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Potherat, A.

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Three dimensionality in quasi-two dimensional flows: recirculations and *Barrel effects*

A. Pothérat
Applied Mathematics Research Centre, Coventry University,
*Priory Street, Coventry CV1 5FB, United Kingdom.**
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A scenario is put forward for the appearance of three-dimensionality both in quasi-2D rotating flows and quasi-2D magnetohydrodynamic (MHD) flows. We show that 3D recirculating flows and currents originate in wall boundary layers and that, unlike in ordinary hydrodynamic flows, they cannot be ignited by confinement alone. They also induce a second form of three-dimensionality with quadratic variations of velocities and current across the channel. This scenario explains both the common tendency of these flows to two-dimensionality and the mechanisms of the recirculations through a single formal analogy covering a wide class of flow including rotating and MHD flows. These trans-disciplinary effects are thus active in atmospheres, oceans or the cooling blankets of nuclear fusion reactors.

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I. INTRODUCTION

Rapidly rotating flows and electrically conducting flows in homogeneous static magnetic fields share a remarkable property: both tend to two-dimensionality. In the former, inertial waves propagation promotes invariance along the rotation axis [11]. In the latter, when magnetic field perturbations induced by flow motion are negligible (in the *quasi-static MagnetoHydroDynamic* (MHD) approximation), electric eddy currents damp velocity variations along the magnetic field lines [14].
This feature is crucial because 2D and 3D flows have radically different dissipative and transport properties, especially when turbulent. While in 3D turbulence, energy follows a direct cascade from large to small structures where it is dissipated by viscous friction, 2D turbulence proceeds through an inverse cascade that accumulates energy in large structures where it is dissipated by friction on the boundaries [6, 20]. In planar fluid layers, this mechanism is suppressed as three-dimensionality emerges [16] and its suppression or appearance are respectively determined by the presence or absence of fluid motion along the third component [5, 22]. The *a priori* distinct questions of the number of components of velocity field (2C/3C) and the of spatial directions in which it varies are thus tightly linked and determine whether flows obey 2D or 3D dynamics, how and when they may switch between them.
The fundamental difference between 2D and 3D states places these questions at the centre a vast array of problems across disciplines: atmospheric and oceanic flows, electromagnetic flow control in metallurgical processes, even the dynamo problem. Despite their importance, the transition mechanisms between 2D and 3D states, in rotating, MHD or other flows are not understood yet, and an important gap exists between theory and experiments. Numerical simulations show that with periodic or free-slip boundaries, instabilities can break down strictly 2D MHD structures and lead to 2D-3D intermittency [4, 21]. In rotating flows, non-linear transfer occurs from 3D to strictly 2D modes [17]. In experiments, shallow layers cannot be strictly 2D but only quasi-2D, either because of the three-dimensionality introduced by viscous boundary layers or because of variations of the same order in the core flow: firstly, such quasi-2D flows ignite 3D recirculations, that are crucial to the 2D-3D transition because of their influence on the inverse cascade and also because they nurture small-scale 3D turbulence [19]. Secondly, in MHD channel flows, eddy currents between the boundary layer and the core were shown to induce a quadratic three-dimensionality [10, 12], an effect recently observed experimentally, too [8]. Currently, the mechanisms that promote one or the other of these effects are unknown. Viscous boundary layers most certainly play a role in promoting 3D recirculations in decaying and steady vortical flows [15], with and without background rotation [1]. Nevertheless, in the absence of Coriolis or Lorentz force, confinement alone suffices to trigger them [3], even in the absence of boundary layer friction. These results stress the need for a clarification of the roles both of external forces such as the Lorentz or the Coriolis force on the one hand, and of confining boundaries on the other hand.
In this Letter, we examine the simple configuration of a symmetric, plane channel in a transverse field, bounded by two no-slip walls distant of $2H$ (fig. 1), to reveal how walls induce three-dimensionality in quasi-2D flows. The governing equations are written in a general form that defines a large class of flow encompassing not only quasi-static MHD and rotating flows, but also rotating-MHD flows relevant to geophysical dynamos. In doing so, a formal analogy is defined within this class of flows, which share a common tendency to two-dimensionality. A hierarchy of mechanisms is singled out that both explains how 3-Component (3C) motions are ignited and how three-dimensionality appears as a variation of physical quantities across the channel.

