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Exact solution of the dimer model on the generalized finite checkerboard latticeN. Sh. Izmailian,^{1,2,*} Chin-Kun Hu,^{3,4,†} and R. Kenna^{1,‡}¹*Applied Mathematics Research Centre, Coventry University, Coventry CV1 5FB, United Kingdom*²*Yerevan Physics Institute, Alikhanian Brothers 2, 375036 Yerevan, Armenia*³*Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan*⁴*National Center for Theoretical Sciences, National Tsing Hua University, Hsinchu 30013, Taiwan*

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We present the exact closed-form expression for the partition function of a dimer model on a generalized finite checkerboard rectangular lattice under periodic boundary conditions. We investigate three different sets of dimer weights, each with different critical behaviors. We then consider different limits for the model on the three lattices. In one limit, the model for each of the three lattices is reduced to the dimer model on a rectangular lattice, which belongs to the $c = -2$ universality class. In another limit, two of the lattices reduce to the anisotropic Kasteleyn model on a honeycomb lattice, the universality class of which is given by $c = 1$. The result that the dimer model on a generalized checkerboard rectangular lattice can manifest different critical behaviors is consistent with early studies in the thermodynamic limit and also provides insight into corrections to scaling arising from the finite-size versions of the model.

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I. INTRODUCTION

Universality and scaling are two key concepts in the study of phase transitions and critical phenomena [1,2]. For two-dimensional critical systems, one can use the central charge c to classify universality classes [3–5]. Using Monte Carlo methods [6–9] and analytic methods [10,11], it has been found that many critical systems have nice universal scaling behaviors [12–21]. In the present paper, we obtain exact solutions of the dimer model on the generalized finite checkerboard lattice. Our results will be useful for further study on finite-size corrections and scaling.

The study of classical dimer models has a long history. Such models are of interest as direct representations of physical systems, such as of diatomic molecules on a lattice. But they are also important because of their equivalence to statistical mechanical models with different universality classes. As far as we know there are at least three different universality classes for dimer models: the Ising universality class with central charge $c = 1/2$ (the dimer model on a so-called 3×12 lattice [22]), the Kasteleyn universality class with central charge $c = 1$ (the dimer model on a honeycomb lattice [23]), and the universality class with central charge $c = -2$ (the dimer model on a square lattice [24,25]). Thus, it appears that the dimer model itself has not a single critical behavior but several critical behaviors associated with different universality classes. A similar situation occurs in the staggered vertex model [26,27] as well as in a one-dimensional anisotropic XY model with alternating nearest-neighbor interactions [28]. These may have multiple phase transitions [26–28].

Here we focus on the dimer model on a checkerboard lattice (see Fig. 1), a setup which provides a tool to study the evolution of physical properties such as system transits between different

geometries. The checkerboard lattice is a simple rectangular lattice with anisotropic dimer weights x_1 , x_2 , y_1 , and y_2 . Each weight a is simply the Boltzmann factor $e^{-E_a/kT}$ for a dimer on a bond of type a with energy E_a . There are three possible classifications of the dimer weights on the bonds of the lattice shown in the Fig. 1. We denote them as checkerboard lattices A, B, and C. Using Baxter's method [29] of converting the dimer model into a vertex model, Wu and Lin [30] have shown that the dimer model on the checkerboard lattice B can be converted to the staggered ice-rule model with some set of vertex weights. Using Baxter's method one can also show the equivalence of the checkerboard lattices A and C to the staggered ice-rule model with different sets of vertex weights.

The dimer model on the checkerboard-A lattice was first introduced by Kasteleyn [31], who showed that the model exhibits a phase transition. Since Kasteleyn did not publish the exact expression for the partition function, and since it has not been given in the existing literature, we supply it here for finite $2M \times 2N$ lattices with periodic boundary conditions using the Pfaffian method. Additionally, we present the exact solutions for the dimer model on checkerboard-B and checkerboard-C lattices using the Pfaffian method. The latter solution recovers that of Cohn, Kenyon, and Propp [32] in the case $M = N$. Another checkerboard lattice is the so-called generalized- K model [33,34], which is characterized by two parameters x and y . This can be considered as a special case of checkerboard-B and checkerboard-C lattices for the case $x_1 = x_2 = x$, $y_2 = 1$, and $y_1 = y$ and we investigate this limit also.

About 50 years ago Kasteleyn [31] introduced the dimer model on an anisotropic honeycomb lattice. This has been called the K model [35], or the K_1 model in our notation. We should emphasize that the dimer model on an anisotropic honeycomb lattice at the critical point has anisotropic scaling.

The anisotropic honeycomb lattice can be described as a bricked structure (see Fig. 2) and the bonds in three principal directions have activity x_1 , x_2 , and y_1 . There is another possible arrangement of the bonds in the principal directions, namely, the bonds having activity y_1 in one direction, while in the other two directions the bonds have activities x_1 and x_2 alternately.

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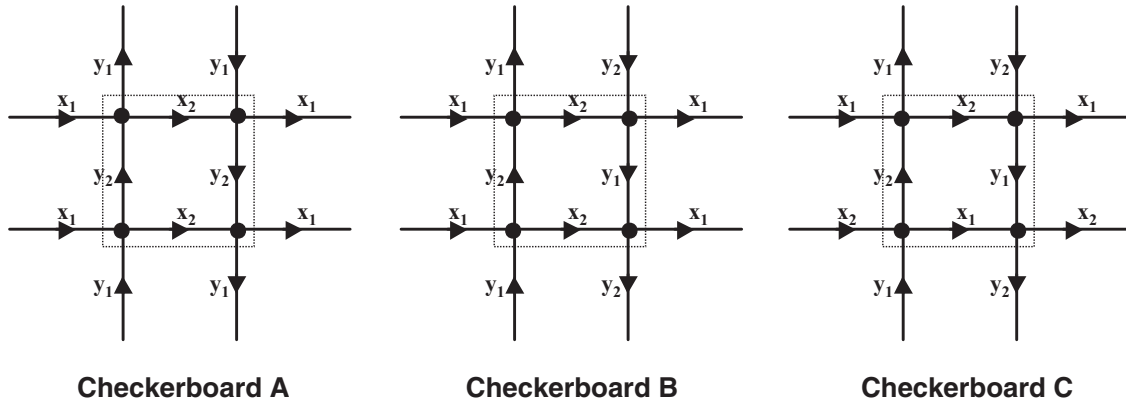


FIG. 1. The unit cell for dimer models on the checkerboard lattices A, B, and C in solid lines. The choice of arrows satisfies Kasteleyn’s clockwise-odd theorem [43].

