two-variable function  $F(\tau, M)$ , when appropriately rescaled, is expressed in terms of a single scaling variable leading to the *scaling function*  $f_{\pm}(x)$ . The expression (2) gives an example of the scaling function for the Helmholtz potential  $F(\tau, M)$ . Together with other scaling functions - for the equation of state and for thermodynamic functions - this appears to give a suitable and comprehensive description of critical phenomena [7, 8]. These scaling functions are also universal in the sense explained above.

For systems on scale-free networks the principle of universality is extended: there, the node-degree distribution exponent  $\lambda$  in Eq.(1) appears to be a global variable and plays a crucial role, similar to the dimension  $d$  for systems on lattices (see e.g. [9–14]). All systems that belong to a given universality class are governed by the same values of critical exponents and critical amplitude ratios and share the same universal form of scaling functions. Recently, the scaling function formalism has been applied to describe the critical behavior of magnetic systems with the structure of a scale-free network [14]. There, scaling functions for the magnetic equation of state and isothermal susceptibility were derived. In this paper we are interested in the entropic form of the equation of state. In particular this opens the possibility to derive scaling functions for the heat capacity. These are of wide and fundamental interest in the theory of critical phenomena and have been the subject of thorough theoretical and experimental studies for various magnetic, fluid, and

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superconducting systems [16–23]. A particular point of interest concerns the existence or non-existence of the moments of the distribution (1) on the universal behavior.

The set-up of the the paper is as follows: in section II we define the notations and derive expressions for the entropic equation of state, heat capacity and magnetocaloric coefficient scaling functions for systems on scale-free networks. These expressions are further analyzed and compared with corresponding functions for bulk systems in section III. The paper concludes with a summary and an outlook in section IV.

## II. FREE ENERGY AND SCALING FUNCTIONS

The critical behavior of a many-particle system interacting on a scale-free network crucially depends on the node degree distribution  $P(k)$  via the decay exponent  $\lambda$  in Eq.(1) [1]. In particular, for an infinite network the value of  $\lambda$  determines the order of the first diverging moment, this order being the lowest integer  $j \geq \lambda - 1$ . This is reflected by the phase transition scenario. For low values of  $\lambda \leq 3$  the system remains ordered for any finite temperature, whereas for  $\lambda > 3$  a finite temperature, order-disorder phase transition occurs. Moreover, critical exponents that govern a second-order phase transition in a scale-free network attain their usual mean-field values for high  $\lambda > 5$  and demonstrate non-trivial  $\lambda$ -dependence in the region  $3 < \lambda < 5$ . Logarithmic corrections to scaling laws appear at  $\lambda = 5$ : this resembles phenomena that occur at marginal space or order parameter dimensions in bulk systems [24].

A starting point for our analysis will be the expression for the free energy of a system with scalar order parameter on a scale-free network. To be specific, from now on we will consider ferromagnetic ordering and the spontaneous magnetization  $M$  as the order parameter with the conjugate magnetic field  $H$ . In this case the corresponding microscopic degrees of freedom are the Ising spins. However, generalization to models of more complicated symmetry is straightforward [12, 13]. Due to the fact that the type of networks under discussion are assumed to have a local tree-like structure, the mean-field approximation is asymptotically exact in the sense that thermal fluctuations can be neglected. This leads to a form of the free energy also found by other techniques. The lowest order contributions to the singular part of the Helmholtz free energy in the vicinity of  $T_c$  are [9–11]

$$F(M, T) = \frac{a}{2}(T - T_c)M^2 + \frac{b}{4}M^4, \quad \lambda > 5, \quad (3)$$

$$F(M, T) = \frac{a}{2}(T - T_c)M^2 + \frac{b}{4}M^{\lambda-1}, \quad 3 < \lambda < 5. \quad (4)$$

The parameters  $a, b > 0$  and the critical temperature  $T_c$  are  $\lambda$ -dependent. This dependence can be made explicit using microscopic approaches [9, 10, 12] or may be

	$\alpha$	$\beta$	$\gamma$	$\delta$	$\omega$	$\alpha_c$	$\gamma_c$	$\omega_c$
$\lambda \geq 5$	0	1/2	1	3	1/2	0	2/3	1/3
$3 < \lambda < 5$	$\frac{\lambda-5}{\lambda-3}$	$\frac{1}{\lambda-3}$	1	$\lambda-2$	$\frac{\lambda-4}{\lambda-3}$	$\frac{\lambda-5}{\lambda-2}$	$\frac{\lambda-3}{\lambda-2}$	$\frac{\lambda-4}{\lambda-2}$

TABLE I: Critical exponents governing temperature and field dependencies of the thermodynamic quantities for different values of  $\lambda$ .