* alban.potherat@grenoble-inp.org

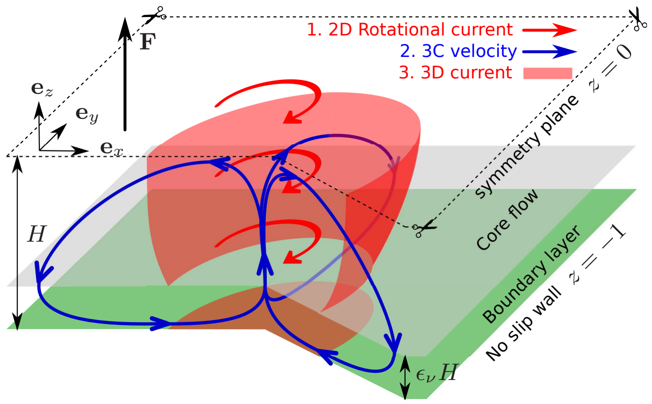

FIG. 1. Channel flow configuration. The three steps of the first Barrel effect are schematically represented. In the second type of Barrel effect, active in MHD flows, the steps are identical with the roles of the current and the velocity swapped.

II. GOVERNING EQUATIONS

Following the formalism of rotating and quasi-static MHD flows, we first define a static homogeneous field $\mathbf{F} = F\mathbf{e}_z$ (vertical for convenience), orthogonal to the channel, that pervades the fluid domain and interacts with a current $\mathbf{c}$ to exert a force $\mathbf{c}\times\mathbf{F}$ on the flow. In rotating flows (*resp.* quasi-static MHD), $\mathbf{F}$ is twice the background rotation vector (*resp.* the magnetic field) and $\mathbf{c}$ is the flow current (*resp.* electric current), that appear in the Coriolis (*resp.* Lorentz) force. The current is linked to the velocity field $\mathbf{u}$ by a phenomenological law such as Ohm's law in MHD, that takes the general form $\mathbf{c} = \mathcal{F}(\mathbf{u})$. The corresponding non-dimensional equations for an incompressible fluid (density ρ, viscosity ν) depend on two dimensionless groups, built on H, the reference velocity U and reference current C: $\epsilon = \rho U^2/(HCF)$ and $\epsilon_\nu^2 = \rho\nu U/(H^2CF)$ respectively express the ratio of inertia and viscous forces to $\mathbf{c}\times\mathbf{F}$, and both are smaller than unity. This class of analogy is summarised in table I. We focus on regimes where deviations to the leading order play a key role, and in particular those due to inertia, *i.e* typically $\epsilon \lesssim 1$ and $10^{-2} \lesssim \epsilon_\nu << 1$. Denoting any quantity g in the core (*i.e.* outside the wall boundary layers) as $\check{g}$, the equations write:

$$\epsilon\frac{\check{d}}{dt}\check{\mathbf{u}}_\perp + \nabla_\perp\check{p} = \check{\mathbf{c}}_\perp\times\mathbf{e}_z + \epsilon_\nu^2(\partial_{zz}^2+\nabla_\perp^2)\check{\mathbf{u}}_\perp, \quad (1)$$

$$\epsilon\frac{\check{d}}{dt}\check{u}_z + \partial_z\check{p} = \epsilon_\nu^2(\partial_{zz}^2+\nabla_\perp^2)\check{u}_z, \quad (2)$$

$$\partial_z\check{u}_z = -\nabla_\perp\cdot\check{\mathbf{u}}_\perp, \quad (3)$$

$$\partial_z\check{c}_z = -\nabla_\perp\cdot\check{\mathbf{c}}_\perp, \quad (4)$$