We refer to such a dimer model, which was motivated by domain-wall considerations [33,36], as the K_2 model (see Fig. 2). The exact solutions of the K_1 and K_2 models reveal an unusual phase transition when $x_1 = x_2 + y_1, x_2 = x_1 + y_1$, or $y_1 = x_1 + x_2$. The specific heat has a square-root divergence as T is lowered toward T_c , but it is identically zero for all T smaller than T_c . This property occurs in many different kinds of physical systems, and we refer to such singular behavior as a KDP-type phase transition after the ferroelectrics transition in potassium dihydrogen phosphate (KH_2PO_4). The isomorphism of the K_1 model to a ferroelectrics model is discussed in Refs. [37,38]. A second such system is lipid bilayer biomembranes [33] and the corresponding dimer model is a K_2 model. The third system is a domain-wall model of the commensurate-incommensurate (CI) transition [39]. The isomorphism of the K_1 model to a domain-wall model is discussed in Refs. [40,41]. Its exact solution allows the testing of theories of CI-Ising crossover [42].

The K_1 and K_2 models with $x_1 = x_2 = x$ can be considered as a special case of the generalized K model for the case $y = 0$. This generalized K model belongs to the Kasteleyn universality class ($c = 1$) and belongs to another universality class ($c = -2$) for $y = 1$, when it is just the dimer model on the rectangular lattice.

It is interesting to consider different cases of the checkerboard lattices A, B, and C. For $x_1 = x_2$ and $y_1 = y_2$ the partition function for all three models reduces to that for the dimer model on the rectangular lattice with uniform

weights. When one of the weights x_1, x_2, y_1 , or y_2 on the checkerboard-A lattice is equal to zero, the partition function reduces to that for the dimer model on the one-dimensional strip. When one of the weights x_1 or x_2 on the checkerboard-B lattice is equal to zero the partition function again reduces to the partition function for the dimer model on the one-dimensional strip. But when one of the weights y_1 or y_2 is equal to zero the partition function for the checkerboard-B model reduces to the partition function for the dimer model on the honeycomb lattice—the so called K_2 model (see Fig. 2). And for the case when one of the weights x_1, x_2, y_1 , or y_2 is equal to zero the partition function for the checkerboard-C model reduces to the partition function for the dimer model on the honeycomb lattice, the so-called K_1 model (see Fig. 2).

Thus and as was first pointed out in studies directly in the thermodynamic limit [26,29,30] the dimer model on the checkerboard-B and checkerboard-C lattices can show two different types of critical behavior, depending on the parameters (weights) of the model. One critical behavior is typical for the Kasteleyn universality class with the central charge $c = 1$ and another is typical for the universality class with $c = -2$, while the dimer model on the checkerboard-A lattice exhibits the critical behavior typical for the universality class with $c = -2$.

II. THE GENERALIZED FINITE CHECKERBOARD MODEL

The partition function for the dimer model on the checkerboard lattice can be written as

$$Z = \sum x_1^{N_{x_1}} x_2^{N_{x_2}} y_1^{N_{y_1}} y_2^{N_{y_2}},$$

where N_a is the number of dimers of type a and the summation is over all possible dimer configurations on the lattice. Let us consider a $2M \times 2N$ rectangular lattice with periodic boundary conditions in both the horizontal and vertical directions. It is well known that the partition function for a planar dimer model with periodic boundary conditions can be expressed as a linear combination of four Pfaffians [43]:

$$Z = \frac{1}{2}(-G_{0,0} + G_{1/2,0} + G_{0,1/2} + G_{1/2,1/2}). \quad (1)$$

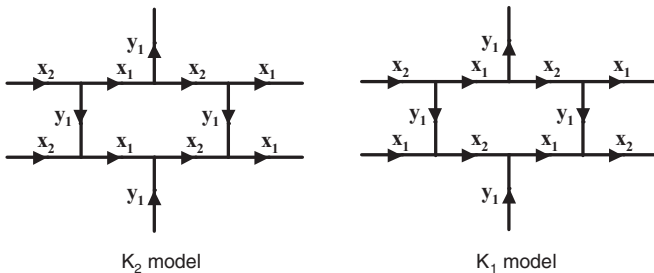


FIG. 2. K_1 and K_2 dimer model on the brick lattice, which is topologically equivalent to the honeycomb lattice.

The Pfaffians $G_{\alpha,\beta}$ with $(\alpha,\beta) = (0,0); (0,1/2); (1/2,0); (1/2,1/2)$ are the square roots of the determinants

$$G_{\alpha,\beta} = \text{Pf}[A_{\alpha,\beta}] = \sqrt{\det A_{\alpha,\beta}},$$

where $\det A_{\alpha,\beta}$ is

$$\det A_{\alpha,\beta} = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} \det \lambda(\phi_{\alpha,n}, \phi_{\beta,m}),$$

and the matrix $\lambda(\phi_{\alpha,n}, \phi_{\beta,m})$ is given by [44]

$$\lambda(\phi_{\alpha,n}, \phi_{\beta,m}) = \sum_{u_1, u_2} a(u_1, u_2) e^{i(u_1 \phi_{\alpha,n} + u_2 \phi_{\beta,m})}. \quad (2)$$

Here, the variables $\phi_{\alpha,n}$ and $\phi_{\beta,m}$ are given by

$$\phi_{\alpha,n} = \frac{2\pi(n+\alpha)}{N}, \quad \phi_{\beta,m} = \frac{2\pi(m+\beta)}{M}, \quad (3)$$

and the matrices $a(u_1, u_2)$ are defined in the following sections.