postulated in a Landau-like approach [11, 13]. For the subsequent analysis we will absorb the parameters into the dimensions of the corresponding observables, passing to dimensionless quantities,

$$f(m, \tau) = \pm \frac{\tau}{2}m^2 + \frac{1}{4}m^4, \quad \lambda > 5, \quad (5)$$

$$f(m, \tau) = \pm \frac{\tau}{2}m^2 + \frac{1}{4}m^{\lambda-1}, \quad 3 < \lambda < 5, \quad (6)$$

with obvious relations between dimension-dependent and dimensionless variables,

$$m = M/M_0, \quad \tau = |T - T_c|/T_c, \quad f = F/F_0, \quad (7)$$

where  $M_0^{\lambda-3} = aT_c/b$  and  $F_0 = aT_cM_0^2$ . Since  $\tau$  measures the absolute distance to the critical point, the free energy has two branches, corresponding to signs '+' and '-' in Eqs. (5) and (6) for  $T > T_c$  and  $T < T_c$ , respectively. It is easy to verify, that a system with the free energy (5), (6) possesses a second-order phase transition at  $\tau = 0$ . Here, we employ the standard notation for critical exponents governing the temperature- and field-dependencies of the thermodynamic functions. For  $h = 0$  and  $T \rightarrow T_c^\pm$ , these are

$$c_h \simeq A^\pm \tau^{-\alpha}, \quad \chi_T \simeq \Gamma^\pm \tau^{-\gamma}, \quad m_T \simeq B_T^\pm \tau^{-\omega} \quad (8)$$

while for  $T \rightarrow T_c^-$  one also has

$$m \simeq B\tau^\beta. \quad (9)$$

On the other hand, for  $\tau = 0$ , the standard definitions are

$$c_h \sim A_c h^{-\alpha_c}, \quad h \simeq D_c m |m|^{\delta-1}, \quad (10)$$

$$\chi_T \simeq \Gamma_c h^{-\gamma_c}, \quad m_T \simeq B_T^c h^{-\omega_c}. \quad (11)$$

(See section II C for the definition of the magnetocaloric coefficient  $m_T$ .) The values of these critical exponents are summarized in Table I [9–12, 14]. It is worth noting here that one way to derive the listed exponents is to consider the naïve dimensions of different terms in the Landau free energy, similar to the standard field theoretical procedure (see e.g. [15]). With the values of critical exponents to hand, one can rewrite the singular part of the Helmholtz potential in the form of a generalized homogeneous function (2) [4]:

$$f(m, \tau) = \tau^2 f_\pm(x), \quad x = m/\tau^{\frac{1}{2}}, \quad \lambda > 5, \quad (12)$$

$$f(m, \tau) = \tau^{\frac{\lambda-1}{\lambda-3}} f_\pm(x), \quad x = m/\tau^{\frac{1}{\lambda-3}}, \quad 3 < \lambda < 5, \quad (13)$$

where the free energy scaling functions are given by [14]

$$f_{\pm}(x) = \pm \frac{1}{2}x^2 + \frac{1}{4}x^4, \quad \lambda > 5, \quad (14)$$

$$f_{\pm}(x) = \pm \frac{1}{2}x^2 + \frac{1}{4}x^{\lambda-1}, \quad 3 < \lambda < 5. \quad (15)$$

Assuming that the Helmholtz potential is a complete differential

$$dF = -SdT + HdM \quad (16)$$

one can further proceed with an analysis based on the magnetic form of the equation of state,

$$H = \left. \frac{\partial F}{\partial M} \right|_T \quad (17)$$

or entropic form of the equation of state [25],

$$S = - \left. \frac{\partial F}{\partial T} \right|_M. \quad (18)$$

As we have noted in the introduction, the scaling functions for the magnetic equation of state (both in the Widom-Griffith [7] and Stanley-Hankey [6] forms) and isothermal susceptibility have recently been reported elsewhere [14]. Here, we will proceed by analyzing the entropic equation of state (18) and heat capacity scaling functions.

In terms of dimensionless variables Eqs. (17), (18) take on the form

$$h(m, \tau) = \left. \frac{\partial f(m, \tau)}{\partial m} \right|_{\tau}, \quad s(m, \tau) = \mp \left. \frac{\partial f(m, \tau)}{\partial \tau} \right|_m \quad (19)$$

with field  $h$  and entropy  $s$  measured in units of  $F_0/M_0$  and  $F_0/T_c$  correspondingly. As before, and throughout, the index  $\pm$  refers to temperatures above and below the critical point  $T_c$ .