$$\check{\mathbf{c}} = \mathcal{F}(\check{\mathbf{u}}). \quad (5)$$

Vector quantities and operators are split into their components along $\mathbf{e}_z$ (subscript z) and in the (x,y) plane (subscript $\perp$). In rotating flows, $\check{\mathbf{c}}$ disappears from the governing equations, as (5) simply becomes $\check{\mathbf{c}} = \check{\mathbf{u}}$. In low-$Rm$ MHD, on the other hand, (5) represents Ohm's law:

$$\check{\mathbf{c}} = -\nabla\check{\phi} + \check{\mathbf{u}}\times\mathbf{e}_z, \quad (6)$$

where the core electric potential $\check{\phi}$ is usually obtained as solution of the Poisson equation that follows from substituting (6) into (4) [10].

III. TENDENCY TO TWO-DIMENSIONALITY IN THE CORE FLOW

We shall first clarify the interplay between the respective dynamics of $(\mathbf{c}_\perp, c_z)$ and $(\mathbf{u}_\perp, u_z)$ in the core. At the leading order $(\epsilon^0, \epsilon_\nu^0)$ when $\epsilon\to 0, \epsilon_\nu\to 0$, (1) and (2) readily imply that $\check{\mathbf{c}}_\perp$ and $\check{p}$ are independent of z (*i.e.* 2D). Denoting leading order quantities with the superscript g^0,

$$\check{\mathbf{c}}_\perp^0 = \mathbf{e}_z\times\nabla_\perp\check{p}^0, \quad (7)$$

$$\partial_z\check{p}^0 = 0. \quad (8)$$

The problem symmetry and (4) further imply that the current is exclusively horizontal:

$$\check{c}_z^0 = 0. \quad (9)$$

Two-dimensionality of the core velocity isn't ensured at this point, but depends on the nature of (5). For the time being, we shall assume that z-dependence doesn't explicitly appear in $\mathcal{F}$. In this case, $\check{\mathbf{u}}_\perp^0$ is indeed 2D. When applied to rotating and quasi-static MHD flows, this result recovers the property that both are quasi-2D in the absence of inertial and viscous effects in the core. The $z-$dependence of $\check{\mathbf{c}}_\perp$ appears from (1) and (2):

$$\partial_z\check{\mathbf{c}}_\perp = \left(\epsilon\frac{\check{d}}{dt} - \epsilon_\nu^2\nabla^2\right)[\nabla_\perp\check{u}_z - \partial_z\check{\mathbf{u}}_\perp]\times\mathbf{e}_z. \quad (10)$$

Since the leading order core velocity is 2D, $\check{\mathbf{c}}_\perp$ must in fact be 2D at least up to $\mathcal{O}(\epsilon,\epsilon_\nu^2)$, so that using (4) and the problem symmetry yields:

$$\check{c}_z = -z\check{c}_z(-1) + \mathcal{O}(\epsilon,\epsilon_\nu^2). \quad (11)$$

Similarly, the problem symmetry and (3) imply that

$$\check{u}_z = -z\check{u}_z(-1) + \mathcal{O}(\epsilon,\epsilon_\nu^2). \quad (12)$$

From (10) and (11), the appearance of $z-$ variations in $\check{\mathbf{c}}_\perp$ and $\check{c}_z$ is determined by the flow and the current injected into the core from the boundary layers that develop along the walls $\check{u}_z(-1)$ and $\check{c}_z(-1)$. Only then, can $z-$ variations in $\check{\mathbf{u}}_\perp$ appear through (5), at the same order as $\check{\mathbf{c}}_\perp$. Therefore, we now need to analyse the wall boundary layers.