Our main results can be summarized as follows. We find that the partition function for the dimer model on checkerboard lattices under periodic boundary conditions is given by

$$Z = \frac{(x_1 x_2)^{MN}}{2} (-Z_{0,0}^2 + Z_{1/2,0}^2 + Z_{0,1/2}^2 + Z_{1/2,1/2}^2), \quad (4)$$

where $Z_{\alpha,\beta}^2$ with $(\alpha,\beta) = (0,0), (0,1/2), (1/2,0), (1/2,1/2)$ are partition functions with twisted boundary conditions, which, for the checkerboard A model are given by

$$\begin{aligned} Z_{\alpha,\beta}^2 &= (x_1 x_2)^{-MN} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} (|x_1 e^{i\frac{\phi_{\alpha,n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha,n}}{2}}|^2 \\ &\quad + |y_1 e^{i\frac{\phi_{\beta,m}}{2}} - y_2 e^{-i\frac{\phi_{\beta,m}}{2}}|^2) \\ &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[\left| \sin \left(\frac{\phi_{\alpha,n}}{2} + \frac{i}{2} \ln \frac{x_2}{x_1} \right) \right|^2 \right. \\ &\quad \left. + z^2 \left| \sin \left(\frac{\phi_{\beta,m}}{2} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) \right|^2 \right] \\ &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[t^2 + \sin^2 \frac{\phi_{\alpha,n}}{2} + z^2 \sin^2 \frac{\phi_{\beta,m}}{2} \right], \quad (5) \end{aligned}$$

for the checkerboard B model are given by

$$\begin{aligned} Z_{\alpha,\beta}^2 &= (x_1 x_2)^{-MN} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} (|x_1 e^{i\frac{\phi_{\alpha,n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha,n}}{2}}|^2 \\ &\quad - (y_1 e^{i\frac{\phi_{\beta,m}}{2}} - y_2 e^{-i\frac{\phi_{\beta,m}}{2}})^2) \\ &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[\left| \sin \left(\frac{\phi_{\alpha,n}}{2} + \frac{i}{2} \ln \frac{x_2}{x_1} \right) \right|^2 \right. \\ &\quad \left. + z^2 \sin^2 \left(\frac{\phi_{\beta,m}}{2} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) \right], \quad (6) \end{aligned}$$

and for the checkerboard C model are given by

$$\begin{aligned} Z_{\alpha,\beta}^2 &= (x_1 x_2)^{-MN} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} (|x_1 e^{i\frac{\phi_{\alpha,n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha,n}}{2}}|^2 \\ &\quad + (y_1 e^{i\frac{\phi_{\beta,m}}{2}} - y_2 e^{-i\frac{\phi_{\beta,m}}{2}})^2) \end{aligned}$$

$$\begin{aligned} &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left| \sin^2 \left(\frac{\phi_{\alpha,n}}{2} + \frac{i}{2} \ln \frac{x_2}{x_1} \right) \right. \\ &\quad \left. + z^2 \sin^2 \left(\frac{\phi_{\beta,m}}{2} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) \right|, \quad (7) \end{aligned}$$

where $\phi_{\alpha,n}$ and $\phi_{\beta,m}$ are given by Eq. (3) and

$$z^2 = \frac{y_1 y_2}{x_1 x_2}, \quad t^2 = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4x_1 x_2}. \quad (8)$$

In the next three sections, we derive these main results for checkerboards A, B, and C, respectively.

III. CHECKERBOARD-A MODEL

The dimer model on the checkerboard-A lattice was first introduced by Kasteleyn who showed that the model exhibits a phase transition [31]. However, Kasteleyn never published the exact expressions for the partition function of that model, so we do so here for the case of periodic boundary conditions.

Orient the lattice edges as shown in Fig. 1 for which it is known [45] that the Kasteleyn clockwise-odd sign rule [43] is satisfied. Let us divide the lattice into (non-overlapping) unit cells, each of four lattice sites as shown in Fig. 1. We represent the location of each unit cell by the indices p_1 and p_2 , representing the x and y coordinates of the center of the cell. Sites within each cell are then identified by an index Γ , which takes four values indicating their locations relative to the center: L_{up} (left up); R_{up} (right up), L_{down} (left down), and R_{down} (right down). We define matrix $A(p_1, p_2; p'_1, p'_2)$, the elements of which represent the directed weights connecting the lattice points of unit cell (p_1, p_2) with those of unit cell (p'_1, p'_2) . Other elements of the matrix $A(p_1, p_2; p'_1, p'_2)$ are zero. The matrix $a(0,0)$ corresponds to the bonding between two points in the same unit cell:

$$a(0,0) \equiv A(p_1, p_2; p_1, p_2)$$

$$= \begin{matrix} & L_{\text{up}} & R_{\text{up}} & L_{\text{down}} & R_{\text{down}} \\ \begin{matrix} L_{\text{up}} \\ R_{\text{up}} \\ L_{\text{down}} \\ R_{\text{down}} \end{matrix} & \begin{pmatrix} 0 & x_2 & -y_2 & 0 \\ -x_2 & 0 & 0 & y_2 \\ y_2 & 0 & 0 & x_2 \\ 0 & -y_2 & -x_2 & 0 \end{pmatrix} \end{matrix}.$$

Matrix $a(1,0)$ corresponds to the bonding between two points in a given unit cell and the one to the right:

$$a(1,0) \equiv A(p_1, p_2; p_1 + 1, p_2)$$

$$= \begin{matrix} & L_{\text{up}} & R_{\text{up}} & L_{\text{down}} & R_{\text{down}} \\ \begin{matrix} L_{\text{up}} \\ R_{\text{up}} \\ L_{\text{down}} \\ R_{\text{down}} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 \end{pmatrix} \end{matrix}.$$