Since the free energy (3), (4) is explicitly a linear function of  $\tau$ , one obtains the usual mean field result for the heat capacity at constant magnetization:

$$C_M = T \left. \frac{\partial S}{\partial T} \right|_M = 0. \quad (20)$$

To find the dimensionless constant-magnetic-field heat capacity [25],

$$c_h = \pm \frac{T}{T_c} \left. \frac{\partial s(\tau, m)}{\partial \tau} \right|_h = (\tau \pm 1) \left. \frac{\partial s(\tau, m)}{\partial \tau} \right|_h, \quad (21)$$

one can consider the entropy as a function of magnetic field and temperature  $s(\tau, m(\tau, h))$  which leads to

$$c_h = (\tau \pm 1) \left[ \left. \frac{\partial s}{\partial \tau} \right|_m + \left. \frac{\partial s}{\partial m} \right|_{\tau} \left. \frac{\partial m(\tau, h)}{\partial \tau} \right|_h \right]. \quad (22)$$

Noting from (3), (4), that  $\partial s / \partial m|_{\tau} = -m$  and  $\partial s / \partial \tau|_m = 0$  one finally arrives at the expression for the heat capacity,

$$c_h = (1 \pm \tau) C_h(\tau, m) \quad (23)$$

with function  $C_h$  given by

$$C_h(\tau, m) = \mp m \left. \frac{\partial m(\tau, h)}{\partial \tau} \right|_h. \quad (24)$$

Let us now consider separately the cases of fast ( $\lambda > 5$ ) and slower ( $3 < \lambda < 5$ ) decay of the node degree distribution (1).

### A. $\lambda > 5$

The free energy (3) leads to the expression for the entropy,

$$s(\tau, m) = -\frac{m^2}{2}, \quad (25)$$

which can be easily recast in a scaling form

$$s(\tau, m) = \tau \mathcal{S}(x), \quad (26)$$

where the scaling variable  $x = m/\tau^{\beta} = m/\tau^{1/2}$  and the entropy scaling function  $\mathcal{S}(x)$  is

$$\mathcal{S}(x) = -\frac{x^2}{2}. \quad (27)$$

To obtain the heat capacity (22) we first write the magnetic equation of state (19)

$$h = \pm \tau m + m^3 \quad (28)$$

and differentiate it with respect to  $\tau$  to obtain:

$$\left. \frac{\partial m}{\partial \tau} \right|_h = \frac{\mp m}{\pm \tau + 3m^2}. \quad (29)$$

Substituting this into (24) leads to the representation of  $C_h$  in the form of a generalized homogeneous function,

$$C_h(\tau, m) = \mathcal{C}_{\pm}(x), \quad (30)$$

with the scaling variable  $x$  defined above and the heat capacity scaling function

$$\mathcal{C}_{\pm}(x) = \frac{x^2}{3x^2 \pm 1}. \quad (31)$$

Note, that in (30) the heat capacity exponent vanishes,  $\alpha = 0$ .

### B. $3 < \lambda < 5$

A particular feature of the entropy of a system on a scale-free network is that its dependence on magnetization both for  $3 < \lambda < 5$  and for  $\lambda > 5$  is given by Eq. (25). In terms of the scaling function for  $3 < \lambda < 5$  it reads

$$s(\tau, x) = \tau^{2/(\lambda-3)} \mathcal{S}(x), \quad (32)$$

where the entropy scaling function does not change and is given by Eq. (27). The power of  $\tau$  is equal to  $1 - \alpha$  and the scaling variable is now

$$x \equiv m/\tau^\beta = m/\tau^{1/(\lambda-3)}. \quad (33)$$

However, the magnetic equation of state (19) for the Helmholtz function (6) becomes  $\lambda$ -dependent:

$$h = \pm\tau m + \frac{\lambda-1}{4}m^{\lambda-2}. \quad (34)$$

As in the previous subsection we obtain from this the derivative  $\partial m/\partial\tau|_h$ , and by substitution into Eq.(24) we arrive at the representation of  $C_h$  in the form of the generalized homogeneous function

$$C_h(\tau, m) = \tau^{\frac{5-\lambda}{\lambda-3}} \mathcal{C}_\pm(x) \quad (35)$$

where the scaling variable  $x$  is given by (33) and the heat capacity scaling function attains a non-trivial  $\lambda$ -dependence

$$\mathcal{C}_\pm(x) = \frac{x^2}{\frac{(\lambda-1)(\lambda-2)}{4}x^{\lambda-3} \pm 1}. \quad (36)$$

Note that on the basis of the scaling functions  $\mathcal{S}(x)$  in Eq.(27) and  $\mathcal{C}_\pm(x)$  in Eqs.(31) and (36), one easily obtains the corresponding scaling functions with respect to the rescaled magnetic field

$$y \equiv h/\tau^{\beta\delta}. \quad (37)$$

The connection between the variables  $x$  and  $y$  results from the magnetic equations of state (28) and (34), and is given by

$$y = \pm x + x^3, \quad \lambda > 5, \quad (38)$$

$$y = \pm x + \frac{\lambda-1}{4}x^{\lambda-2}, \quad 3 < \lambda < 5. \quad (39)$$