	Field $\mathbf{F}$	current $\mathbf{c} = \mathcal{F}(\mathbf{u})$	ϵ	ϵ_ν	Wall boundary layer ($\hat{\mathbf{u}}_\perp^0 =$)
Quasi-static MHD	magnetic field $\mathbf{B}$	electric current $\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u}\times\mathbf{B})$	Stuart number $N = \epsilon^{-1} = \frac{\sigma B^2 H}{\rho U}$	Hartmann number $Ha = \epsilon_\nu^{-1} = BH\sqrt{\frac{\sigma}{\rho\nu}}$	Hartmann layer [9] $\check{\mathbf{u}}_\perp^0(-1)(1-e^{-\zeta})$
Rotating flows	double rotation $2\mathbf{\Omega}$	flow current $\rho\mathbf{u}$	Rossby number $Ro = \epsilon = \frac{U}{2H\Omega}$	Ekman number $E = \epsilon_\nu^2 = \frac{\nu}{2H^2\Omega}$	Ekman layer [11] $\check{\mathbf{u}}_\perp^0(-1)(1-e^{-\frac{\zeta}{\sqrt{2}}}\cos\frac{\zeta}{\sqrt{2}}) + \mathbf{e}_z\times\check{\mathbf{u}}_\perp^0(-1)e^{-\frac{\zeta}{\sqrt{2}}}\sin\frac{\zeta}{\sqrt{2}}$

TABLE I. Analogy table giving the expressions of generic dimensional quantities $\mathbf{F}$, $\mathbf{c}$, and non-dimensional numbers ϵ_ν and ϵ for quasi-static MHD and rotating flows. σ and ϕ are the electric conductivity of the fluid and the electric potential.

IV. RECIRCULATIONS DRIVEN IN THE BOUNDARY LAYER

In the vicinity of walls, horizontal viscous friction must balance $\mathbf{c}\times\mathbf{F}$ to achieve the no slip boundary condition, and this imposes the thickness of the resulting boundary layer to scale as ϵ_ν (respectively the *Ekman* and *Hartmann* layers in rotating and MHD flows). The governing equations are accordingly rewritten in the boundary layer near the wall $z=-1$ using stretched variable $\zeta = \epsilon_\nu^{-1}(z+1)$, to reflect that viscous friction becomes $\mathcal{O}(1)$. Denoting any quantity g in this region as $\hat{g}$,

$$\epsilon\frac{\hat{d}}{dt}\hat{\mathbf{u}}_\perp + \nabla_\perp\hat{p} = \hat{\mathbf{c}}_\perp\times\mathbf{e}_z + (\partial^2_{\zeta\zeta} + \epsilon_\nu^2\nabla_\perp^2)\hat{\mathbf{u}}_\perp, \quad (13)$$

$$\epsilon\frac{\hat{d}}{dt}\hat{u}_z + \partial_z\hat{p} = (\partial^2_{\zeta\zeta} + \epsilon_\nu^2\nabla_\perp^2)\hat{u}_z, \quad (14)$$

$$\partial_\zeta\hat{u}_z = -\epsilon_\nu\nabla_\perp\cdot\hat{\mathbf{u}}_\perp, \quad (15)$$

$$\partial_\zeta\hat{c}_z = -\epsilon_\nu\nabla_\perp\cdot\hat{\mathbf{c}}_\perp, \quad (16)$$

$$\hat{\mathbf{c}} = \mathcal{F}(\hat{\mathbf{u}}). \quad (17)$$

The no slip condition at the wall is written

$$\hat{\mathbf{u}}_\perp(-1) = 0, \quad (18)$$

while the boundary conditions for $\hat{\mathbf{c}}$ and $\hat{u}_z$ shall be left unspecified for the time being, thus allowing for possible wall-injection of current or mass into the fluid. Core and boundary layer variables must also satisfy a matching condition. Since they are not explicitly expanded in powers of ϵ, ϵ_ν, the general matching condition for any quantity g put forward by [7] simplifies to [13]:

$$\lim_{z\to-1}\check{g}(z+1) = \lim_{\zeta\to\infty}\hat{g}(\zeta). \quad (19)$$