Similarly $a(0, 1)$ corresponds to the bonding between two points in the given unit cell and the one above it:

$$a(0, 1) \equiv A(p_1, p_2; p_1, p_2 + 1)$$

$$= \begin{matrix} & L_{\text{up}} & R_{\text{up}} & L_{\text{down}} & R_{\text{down}} \\ \begin{matrix} L_{\text{up}} \\ R_{\text{up}} \\ L_{\text{down}} \\ R_{\text{down}} \end{matrix} & \begin{pmatrix} 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & -y_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}.$$

We note also that $a(-1, 0) = -a^T(1, 0)$ and $a(0, -1) = -a^T(0, 1)$. Then the determinant of matrix $A(p_1, p_2; p'_1, p'_2)$ can be written as

$$\det A = \prod_{n=1}^N \prod_{m=1}^M \det \lambda(\phi_{\alpha, n}, \phi_{\beta, m}),$$

where $\lambda(\phi_{\alpha, n}, \phi_{\beta, m})$ is given by Eq. (2):

$$\begin{aligned} \lambda(\phi_{\alpha, n}, \phi_{\beta, m}) &= \sum_{u_1, u_2} a(u_1, u_2) e^{i(u_1 \phi_{\alpha, n} + u_2 \phi_{\beta, m})} \\ &= a(0, 0) + a(1, 0) e^{i\phi_{\alpha, n}} + a(-1, 0) e^{-i\phi_{\alpha, n}} \\ &\quad + a(0, 1) e^{i\phi_{\beta, m}} + a(0, -1) e^{-i\phi_{\beta, m}}. \end{aligned} \tag{9}$$

In matrix form $\lambda(\phi_{\alpha, n}, \phi_{\beta, m})$ can be written as

$$\lambda(\phi_{\alpha, n}, \phi_{\beta, m}) = \begin{pmatrix} 0 & x_2 - x_1 e^{-i\phi_{\alpha, n}} & -y_2 + y_1 e^{i\phi_{\beta, m}} & 0 \\ -x_2 + x_1 e^{i\phi_{\alpha, n}} & 0 & 0 & y_2 - y_1 e^{i\phi_{\beta, m}} \\ y_2 - y_1 e^{-i\phi_{\beta, m}} & 0 & 0 & x_2 - x_1 e^{-i\phi_{\alpha, n}} \\ 0 & -y_2 + y_1 e^{-i\phi_{\beta, m}} & -x_2 + x_1 e^{i\phi_{\alpha, n}} & 0 \end{pmatrix}. \tag{10}$$

Therefore

$$\det \lambda(\phi_{\alpha, n}, \phi_{\beta, m}) = |g(\phi_{\alpha, n}, \phi_{\beta, m})|^2,$$

where

$$\begin{aligned} g(\phi_{\alpha, n}, \phi_{\beta, m}) &= (x_1 e^{i\frac{\phi_{\alpha, n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha, n}}{2}})(x_1 e^{-i\frac{\phi_{\alpha, n}}{2}} - x_2 e^{i\frac{\phi_{\alpha, n}}{2}}) + (y_1 e^{i\frac{\phi_{\beta, m}}{2}} - y_2 e^{-i\frac{\phi_{\beta, m}}{2}})(y_1 e^{-i\frac{\phi_{\beta, m}}{2}} - y_2 e^{i\frac{\phi_{\beta, m}}{2}}) \\ &= |x_1 e^{i\frac{\phi_{\alpha, n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha, n}}{2}}|^2 + |y_1 e^{i\frac{\phi_{\beta, m}}{2}} - y_2 e^{-i\frac{\phi_{\beta, m}}{2}}|^2. \end{aligned} \tag{11}$$

The Pfaffians $G_{\alpha, \beta}$ with $(\alpha, \beta) = (0, 0); (0, 1/2); (1/2, 0); (1/2, 1/2)$ for the checkerboard-A model on the $2M \times 2N$ rectangular lattice can be written in the form

$$G_{\alpha, \beta} = \prod_{n=1}^N \prod_{m=1}^M (|x_1 e^{i\frac{\phi_{\alpha, n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha, n}}{2}}|^2 + |y_1 e^{i\frac{\phi_{\beta, m}}{2}} - y_2 e^{-i\frac{\phi_{\beta, m}}{2}}|^2),$$

where $\phi_{\alpha, n}$ and $\phi_{\beta, m}$ are given by Eq. (3) and $(\alpha, \beta) = (0, 0), (0, 1/2), (1/2, 0), (1/2, 1/2)$. Then, the partition function for the checkerboard-A model can be rewritten in the form given by Eq. (4) with

$$\begin{aligned} Z_{\alpha, \beta}^2 &= \frac{G_{\alpha, \beta}}{(x_1 x_2)^{MN}} = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[\left| \sin \left(\frac{\phi_{\alpha, n}}{2} + \frac{i}{2} \ln \frac{x_2}{x_1} \right) \right|^2 + z^2 \left| \sin \left(\frac{\phi_{\beta, m}}{2} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) \right|^2 \right] \\ &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[t^2 + \sin^2 \frac{\phi_{\alpha, n}}{2} + z^2 \sin^2 \frac{\phi_{\beta, m}}{2} \right] \\ &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[t^2 + \sin^2 \left(\frac{\pi(n + \alpha)}{N} \right) + z^2 \sin^2 \left(\frac{\pi(m + \beta)}{M} \right) \right], \end{aligned} \tag{12}$$

in which t^2 and z^2 are given by Eq. (8). With the help of the identity [46]

$$4|\sinh(N\omega + i\pi\alpha)|^2 = 4[\sinh^2 N\omega + \sin^2 \pi\alpha] = \prod_{n=0}^{N-1} 4 \left[\sinh^2 \omega + \sin^2 \left(\frac{\pi(n + \alpha)}{N} \right) \right], \tag{13}$$

$Z_{\alpha,\beta}$ can be transformed into a simpler form,

$$Z_{\alpha,\beta} = \prod_{m=0}^{M-1} 2 \left| \sinh \left[N \omega_1 \left(\frac{\pi(m+\beta)}{M} \right) + i\pi\alpha \right] \right|, \quad (14)$$

where $\omega_1(k) = \operatorname{arcsinh} z \sqrt{t^2 + z^2 \sin^2 k}$.