Solving the above equations with respect to  $x$  and substituting the result  $x(y)$  into the functions  $\mathcal{S}(x)$  and  $\mathcal{C}_\pm(x)$  leads to the scaling functions  $\mathcal{S}(y)$  and  $\mathcal{C}_\pm(y)$ . The behavior of the above scaling functions will be analyzed in the next section. These functions together with the scaling functions for the magnetic equation of state  $h = \tau^{\beta\delta}H_\pm(m/\tau^\beta)$  and isothermal susceptibility  $\chi_T = \tau^{-\gamma}\chi_\pm(m/\tau^\beta)$  [14] are summarized in table II.

### C. Isothermal magnetocaloric effect, $\lambda > 3$

Before we proceed with the discussion of the peculiarities of the entropic equation of state and of the thermodynamic functions following from it, let us introduce an additional observable – the isothermal magnetocaloric coefficient. It serves as a direct measure of the heat released by the system due to the magnetocaloric effect

	$3 < \lambda < 5$	$\lambda > 5$
$f_\pm$	$\pm x^2/2 + x^{\lambda-1}/4$	$\pm x^2/2 + x^4/4$
$H_\pm$	$\frac{\lambda-1}{4}x^{\lambda-2} \pm x$	$x^3 \pm x$
$\chi_\pm$	$\frac{1}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{1}{3x^2 \pm 1}$
$\mathcal{S}$	$-x^2/2$	$-x^2/2$
$\mathcal{C}_\pm$	$\frac{x^2}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{x^2}{3x^2 \pm 1}$
$\mathcal{M}_\pm$	$\frac{x}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{x}{3x^2 \pm 1}$
$A^+/A^-$	0	0
$\Gamma^+/\Gamma^-$	$\lambda - 3$	2
$R_\chi$	1	1
$R_C$	0	0
$R_A$	$\frac{1}{\lambda-2}[\frac{4}{\lambda-1}]^{\frac{\lambda-5}{(\lambda-2)(\lambda-3)}}$	1/3

TABLE II: Scaling functions and amplitude ratios near the critical point in scale-free networks. The scaling variable is  $x = m/\tau^\beta$ . The ratio  $\Gamma^+/\Gamma^-$  is taken from Ref. [13] and scaling functions  $f_\pm$ ,  $H_\pm$ , and  $\chi_\pm$  follow from Ref. [14].

upon an isothermal increase of the magnetic field and is defined as (see e.g. [26])

$$M_T = -T \frac{\partial M}{\partial T} \Big|_H. \quad (40)$$

In contradistinction to the heat capacity, which often does not diverge or is a weakly divergent quantity for many 3d systems, the magnetocaloric coefficient is frequently strongly divergent at second-order phase transitions [22, 26] and therefore it is instructive to analyze how this behavior is modified by a scale-free network. Using Maxwell relations,  $M_T$  can be obtained both from the magnetic or from the entropic equations of state, Eqs. (17), (18). Therefore an equivalent representation to the one given in (40) representation reads:

$$M_T = -T \frac{\partial S}{\partial H} \Big|_T. \quad (41)$$

Analogous to the first equation in (7), we define the dimensionless isothermal magnetocaloric coefficient as

$$m_T = \frac{M_T}{M_0}. \quad (42)$$

From the above representation this is

$$m_T = -(1 \pm \tau) \frac{\partial s(m, \tau)}{\partial m} \Big|_\tau \frac{\partial m(\tau, h)}{\partial h} \Big|_\tau. \quad (43)$$

Recognizing that the last term in (43) is a dimensionless isothermal susceptibility  $\chi_T(\tau, m)$  and writing it in the scaling form

$$\chi_T = \tau^{-\gamma} \chi_\pm(x), \quad x = m/\tau^\beta, \quad (44)$$

we arrive at the scaling representation for the dimensionless isothermal magnetocaloric coefficient  $m_T$

$$m_T = (1 \pm \tau) \tau^{-\omega} \mathcal{M}_\pm(x), \quad (45)$$

with the scaling function

$$\mathcal{M}_{\pm}(x) = x\chi_{\pm}(x), \quad (46)$$

and a scaling relation for the isothermal magnetocaloric coefficient critical exponent  $\omega$ ,

$$\omega = 1 - \beta. \quad (47)$$

While the equality (47) is a general one and directly follows from the scaling form of Eq. (43), the relation (46) between functions  $\mathcal{M}_{\pm}(x)$  and  $\chi_{\pm}(x)$  holds only for systems where the entropy scaling function has the simple representation (27).