$\check{c}_z(-1)$ is obtained by integrating $\nabla_\perp\times(13)$ using (16), (19), noting that $\partial_\zeta\check{\omega}_z(-1) = \epsilon_\nu\partial_z\check{\omega}_z(-1)$:

$$\check{c}_z(-1) = c_z^W + \epsilon_\nu\partial_\zeta\omega_z^W + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2), \quad (20)$$

where the superscript W refers to values taken at the wall. The current injected at the wall c_z^W would be determined by a boundary condition for the current, here unspecified. Eqs. (20) and (11) reveal the two main mechanisms that can feed eddy currents in the core: injection of current at the wall and rotational wall friction contribute respectively at the leading order, and at $\mathcal{O}(\epsilon_\nu)$. Importantly, (10) implies that although these recirculations make $\check{\mathbf{c}}$ 3C, they do not directly affect the $z-$ dependence of $\check{\mathbf{c}}_\perp(z)$, which remains 2D up to $\mathcal{O}(\epsilon, \epsilon_\nu^2)$ at least.
Strikingly, if $c_z^W \neq 0$, (20) contradicts (9): a linearly z-dependent vertical current in the core leads to diverging horizontal currents that induce a rotational force $\check{\mathbf{c}}_\perp\times\mathbf{e}_z$, which cannot be balanced by the sole leading order pressure gradient in (7). To resolve this paradox, viscous or inertial effects must exist in the core to oppose $\check{\mathbf{c}}_\perp\times\mathbf{e}_z$. These can therefore not be neglected and, in fact, prevent two-dimensionality at the leading order. Quasi-static MHD provides a well understood manifestation of this effect: MHD flows are often driven by injecting electric current through point-electrodes embedded in an otherwise electrically conducting wall [18]. Above such an electrode develops a vortex of rotation axis $\mathbf{e}_z$, with a viscous core that is 3D at the leading order. Injecting current at the wall therefore directly prevents quasi-two dimensionality, at least locally. Having now singled out this important effect, we shall assume $c_z^W = 0$, unless otherwise specified, for the reminder of this Letter and focus on higher order effects.
At order $\mathcal{O}(\epsilon_\nu)$, (20) generalises two classic properties of Ekman and Hartmann boundary layers: in rotating flows, friction in the Ekman layer gives rise to secondary flows in the core $\check{u}_z(-1) \simeq \frac{\sqrt{2}}{2}E^{1/2}\check{\omega}_z^0(-1)$, by *Ekman pumping* [11]. In quasi-static MHD, $\partial_\zeta\hat{\mathbf{u}}_\perp^W = \check{\mathbf{u}}_\perp(-1) + \mathcal{O}(\epsilon_\nu, \epsilon)$, so (20) expresses that vorticity in the core drives an electric current $\check{j}_z(-1) \simeq Ha^{-1}\omega_z^0(-1)$ out of the Hartmann layers [9] (these two results are recovered using the leading order solutions of (13-17), $\hat{\mathbf{u}}_\perp$, given in table I).

We shall now turn our attention to the determination of $\check{u}_z(-1)$, which controls $\partial_z\check{\mathbf{c}}_\perp(z)$. First, from (14) and (15), the pressure is constant across the boundary layer:

$$\nabla_\perp\hat{p}(\zeta) = \nabla_\perp\check{p}(-1) + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2). \quad (21)$$

The horizontal pressure gradient in the boundary layer thus results from the balance of forces in the core. (21)

and (13) express that each of these forces alters the local balance between viscous forces and $\hat{\mathbf{c}} \times \mathbf{e}_z$ in the boundary layer, and thereby drives horizontal jets. Should these jets be horizontally divergent, they in turn induce a vertical flow from the boundary layer to the core. The equation for $\hat{u}_z$ thus follows from $\nabla_\perp \cdot$(13) and (15), and $\check{u}_z(-1)$ is obtained by integration, using (19) and (12):

$$(1-\epsilon_\nu)\check{u}_z(-1) = u_z^W + \epsilon_\nu u_z^C + \epsilon_\nu \epsilon u_z^I + \mathcal{O}(\epsilon_\nu^3), \qquad (22)$$

$$u_z^C = \int_0^\infty \int_\zeta^\infty \int_{\zeta_2}^\infty \nabla_\perp \times [\hat{\mathbf{c}}_\perp - \check{\mathbf{c}}_\perp(-1)] \cdot \mathbf{e}_z d\zeta d\zeta_2 d\zeta_1,$$

$$u_z^I = \int_0^\infty \int_\zeta^\infty \int_{\zeta_2}^\infty \nabla_\perp \cdot \left[\frac{\check{d}}{dt}\check{\mathbf{u}}_\perp(-1) - \frac{\hat{d}}{dt}\hat{\mathbf{u}}_\perp \right] d\zeta d\zeta_2 d\zeta_1.$$