The free energy per site βF is defined as

$$-\beta F = \frac{1}{S} \ln Z, \quad (15)$$

where S is the area of the lattice, Z is the partition function, and $\beta = k_B T$ with Boltzmann constant k_B and temperature T .

From Eqs. (4) and (12) one can derive free energy per site for the dimer model on the checkerboard-A lattice in the form

$$-\beta F = \frac{1}{8\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \ln 4 \left[t^2 + \sin^2 \frac{\theta}{2} + z^2 \sin^2 \frac{\varphi}{2} \right]. \quad (16)$$

The dimer model on the checkerboard-A lattice therefore has a singularity at $t = 0$ for $t = 0$ ($x_1 = x_2 = x$, $y_1 = y_2 = y$) when the partition function reduces to the partition function on the rectangular lattice with uniform weights given by Eq. (4) with $Z_{\alpha,\beta}$, namely,

$$Z_{\alpha,\beta}^2 = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[\sin^2 \left(\frac{\pi(n+\alpha)}{N} \right) + \frac{y^2}{x^2} \sin^2 \left(\frac{\pi(m+\beta)}{M} \right) \right]. \quad (17)$$

Recently it has been shown that this dimer model on the rectangular lattice with uniform weights belongs to the $c = -2$ universality class [24,25].

IV. THE CHECKERBOARD-B MODEL

Let us now consider the dimer model on the checkerboard-B lattice. The unit cell is depicted in Fig. 1. Following reasoning similar to that of the previous section, we obtain

$$\lambda(\phi_{\alpha,n}, \phi_{\beta,m}) = a(0,0) + a(1,0) e^{i\phi_{\alpha,n}} + a(-1,0) e^{-i\phi_{\alpha,n}} + a(0,1) e^{i\phi_{\beta,m}} + a(0,-1) e^{-i\phi_{\beta,m}}, \quad (18)$$

in which

$$a(0,0) = \begin{pmatrix} 0 & x_2 & -y_2 & 0 \\ -x_2 & 0 & 0 & y_1 \\ y_2 & 0 & 0 & x_2 \\ 0 & -y_1 & -x_2 & 0 \end{pmatrix}, \quad a(1,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 \end{pmatrix}, \quad a(0,1) = \begin{pmatrix} 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & -y_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (19)$$

and $a(-1,0) = -a^T(1,0)$ and $a(0,-1) = -a^T(0,1)$. The matrix form of $\lambda(\phi_{\alpha,n}, \phi_{\beta,m})$ is now given by

$$\lambda(\phi_{\alpha,n}, \phi_{\beta,m}) = \begin{pmatrix} 0 & x_2 - x_1 e^{-i\phi_{\alpha,n}} & -y_2 + y_1 e^{i\phi_{\beta,m}} & 0 \\ -x_2 + x_1 e^{i\phi_{\alpha,n}} & 0 & 0 & y_1 - y_2 e^{i\phi_{\beta,m}} \\ y_2 - y_1 e^{-i\phi_{\beta,m}} & 0 & 0 & x_2 - x_1 e^{-i\phi_{\alpha,n}} \\ 0 & -y_1 + y_2 e^{-i\phi_{\beta,m}} & -x_2 + x_1 e^{i\phi_{\alpha,n}} & 0 \end{pmatrix}. \quad (20)$$

The determinant of $\lambda(\phi_{\alpha,n}, \phi_{\beta,m})$ can be written as

$$\det \lambda(\phi_{\alpha,n}, \phi_{\beta,m}) = |g(\phi_{\alpha,n}, \phi_{\beta,m})|^2,$$

where $g(\phi_{\alpha,n}, \phi_{\beta,m})$ is given by

$$\begin{aligned} g(\phi_{\alpha,n}, \phi_{\beta,m}) &= |x_1 e^{i\frac{\phi_{\alpha,n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha,n}}{2}}|^2 - (y_1 e^{i\frac{\phi_{\beta,m}}{2}} - y_2 e^{-i\frac{\phi_{\beta,m}}{2}})^2 \\ &= 4x_1 x_2 \left| \sin \left(\frac{\phi_{\alpha,n}}{2} + \frac{i}{2} \ln \frac{x_2}{x_1} \right) \right|^2 + 4y_1 y_2 \sin^2 \left(\frac{\phi_{\beta,m}}{2} + i \ln \frac{y_2}{y_1} \right). \end{aligned}$$

Therefore the partition function for the dimer model on the $2M \times 2N$ checkerboard-B lattice can be written in the form given by Eq. (4), with

$$\begin{aligned} Z_{\alpha,\beta}^2 &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left| \sin \left(\frac{\phi_{\alpha,n}}{2} + \frac{i}{2} \ln \frac{x_2}{x_1} \right) \right|^2 + z^2 \sin^2 \left(\frac{\phi_{\beta,m}}{2} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) \Big| \\ &= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left| \frac{(x_1 - x_2)^2}{4x_1 x_2} + \sin^2 \frac{\phi_{\alpha,n}}{2} + z^2 \sin^2 \left(\frac{\phi_{\beta,m}}{2} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) \right|, \end{aligned} \quad (21)$$

with $(\alpha, \beta) = (0,0), (0,1/2), (1/2,0), (1/2,1/2)$. With the help of the identity given by Eq. (13), $Z_{\alpha,\beta}$ can be transformed into a simpler form,

$$Z_{\alpha,\beta}^2 = \prod_{m=0}^{M-1} 4 \left| \sinh \left[N\omega_2 \left(\frac{\pi(m+\beta)}{M} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) + i\pi\alpha \right] \sinh \left[N\omega_2 \left(\frac{\pi(m+\beta)}{M} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) - i\pi\alpha \right] \right|, \quad (22)$$

where

$$\omega_2(k) = \operatorname{arcsinh} \sqrt{\frac{(x_1 - x_2)^2}{4x_1x_2} + z^2 \sin^2 k}.$$