As noticed above, another way to obtain  $m_T$  is to start from the magnetic equation of state using the representation (40). Then one obtains

$$m_T = \mp(1 \pm \tau) \frac{\partial m(\tau, h)}{\partial \tau} \Big|_h. \quad (48)$$

Comparing this expression with the formulas obtained above for the heat capacity (23), (24) one arrives at the relation between the scaling functions  $\mathcal{M}_{\pm}(x)$  and  $\mathcal{C}_{\pm}(x)$ ,

$$\mathcal{C}_{\pm}(x) = x\mathcal{M}_{\pm}(x), \quad (49)$$

which in particular leads to (c.f. (46))

$$\mathcal{C}_{\pm}(x) = x^2\chi_{\pm}(x). \quad (50)$$

The scaling function  $\mathcal{M}_{\pm}(x)$  defined above is displayed for different ranges of the values of  $\lambda$  in table II. For the critical exponents  $\omega$  we get

$$\omega = \frac{\lambda - 4}{\lambda - 3}, \quad 3 < \lambda < 5; \quad \omega = 1/2, \quad \lambda > 5. \quad (51)$$

It is easy to find the scaling relation for the critical exponent  $\omega_c$  that governs the field dependence of  $m_T(\tau = 0, h)$  [22]:

$$\omega_c = \frac{1 - \beta}{\beta\delta}. \quad (52)$$

The values of this exponent read

$$\omega_c = \frac{\lambda - 4}{\lambda - 2}, \quad 3 < \lambda < 5; \quad \omega_c = 1/3, \quad \lambda > 5. \quad (53)$$

Thus while  $c_h$  does not diverge ( $\alpha < 0$ ) for the entire range  $3 < \lambda < 5$ ,  $m_T$  is divergent ( $\omega > 0$ ) over half that range  $4 < \lambda < 5$ , and is a better locator of the phase transition there. The above calculated exponents  $\omega$ ,  $\omega_c$  are displayed together with other exponents in the comprehensive table I, which presents a summary of the data concerning the temperature and field behavior of different thermodynamic quantities in the vicinity of the critical point for different values of  $\lambda$ . In the course of the analysis of different types of critical phenomena in scale-free networks [1], it has been revealed that the onset of divergencies of moments of the node-degree distribution

function  $P(k)$ , Eq. (1) relate to changes in the scaling scenario of these systems. As one can see from (51), (53) the exponents  $\omega$  and  $\omega_c$  change their sign to become negative for  $\lambda < 4$ :  $m_T$  is not divergent at the critical point any more in the region  $3 < \lambda < 4$ . Therefore, along with the two marginal values of  $\lambda = 5$  and  $\lambda = 3$  which correspond to the divergencies of  $\langle k^4 \rangle$  and  $\langle k^2 \rangle$  and define the 'window' of non-trivial critical behavior on a scale-free network. The divergence of the third moment of the node-degree distribution  $\langle k^3 \rangle$  leads to a qualitative change in the critical behavior of the isothermal magnetocaloric coefficient.

### III. DISCUSSION

As one can see from table I, the heat capacity exponent  $\alpha$  is negative in the region  $3 < \lambda < 5$  where a non-trivial  $\alpha(\lambda)$  dependence is observed. This means that the singular part of the heat capacity  $c_h$  vanishes at  $T_c$ . Taken that  $c_h$  vanishes also at  $T = 0$  and that it is a positive smooth function of  $T$  in between, one concludes that it has a maximum at some temperature  $T_0$ , where  $0 < T_0 < T_c$  for any  $3 < \lambda < 5$ . Therefore the energy fluctuations are maximal at  $T_0$  (see [13] for more details). Such behavior is a generic feature of systems with  $\alpha < 0$ , other examples include the three dimensional Heisenberg and planar magnets [27], liquid helium [28], and disordered uniaxial magnets [29]. From Eqs.(31) and (36) one finds that  $c_h(T > T_c, h = 0) = 0$  for any  $\lambda$ . This leads to the following amplitude ratio which holds for all  $\lambda > 3$ :

$$A^+/A^- = 0. \quad (54)$$

Amplitude ratios are known to be universal along with the scaling functions and critical exponents (see e.g. [8]). It is appropriate to adduce here how do these ratios change for systems on scale-free networks. The results are summarized in the lower part of table II. Besides than the heat capacity amplitude ratio (54), the isothermal magnetic-susceptibility amplitude ratio appears to be  $\lambda$ -dependent for  $3 < \lambda < 5$ :  $\Gamma^+/\Gamma^- = \lambda - 3$  [13]. Using the expressions (28), (31), (34), (36) it is straightforward to find for the other amplitudes for  $\lambda > 5$

$$B = D_c = 1, \quad A_c = 1/3, \quad (55)$$

and for  $3 < \lambda < 5$ ,

$$B = \left(\frac{4}{\lambda - 1}\right)^{1/(\lambda - 3)}, \quad D_c = \frac{\lambda - 1}{4}, \\ A_c = \frac{1}{\lambda - 2} \left(\frac{4}{\lambda - 1}\right)^{3/(\lambda - 2)}. \quad (56)$$