Eq. (22) singles out three possible origins of secondary flows between core and boundary layer: the flow directly injected at the wall u_z^W doesn't interact with the boundary layer and is integrally transmitted to the core where it affects the flow at the leading order. The symmetric boundary conditions at the walls imply through (12) that it creates two symmetric recirculations there. Should no flow be injected at the wall ($u_z^W = 0$), secondary flows can result from a rotational current as $\check{u}_z(-1) = \epsilon_\nu u_z^C + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2)$. Unlike wall-injected flow, the corresponding vertical flow builds up from horizontally divergent jets in the boundary layer but recirculates in the core in the same way. It leads to secondary flows at order ϵ_ν there, a scaling they inherit from the thickness of the boundary layer where they are created. Finally, if the core current is curl-free at the leading order, then $u_z^C = 0$. Inertial effects of order ϵ then take over as the dominant mechanism that drives jets in the boundary layers, resulting in a secondary flow $\check{u}_z(-1) = \epsilon\epsilon_\nu u_z^I + \mathcal{O}(\epsilon_\nu^3)$. Importantly, no recirculation occurs if $u_z^W = u_z^C = u_z^I = 0$, so in contrast to quasi-2D flows not subject to an external homogeneous field $\mathbf{F}$ [3], they are not triggered by confinement alone.

In rotating flows, $\mathbf{c} = \mathbf{u}$ and $\nabla \times \mathbf{c}_\perp = \omega_z \mathbf{e}_z \neq 0$ in general (see table I): the mechanism responsible for secondary flows is the Ekman pumping mentioned earlier when analysing eddy currents fed by $\check{c}_z(-1)$. Still, application of (22) again recovers the well-known result that $\check{u}_z(-1) = \epsilon_\nu u_z^C + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2) = \frac{\sqrt{2}}{2} E^{1/2} \check{\omega}_z(-1) + (RoE^{1/2}, E)$[11].

In quasi-static MHD, by contrast, $u_z^C = 0$ so in the inertialess theory of the Hartmann layer, no flow escapes to the core [9]. When inertia is taken into account though, (22) yields $\check{u}_z(-1) = \epsilon\epsilon_\nu u_z^I + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^3) = -(5/6)Ha^{-1}N^{-1}\nabla \cdot \left[\check{\mathbf{u}}_\perp^0(-1) \cdot \nabla \check{\mathbf{u}}_\perp^0(-1)\right]$ [12].

V. THE "BARREL" EFFECTS

Having expressed the current and mass flow that feed into the core, we are now in position to return to the core flow equations analysed at the beginning of this Letter and determine the conditions of appearance of $z-$dependence in core quantities. First, since $\check{\mathbf{c}}_\perp$ appears at a higher order than $\check{\mathbf{u}}_\perp$ in (1-3), and since we further assumed that $z-$ dependence didn't appear explicitly in $\mathcal{F}$, z-dependence cannot appear at lower order in $\check{\mathbf{u}}_\perp$ than in $\check{\mathbf{c}}_\perp$. Consequently, at the first order at which it appears, the second term in the expression of $\partial_z \check{\mathbf{c}}_\perp$ (10) vanishes. By virtue of the symmetric boundary conditions, and depending on (22), three-dimensionality may appear under either of the three forms:

$$\check{\mathbf{c}}_\perp = \check{\mathbf{c}}_\perp(0) + \epsilon \frac{z^2}{2} \frac{d}{dt} \nabla_\perp u_z^W + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2), \qquad (23)$$

$$\check{\mathbf{c}}_\perp = \check{\mathbf{c}}_\perp(0) + \epsilon\epsilon_\nu \frac{z^2}{2} \frac{d}{dt} \nabla_\perp u_z^C + \mathcal{O}(\epsilon^2\epsilon_\nu, \epsilon_\nu^3), \qquad (24)$$

$$\check{\mathbf{c}}_\perp = \check{\mathbf{c}}_\perp(0) + \epsilon^2\epsilon_\nu \frac{z^2}{2} \frac{d}{dt} \nabla_\perp u_z^I + \mathcal{O}(\epsilon_\nu^2). \qquad (25)$$