The free energy per site βF for the dimer model on the checkerboard-B lattice can be written in the form

$$-\beta F = \frac{1}{2} \ln(x_1x_2) + \frac{1}{8\pi^2} \operatorname{Re} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \ln 4 \left[\frac{(x_1 - x_2)^2}{4x_1x_2} + \sin^2 \frac{\theta}{2} + z^2 \sin^2 \left(\frac{\varphi}{2} + \frac{i}{2} \ln \frac{y_2}{y_1} \right) \right]. \quad (23)$$

As for the case the checkerboard-A model, the partition function for the checkerboard-B model has a singularity at $t = 0$. Again at $t = 0 (x_1 = x_2, y_1 = y_2)$ the partition function for the checkerboard-B model reduces to the partition function on the rectangular lattice with uniform weights [see Eq. (17)], which belongs to the $c = -2$ universality class.

There is another point at which the partition function for the checkerboard-B model has a singularity. When y_1 or y_2 is equal to zero the partition function for the checkerboard-B model reduces to the partition function on the honeycomb lattice—the so-called K_2 model. That model belongs to another universality class with central charge $c = 1$. Let us consider the case $y_2 = 0$. Then the partition function with twisted boundary conditions given by Eq. (21) can be rewritten as

$$Z_{\alpha,\beta}^2 = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left| \frac{(x_1 - x_2)^2}{4x_1x_2} + \sin^2 \frac{\phi_{\alpha,n}}{2} - \frac{y_1^2}{4x_1x_2} \exp(i\phi_{\beta,m}) \right|. \quad (24)$$

For even M one can use the identity

$$\prod_{m=0}^{M-1} |a - b \exp(i\phi_{\beta,m})| = \prod_{m=0}^{M-1} |a + b \exp(i\phi_{\beta,m})|, \quad (25)$$

and rewrite Eq. (24) in another equivalent form,

$$Z_{\alpha,\beta}^2 = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left| \frac{(x_1 - x_2)^2}{4x_1x_2} + \sin^2 \frac{\phi_{\alpha,n}}{2} + \frac{y_1^2}{4x_1x_2} \exp(i\phi_{\beta,m}) \right|. \quad (26)$$

Then the free energy per site βF in the thermodynamic limit can be written in the form

$$-\beta F = \frac{1}{2} \ln(x_1x_2) + \frac{1}{8\pi^2} \operatorname{Re} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \ln 4 \left[\frac{(x_1 - x_2)^2}{4x_1x_2} + \sin^2 \frac{\theta}{2} + \frac{y_1^2}{4x_1x_2} e^{i\varphi} \right]. \quad (27)$$

Note that our result for the free energy for the K_2 model recovers the results of Ref. [33]. There it was shown that the K_2 model exhibits a KDP-type phase transition at $y_1 = x_1 + x_2$, $x_1 = y_1 + x_2$, or $x_2 = y_1 + x_1$, and the second derivatives of the free energy in the $\{x_1, x_2, y_1\}$ space exhibit an inverse-root singularity near the phase boundaries.

To summarize, the dimer model on the checkerboard-B lattice exhibits two different types of critical behavior; one is characteristic of the $c = -2$ universality class and another critical behavior is typical for $c = 1$.

V. THE CHECKERBOARD-C MODEL

Let us now consider the dimer model on the checkerboard-C lattice. The unit cell is again depicted in Fig. 1. We obtain

$$\lambda(\phi_{\alpha,n}, \phi_{\beta,m}) = a(0,0) + a(1,0) e^{i\phi_{\alpha,n}} + a(-1,0) e^{-i\phi_{\alpha,n}} + a(0,1) e^{i\phi_{\beta,m}} + a(0,-1) e^{-i\phi_{\beta,m}}, \quad (28)$$

in which

$$a(0,0) = \begin{pmatrix} 0 & x_2 & -y_2 & 0 \\ -x_2 & 0 & 0 & y_1 \\ y_2 & 0 & 0 & x_1 \\ 0 & -y_1 & -x_1 & 0 \end{pmatrix}, \quad a(1,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 \end{pmatrix}, \quad a(0,1) = \begin{pmatrix} 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & -y_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (29)$$

and $a(-1,0) = -a^T(1,0)$ and $a(0,-1) = -a^T(0,1)$. The λ matrix is

$$\lambda(\phi_{\alpha,n}, \phi_{\beta,m}) = \begin{pmatrix} 0 & x_2 - x_1 e^{-i\phi_{\alpha,n}} & -y_2 + y_1 e^{i\phi_{\beta,m}} & 0 \\ -x_2 + x_1 e^{i\phi_{\alpha,n}} & 0 & 0 & y_1 - y_2 e^{i\phi_{\beta,m}} \\ y_2 - y_1 e^{-i\phi_{\beta,m}} & 0 & 0 & x_1 - x_2 e^{-i\phi_{\alpha,n}} \\ 0 & -y_1 + y_2 e^{-i\phi_{\beta,m}} & -x_1 + x_2 e^{i\phi_{\alpha,n}} & 0 \end{pmatrix} \quad (30)$$

and

$$\begin{aligned} g(\phi_{\alpha,n}, \phi_{\beta,m}) &= -(x_1 e^{i\frac{\phi_{\alpha,n}}{2}} - x_2 e^{-i\frac{\phi_{\alpha,n}}{2}})^2 - (y_1 e^{i\frac{\phi_{\beta,m}}{2}} - y_2 e^{-i\frac{\phi_{\beta,m}}{2}})^2 \\ &= 4x_1 x_2 \sin^2\left(\frac{\phi_{\alpha,n}}{2} + i \ln \frac{x_2}{x_1}\right) + 4y_1 y_2 \sin^2\left(\frac{\phi_{\beta,m}}{2} + i \ln \frac{y_2}{y_1}\right). \end{aligned}$$