Now, defining three more amplitude ratios by [8, 18, 19]

$$R_{\chi} = \Gamma^+ D_c B^{\delta - 1}, \quad (57)$$

$$R_c = A^+ \Gamma^+ / B^2, \quad (58)$$

$$R_A = A_c D_c^{-(1 + \alpha_c)} B^{-2/\beta}, \quad (59)$$

and substituting into these ratios the amplitudes (55) and (56), we arrive at their values for the scale-free network, as listed in table II.

Let us concentrate now on the scaling functions. As noted in section II, the entropy scaling function  $\mathcal{S}_{\pm}(x)$  which in the usual Landau theory is given by Eq. (27) keeps its form also in the case of scale-free networks with  $3 < \lambda < 5$ . However, the constant-magnetic-field heat capacity scaling function  $\mathcal{C}_{\pm}$  essentially changes in this region. In Fig. 1 we plot  $\mathcal{C}_{\pm}$  as a function of the scaling variable  $x = m/\tau^{\beta}$  for different values of  $\lambda$ . The most striking feature in the behavior of the scaling function is that its asymptotics for large  $x$  change for  $\lambda < 5$ . Indeed, for  $\lambda > 5$  the asymptotical value is given by a constant,  $\mathcal{C}_{\pm}(x \rightarrow \infty) = 1/3$ , whereas in the range  $3 < \lambda < 5$  the function behaves as a power law,

$$\mathcal{C}_{\pm}(x \rightarrow \infty) = \frac{4}{(\lambda-1)(\lambda-2)} x^{5-\lambda}. \quad (60)$$

In turn, this is reflected in the development of a minimum in the  $\mathcal{C}_{-}$  branch of the function as  $\lambda$  decreases (see the figure).

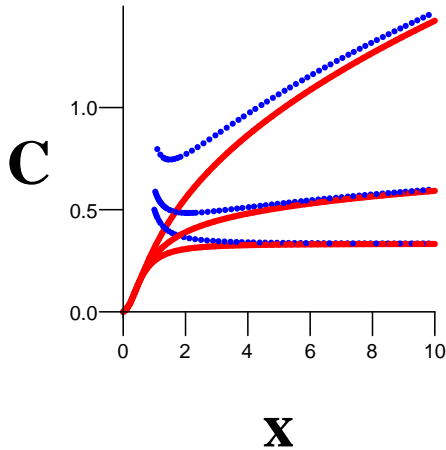


FIG. 1: Heat capacity scaling functions  $\mathcal{C}_{-}(x)$  (dotted curves, blue online) and  $\mathcal{C}_{+}(x)$  (solid curves, red online) as functions of the scaling variable  $x = m/\tau^{\beta}$  at  $\lambda > 5$ ,  $\lambda = 4.8$ , and  $\lambda = 4.5$  (lower, middle and upper pairs of curves, respectively).

Another particular feature of the plots of Fig. 1 is inherent to the presentation of the scaling plots in the  $\mathcal{C}_{-}$ - $x$ -plane and is connected to the presence of a pole in  $\mathcal{C}_{\pm}(x)$  for small  $x$ . As one sees immediately from Eqs. (31) and (36), this pole occurs at  $x = 1/\sqrt{3}$  and  $x = (4/[(\lambda-1)(\lambda-2)])^{1/(\lambda-3)}$  for  $\lambda = 5$  and  $3 < \lambda < 5$ , correspondingly. However, the physical values of the scaling variable  $x$  do not cover the region where the pole occurs. Indeed, from the magnetic equations of state (28), (34) one may obtain the solutions for the magnetization at zero magnetic field  $m(\tau, h = 0)$ . Taking that

a non-zero magnetic field must increase the value of  $m$  one arrives at the following minimal values of the scaling variable  $x$ :

$$x_{\min} = 1, \quad \lambda > 5, \quad (61)$$

$$x_{\min} = \left(\frac{4}{\lambda-1}\right)^{1/(\lambda-3)}, \quad 3 < \lambda < 5. \quad (62)$$

Therefore, the curves for the scaling function  $\mathcal{C}_{-}$  in Fig. 1 originate at the corresponding minimal values of  $x$  defined by the relations (61), (62).