Physically, these equations express that the z-linear vertical flow created in the core by the mass flow ejected from the boundary layers builds up a z-quadratic pressure in the core. A quadratic current must in turn be drawn in the core, for the force $\check{\mathbf{c}}_\perp \times \mathbf{e}_z$ to be able to balance the corresponding quadratic component of the horizontal pressure gradient. The z-dependence of $\check{\mathbf{u}}_\perp$ is determined by the nature of $\mathcal{F}$: in rotating flows for example, $\check{\mathbf{c}}_\perp = \mathcal{F}(\check{\mathbf{u}}_\perp) = \check{\mathbf{u}}_\perp$. Consequently, $\check{\mathbf{u}}_\perp$ also depends quadratically on z, because $\check{\mathbf{c}}_\perp$ does. Core structures are thus not columnar as usually assumed in quasi-2D flows but rather *barrel*-shaped.

The Barrel effect has not yet been observed as such in rotating flows, but was discovered in the MHD numerical simulations of [10] and predicted theoretically by [12]. We shall now show that the same general mechanism is at play in both cases. To this end, we must invoke a second analogy between MHD and rotating flows and release the assumption that $\mathcal{F}$ doesn't explicitly depend on z: in MHD, the non-dimensional expression of $\mathcal{F}$ is given by Ohm's law $\mathbf{c} = \mathbf{j} = -\nabla\phi + \mathbf{u} \times \mathbf{e}_z$ (table I). At the leading order, z-dependence is still absent in $\mathcal{F}$, so all core quantities remain 2D. Application of (11) and (20) provide the expression of the vertical current there: $\check{c}_z = Ha^{-1} z \omega_z^0 + \mathcal{O}(Ha^{-2}, N^{-1})$. According to Ohm's law, this vertical electric current induces a quadratic component of the electric potential $\phi = (z^2/2) Ha^{-1} \check{\omega}_z^0 + \mathcal{O}(Ha^{-2}, N^{-1})$, which introduces an explicit z-dependence in $\mathcal{F}$ at this order. Since, however, (10) still implies that $\check{\mathbf{c}}_\perp$ must be 2D at $\mathcal{O}(Ha^{-1})$, Ohm's law demands that $\check{\mathbf{u}}_\perp$ be quadratic at $\mathcal{O}(Ha^{-1})$. This mechanism is analogous to the Barrel effect identified previously, with j_z, ϕ and Ohm's law taking over the respective roles of u_z, p and (2). The first Barrel effect was driven by vertical *flows* out of the boundary layer into the core, induced by 2D *rotational currents*. The MHD Barrel effect, by contrast, is driven by vertical *currents* out of the boundary layer, due to 2D *vorticity*. This second analogy underlines that although one common formalism explains why rotating and MHD flows are either 2D, quasi-2D or

3C, the origins of their "Barrel" three-dimensionality are still formally analogous but involve a distinct analogy.

VI. INFLUENCE OF THE BOUNDARY CONDITIONS

Up to this point, the analysis has been confined to a symmetric channel flow bounded by no-slip walls. In the quest for a flow closer to strict two-dimensionality, however, these conditions were often altered in a number of experimental and numerical studies of quasi-2D flows. We shall now examine how the driving mechanism for 3D recirculations and barrel effects are affected by these different boundary conditions in two main cases:

A. No slip bottom wall and upper free surface

The upper wall at $z = 1$ is replaced with a non-deformable free surface (*i.e.* it remains flat). This configuration was studied experimentally and numerically by [1]. The author showed in particular that free surface deformation didn't play an important role in the appearance of three-dimensionality, thus justifying the relevance of this boundary condition. Further assuming that no current escapes through the free surface, the boundary conditions at $z = 1$ become:

$$\partial_z \mathbf{u}_\perp(1) = 0 \qquad u_z(1) = 0 \qquad c_z(1) = 0. \tag{26}$$

In the channel with upper and lower no-slip walls, the problem symmetry implied $\partial_z \mathbf{u}_\perp(0) = 0$, $u_z(0) = 0$, and $c_z(0) = 0$, so the flow with a non-deformable free surface is simply that in the lower half of the channel found previously and the physical mechanisms remain the same.