Thus the partition function for the checkerboard-C model on the $2M \times 2N$ rectangular lattice can be written in the form given by Eq. (4) with

$$Z_{\alpha,\beta}^2 = (x_1 x_2)^{-MN} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} |(x_1 e^{i\frac{\pi(n+\alpha)}{N}} - x_2 e^{-i\frac{\pi(n+\alpha)}{N}})^2 + (y_1 e^{i\frac{\pi(m+\beta)}{M}} - y_2 e^{-i\frac{\pi(m+\beta)}{M}})^2| \quad (31)$$

$$= \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left| \sin^2\left(\frac{\pi(n+\alpha)}{N} + \frac{i}{2} \ln \frac{x_2}{x_1}\right) + z^2 \sin^2\left(\frac{\pi(m+\beta)}{M} + \frac{i}{2} \ln \frac{y_2}{y_1}\right) \right|, \quad (32)$$

where $(\alpha, \beta) = (0,0), (0,1/2), (1/2,0), (1/2,1/2)$.

The free energy per site βF for the dimer model on the checkerboard-C lattice can be written in the form

$$-\beta F = \frac{1}{8\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \ln |(x_1 e^{i\frac{\theta}{2}} - x_2 e^{-i\frac{\theta}{2}})^2 + (y_1 e^{i\frac{\varphi}{2}} - y_2 e^{-i\frac{\varphi}{2}})^2| \quad (33)$$

$$= \frac{1}{2} \ln(x_1 x_2) + \frac{1}{8\pi^2} \text{Re} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \ln 4 \left[\sin^2\left(\frac{\theta}{2} + \frac{i}{2} \ln \frac{x_2}{x_1}\right) + z^2 \sin^2\left(\frac{\varphi}{2} + \frac{i}{2} \ln \frac{y_2}{y_1}\right) \right]. \quad (34)$$

Note that our result for the partition function for the checkerboard-C model for the case $M = N$ reproduces the result of Cohn, Kenyon and Proop [32]. In our notation, that result reads as

$$Z_{\alpha,\beta}^2 = (x_1 x_2)^{-N^2} \prod_{n=0}^{N-1} \prod_{m=0}^{N-1} |(x_1 e^{i\frac{\pi(n+\alpha)}{N}} + x_2 e^{-i\frac{\pi(n+\alpha)}{N}})^2 + (y_1 e^{i\frac{\pi(m+\beta)}{N}} + y_2 e^{-i\frac{\pi(m+\beta)}{N}})^2|. \quad (35)$$

The only difference is the sign inside the brackets (see Eqs. (7.7) and (7.8) of Ref. [32]). It is easy to see that for even N the identity

$$\prod_{n=0}^{N-1} |(x_1 e^{i\frac{\pi(n+\alpha)}{N}} + x_2 e^{-i\frac{\pi(n+\alpha)}{N}})^2 + f| = \prod_{n=0}^{N-1} |(x_1 e^{i\frac{\pi(n+\alpha)}{N}} - x_2 e^{-i\frac{\pi(n+\alpha)}{N}})^2 + f|$$

holds, which is enough to prove the equivalence between our result given by Eq. (31) for $M = N$ and the result of Ref. [32] given by Eq. (35).

As for the case of the checkerboard-A and checkerboard-B models, the partition function for the checkerboard-C model has a singularity at $t = 0$ ($x_1 = x_2, y_1 = y_2$). Again the partition function for the checkerboard-C model reduces to the partition function on the rectangular lattice with uniform weights [see Eq. (17)], which belongs to the $c = -2$ universality class.

There is another point at which the partition function for the checkerboard-C model has a singularity. When one of the weights x_1, x_2, y_1 , or y_2 is equal to zero the partition function for the checkerboard-C model reduces to the partition function on the honeycomb lattice (the so-called K_1 model), which belongs to another universality class with central charge $c = 1$. Let us consider the case $y_2 = 0$. The partition function with twisted boundary conditions given by Eq. (32) can be rewritten as

$$Z_{\alpha,\beta}^2 = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left| \frac{1}{2} - \frac{x_1}{4x_2} \exp(i\phi_{\alpha,n}) - \frac{x_2}{4x_1} \exp(-i\phi_{\alpha,n}) - \frac{y_1^2}{4x_1 x_2} \exp(i\phi_{\beta,m}) \right|.$$

Then the free energy per site βF for the dimer K_1 model can be written in the form

$$-\beta F = \frac{1}{8\pi^2} \text{Re} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \ln |(x_1 e^{i\frac{\theta}{2}} - x_2 e^{-i\frac{\theta}{2}})^2 + y_1^2 e^{i\varphi}|. \quad (36)$$

The K_1 model has been studied in detail in Refs. [37,38]. There it was shown that the K_1 model also exhibits a KDP-type phase transition at $y_1 = x_1 + x_2$, $x_1 = y_1 + x_2$, or $x_2 = y_1 + x_1$, where the specific heat has a square-root singularity near the phase boundaries.

Thus as in the case of the dimer model on the checkerboard-B lattice, the dimer model on the checkerboard-C lattice also has two different types of critical behavior, one typical for the $c = -2$ universality class and the other typical for the $c = 1$ universality class.

VI. SUMMARY

We considered the dimer model on generalized finite checkerboard rectangular lattices. We obtained the closed-form expressions for the partition function of the dimer models on $2M \times 2N$ checkerboard rectangular lattices A, B, and C under periodic boundary conditions. We also considered the different limits of each model. When dimer weights $x_1 = x_2$ and $y_1 = y_2$ the partition function for all three models reduces to that of the dimer model on the rectangular lattice with uniform weights, which belongs to the $c = -2$ universality class. In another limit, namely, when one of the weights y_1 or y_2 is zero, the partition function for the checkerboard-B model reduces to that of the dimer model on the honeycomb lattice, the so-called K_2 model. When one of the weights x_1 , x_2 , y_1 , or y_2 is equal to zero the partition function for the checkerboard-C model reduces to that of the dimer model on the honeycomb lattice, the so-called K_1 model. The K_1 and K_2 models exhibit a KDP-type phase transition and belong to the Kasteleyn universality class with the central charge $c = 1$.