In turn, as explained in section II, one may re-express the scaling function  $\mathcal{C}_{\pm}$  in terms of the scaled magnetic field  $y$  using Eq. (37). Corresponding plots for the scaling function  $\mathcal{C}_{\pm}(y)$  in this variable are given in Fig. 2 for different values of  $\lambda$ . Again, one observes a change in the asymptotics of the scaling function: instead of a constant at  $\lambda > 5$ , for  $3 < \lambda < 5$  the asymptotic functional dependence is given by

$$\mathcal{C}_{\pm}(y \rightarrow \infty) = \frac{1}{(\lambda-2)} \left(\frac{4}{(\lambda-1)}\right)^{\frac{3}{\lambda-2}} y^{\frac{5-\lambda}{\lambda-2}}. \quad (63)$$

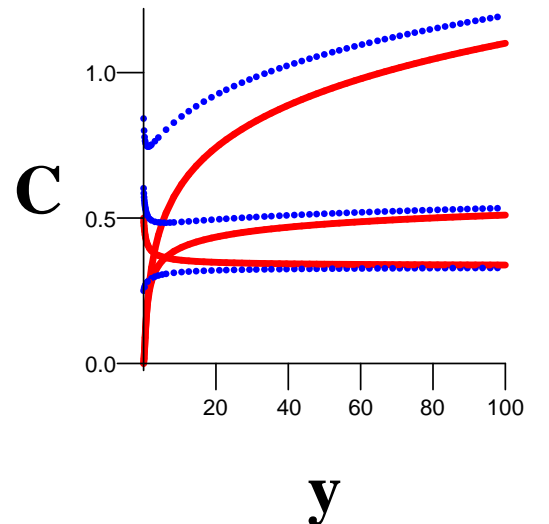


FIG. 2: Heat capacity scaling functions  $\mathcal{C}_{-}(y)$  (dotted curves, blue online) and  $\mathcal{C}_{+}(y)$  (solid curves, red online) as functions of the scaling variable  $y = h/\tau^{\beta}\delta$  at  $\lambda > 5$ ,  $\lambda = 4.8$ , and  $\lambda = 4.5$  (lower, middle and upper pairs of curves, respectively).

In figures 3 and 4 we give the plots of the scaling functions  $\mathcal{M}_{\pm}$  for the isothermal magnetocaloric coefficient in the scaling variables  $x = m/\tau^{\beta}$  and  $y = h/\tau^{\beta}\delta$ , respectively. As discussed at the end of the previous section,  $m_T$  changes its behavior at  $\lambda = 4$ . This feature is also reflected in the behavior of the scaling functions: their asymptotics change at  $\lambda = 4$ . Indeed, for  $\lambda > 5$  the function decays as  $\mathcal{M}_{\pm}(x \rightarrow \infty) \sim 1/3x$  whereas from the

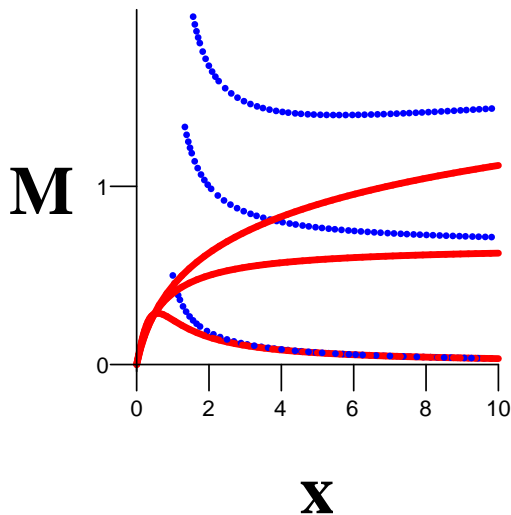


FIG. 3: Scaling functions for the isothermal magnetocaloric coefficient.  $\mathcal{M}_-(x)$  (dotted curves, blue online) and  $\mathcal{M}_+(x)$  (solid curves, red online) as functions of the scaling variable  $x = m/\tau^\beta$  at  $\lambda > 5$ ,  $\lambda = 4$ , and  $\lambda = 3.8$  (lower, middle and upper pairs of curves, respectively). The decay which is observed for  $\lambda > 4$  switches to power law growth for  $\lambda > 4$ . The functions approach the constant value  $\mathcal{M}_\pm = 2/3$  for  $\lambda = 4$ .