B. Upper and a lower non-deformable free surfaces

This second, more ideal variant was investigated numerically with and without background rotation [1, 3]. (18) is replaced with

$$\partial_z \mathbf{u}_\perp(-1) = 0. \tag{27}$$

The direct consequence is that the viscous boundary layer at $z = -1$ is no longer present, so this boundary condition applies directly to the core flow (*i.e.* $\check{\mathbf{u}} = \mathbf{u}$ and $\check{\mathbf{c}} = \mathbf{c}$). Again, the boundary condition at $z = -1$ for u_z and c_z can be left unspecified, to allow for possible injection of mass and current through the bottom free surface (The injection can be made symmetric as for the channel flow, by relaxing the condition $c_z(1) = 0$ locally). This configuration is also symmetric with respect to the $z = 0$ plane so equations (11) and (12) for the core remain valid. From (10), injecting current at $z = -1$ therefore still leads to current recirculations and to three-dimensionality, both at the leading order, and locally prevents quasi-two dimensionality. Injecting mass also feeds recirculations at the leading order, but incurs a barrel effect at $\mathcal{O}(\epsilon)$, also from (10). If $u_z(-1) = c_z(-1) = 0$, it follows from (11) and (12) that $c_z = u_z = 0$: both vertical flows and vertical current become severed from their only source in (11) and (12), so no recirculation can occur, and the flow remains 2C. Since the barrel effects are driven by these recirculations, they are shut down too and the flow is indeed strictly 2D. Such conditions are ideal but they reveal the crucial role played by the boundary layers as sources of three-dimensionality in the presence of an external field $\mathbf{F}$. This result comes in stark contrast to similar configurations without external field where confinement alone incurs three-dimensionality [3]. In practice, [2] proposed to reduce the influence of the boundary layer by resting the active fluid layer on another one, acting as buffer. The interface between the two fluids was assumed non-deformable but the boundary condition for the bottom boundary of the active layer wasn't exactly that of a free surface. Local viscous friction still existed and a viscous boundary layer developed there, albeit incurring less lower friction than if a solid wall were present. The authors noticed reduced but persistent three-dimensionality. Our analysis shows that in the presence of an external field, the same effect is present, as reducing the friction at the bottom boundary in (20) would immediately damp both the current recirculations through (11) and the subsequent barrel effect.

VII. CONCLUSION

In this Letter, a quantitative analysis of the conditions of appearance of three-dimensionality in quasi-2D flows under the influence of an external field was conducted. It was shown that in the absence of any three-dimensional forces, three-dimensionality appears under the two forms of 3D recirculations and transverse variation of fluid quantities. We called the latter "Barrel effect" and proved that it was driven by the former. Although a second order correction in the asymptotic sense, the Barrel effect becomes crucial at the transition between 2D and 3D regimes. In turbulent flows, three-dimensionality is indeed driven by inertia and therefore appears when ϵ becomes closer to unity. This implies that three-dimensional instabilities that lead to a fully 3D state may not develop on a base flow made of columnar structures, as in strictly 2D flows, but rather on a flow with a quadratic profile that cannot be neglected and whose stability properties are currently unknown. Finally, this Letter stresses that in quasi-2D fluid layers pervaded by a uniform transverse field $\mathbf{F}$ but subject to no explicitly 3D force, 3D recirculations and flow three-dimensionality are linked and occur exclusively because of the presence of boundary layers along the confining walls. This complements the findings of [1, 3], who showed that in the same configuration, with and without background rotation, three-dimensionality could still be observed in the absence of boundary layer friction either if the forcing was itself 3D, or without background rotation, because of confinement alone.

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