Thus, the dimer model on the checkerboard-B and checkerboard-C lattices can show two different critical behaviors, depending on the parameters (weights) of the model. In one limit the model reduces to the dimer model on a rectangular lattice, which belongs to the $c = -2$ universality class. In another limit it reduces to the anisotropic Kasteleyn model on a honeycomb lattice, which belongs to another universality class with $c = 1$. The result that the dimer model on a generalized checkerboard rectangular lattice can manifest different critical behaviors is consistent with early studies in the thermodynamic limit [26,29,30].

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- [1] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York, 1971).
 - [2] For a recent review, see C.-K. Hu, *Chin. J. Phys.* **52**, 1 (2014).
 - [3] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Nucl. Phys. B* **241**, 333 (1984).
 - [4] D. Friedan, Z. Qiu, and S. Shenker, *Phys. Rev. Lett.* **52**, 1575 (1984).
 - [5] P. A. Pearce and A. Klumper, *Phys. Rev. Lett.* **66**, 974 (1991).
 - [6] N. Metropolis and S. Ulam, *J. Am. Stat. Assoc.* **44**, 335 (1949).
 - [7] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, *J. Chem. Phys.* **21**, 1087 (1953).
 - [8] C.-K. Hu, *Phys. Rev. B* **46**, 6592 (1992).
 - [9] C.-K. Hu, *Phys. Rev. Lett.* **69**, 2739 (1992).
 - [10] M. C. Wu and C.-K. Hu, *J. Phys. A* **35**, 5189 (2002).
 - [11] E. V. Ivashkevich, N. S. Izmailian, and C.-K. Hu, *J. Phys. A* **35**, 5543 (2002).
 - [12] C.-K. Hu, *J. Phys. A* **27**, L813 (1994).
 - [13] C.-K. Hu, C. Y. Lin, and J. A. Chen, *Phys. Rev. Lett.* **75**, 193 (1995).
 - [14] C.-K. Hu and C. Y. Lin, *Phys. Rev. Lett.* **77**, 8 (1996).
 - [15] F. G. Wang and C.-K. Hu, *Phys. Rev. E* **56**, 2310 (1997).
 - [16] Y. Okabe, K. Kaneda, M. Kikuchi, and C.-K. Hu, *Phys. Rev. E* **59**, 1585 (1999).
 - [17] Y. Tomita, Y. Okabe, and C.-K. Hu, *Phys. Rev. E* **60**, 2716 (1999).
 - [18] N. S. Izmailian and C.-K. Hu, *Phys. Rev. Lett.* **86**, 5160 (2001).
 - [19] M. C. Wu, C.-K. Hu, and N. S. Izmailian, *Phys. Rev. E* **67**, 065103 (2003).
 - [20] N. S. Izmailian and R. Kenna, *Phys. Rev. E* **84**, 021107 (2011).
 - [21] N. Sh. Izmailian, K. B. Oganessian, M.-C. Wu, and C.-K. Hu, *Phys. Rev. E* **73**, 016128 (2006).
 - [22] M. E. Fisher, *J. Math. Phys.* **7**, 1776 (1966).
 - [23] F. Y. Wu, *Int. J. Mod. Phys. B* **20**, 5357 (2006).
 - [24] N. Sh. Izmailian, V. B. Priezzhev, P. Ruelle, and C.-K. Hu, *Phys. Rev. Lett.* **95**, 260602 (2005).
 - [25] N. Sh. Izmailian, V. B. Priezzhev, and P. Ruelle, *SIGMA* **3**, 001 (2007).
 - [26] C. S. Hsue, K. Y. Lin, and F. Y. Wu, *Phys. Rev. B* **12**, 429 (1975).
 - [27] J. E. Sacco and F. Y. Wu, *J. Phys. A: Math. Gen.* **8**, 1780 (1975).
 - [28] J. H. H. Perk, H. W. Capel, M. J. Zuilhof, and Th. J. Siskens, *Phys. A* **81**, 319 (1975).
 - [29] R. J. Baxter, *Ann. Phys. (N. Y.)* **70**, 193 (1972).
 - [30] F. Y. Wu and K. Y. Lin, *Phys. Rev. B* **12**, 419 (1975).
 - [31] P. W. Kasteleyn, *J. Math. Phys.* **4**, 287 (1963).
 - [32] H. Cohn, R. Kenyon, and J. Propp, *J. Am. Math. Soc.* **14**, 297 (2001).
 - [33] J. F. Nagle, *J. Chem. Phys.* **58**, 252 (1973).
 - [34] S. M. Bhattacharjee and J. F. Nagle, *Phys. Rev. A* **31**, 3199 (1985).
 - [35] J. F. Nagle, *Phys. Rev. Lett.* **34**, 1150 (1975).
 - [36] C. S. O. Yokoi, J. F. Nagle, and S. R. Salinas, *J. Stat. Phys.* **44**, 729 (1986).
 - [37] F. Y. Wu, *Phys. Rev.* **168**, 539 (1968).
 - [38] F. Y. Wu, *Phys. Rev. B* **3**, 3895 (1971).

- [39] P. Bak, *Rep. Prog. Phys.* **45**, 587 (1982).
- [40] S. M. Bhattacharjee, J. F. Nagle, D. A. Huse, and M. E. Fisher, *J. Stat. Phys.* **32**, 361 (1983).
- [41] M. E. Fisher, *J. Stat. Phys.* **34**, 667 (1984).
- [42] D. A. Huse and M. E. Fisher, *Phys. Rev. B* **29**, 239 (1984).
- [43] P. W. Kasteleyn, *Physica (Utrecht)* **27**, 1209 (1961).
- [44] E. W. Montroll, Lattice statistics, in *Applied Combinatorial Mathematics*, edited by E. F. Beckenbach (Wiley, New York, 1964).
- [45] T. T. Wu, *J. Math. Phys.* **3**, 1265 (1962).
- [46] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).