asymptotic behavior in the region  $3 < \lambda < 5$ ,

$$\mathcal{M}_\pm(x \rightarrow \infty) = \frac{4}{(\lambda-1)(\lambda-2)} x^{4-\lambda}, \quad (64)$$

one concludes, that for  $\lambda < 4$  the power law decay switches to a power law growth, while  $\mathcal{M}_\pm(x \rightarrow \infty) = \text{const}$  for the marginal value  $\lambda = 4$ . The corresponding asymptotics in the variables  $y$  is of the form

$$\begin{aligned} \mathcal{M}_\pm(y \rightarrow \infty) &= \frac{1}{3y^{1/3}}, \quad \lambda > 5, \\ \mathcal{M}_\pm(y \rightarrow \infty) &= \frac{1}{(\lambda-2)} \left( \frac{4}{(\lambda-1)} \right)^{\frac{2}{\lambda-2}} y^{\frac{4-\lambda}{\lambda-2}}, \\ & \quad 3 < \lambda < 5. \end{aligned} \quad (65)$$

#### IV. CONCLUSIONS

Usually the universality of critical phenomena is attributed to the presence of only a few relevant global parameters. Different systems that share the same values of these global variables manifest the same criticality. A classical example is given by the famous 3d Ising model universality class that is inherent to the critical behaviour of such differing systems as uniaxial magnets, simple

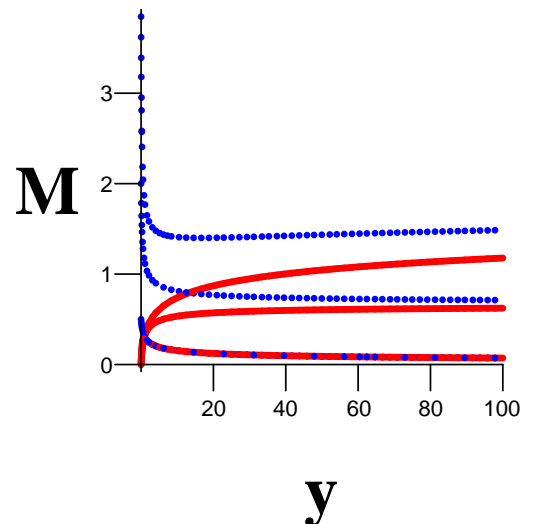


FIG. 4: Isothermal magnetocaloric coefficient scaling functions  $\mathcal{M}_-(y)$  (dotted curves, blue online) and  $\mathcal{M}_+(y)$  (solid curves, red online) as functions of the scaling variable  $y = m/\tau^{\beta\delta}$  at  $\lambda > 5$ ,  $\lambda = 4$ , and  $\lambda = 3.8$  (lower, middle and upper pairs of curves, respectively).

fluids or binary alloys. The critical behavior in all these systems is quantitatively described by the same values of the critical exponents, amplitude ratios, and by the same form of the scaling functions. As this paper demonstrates, in particular the usual ‘Euclidean space’ understanding of universality of critical phenomena breaks down if the critical behavior occurs on a scale-free network. The presence of high-degree vertices (hubs) may lead to substantial changes in ordering processes. The parameter that controls the ‘importance’ of the hubs is the node-degree distribution exponent  $\lambda$ , Eq. (1), and it is this parameter that plays the role of a global variable as far as the critical behavior is considered.

In particular, the universal quantities that govern criticality become  $\lambda$ -dependent for small enough  $\lambda$  and in this way the network structure is felt. However, the presence of magnetization is necessary to ‘feel’ the network structure. To give an example, the structure matters for  $T < T_c$  at any  $h$  and for  $T > T_c$  for  $h \neq 0$  (c.f. that amplitude  $\Gamma_-$  is  $\lambda$ -dependent, whereas  $\Gamma_+$  is not). Another interesting observation is that the fluctuation in network structure only enters via the magnetization and since the entropy  $S$  is measured at constant magnetization, it is given by a usual Landau-like mean field value for any  $\lambda > 3$ . This makes a difference between the global parameters  $\lambda$  and dimension  $d$  regarding its influence of the fluctuations on calculating the singular behaviour of the physical quantities: no renormalization-group calculation is necessary.



In this paper we completed the quantitative description of critical behavior in scale-free networks by calculating the entropic equation of state, the resulting scaling functions as well as the universal amplitude ratios. The corresponding formulas, together with other data for the critical exponents and amplitudes are summarized in Tables I and II. They constitute a comprehensive list of observables that describe the scaling and characterize the criticality in scale-free networks.

The starting point for our study was the asymptotic form of the free energy in the vicinity of a critical point in a scale-free network, Eqs. (3), (4). The validity of this expression has been proven at different levels of rigor using microscopic approaches based on the recursion relations [10] the replica method [9] or phenomenological Landau approaches [11, 13] as well as mean field theory [12]. It is

instructive to note here that, because the networks under discussion have a local tree-like structure, the mean-field approximation is asymptotically exact. One of the consequences of this fact is that the values of the exponents do not change if an  $O(m)$ -symmetrical order parameter is considered (see e.g. [13]), as it is usual in the Landau theory.

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