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Characterizing the Causal Automorphisms of $2-d$ Minkowski space

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Abstract

We present a simple characterization of the causal automorphisms of $2-d$ Minkowski space, and relate it to the characterization provided by Kim [1]

1 Introduction

Let M be a space-time, i.e. a smooth Lorentz manifold. For $x, y \in M$ we say that x chronologically precedes y , denoted $y \in I^+(x)$ if there is a smooth, future directed, timelike curve from x to y , and that x causally precedes y , $y \in J^+(x)$, if there is a smooth, future directed, causal curve from x to y [2]. A bijection from M to itself which preserves these causal relations is called a causal automorphism. The set of causal automorphisms is clearly a group under composition, and can therefore be thought of as the symmetry group of the causal structure of M .

It has been known since 1964 that in more than two space-time dimensions the causal automorphisms of Minkowski space are generated by the inhomogeneous Lorentz group together with the dilatations [3]. However, this is not the case in two space-time dimensions. More recently, in 2009, Kim [4] showed that the group of causal automorphisms of two-dimensional Minkowski space, \mathbb{M}^2 , was infinite-dimensional, and then in 2010 Kim [1] also showed that it was characterized by two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the rather non-obvious conditions that

1. f is a homeomorphism

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2. g is continuous
3. $\sup(g \pm f) = \infty$
4. $\inf(g \pm f) = -\infty$
5. $\left| \frac{g(t+\delta t) - g(t)}{f(t+\delta t) - f(t)} \right| < 1$ for all t and δt .

A causal automorphism is specified by such a pair f, g as follows: put the usual Minkowskian coordinates (x, t) on \mathbb{M}^2 , and send the point $(x, 0)$ to $(f(x), g(x))$. The constraints on f and g ensure that this is again a Cauchy surface. By considering causally admissible sets—a subset of a Cauchy surface is causally admissible if it comprises all the points causally related to some point in space-time—Kim shows that this then determines a unique causal automorphism on \mathbb{M}^2 , and that all causal automorphisms are obtained in this way. Specifically, if (x, t) are the usual Minkowskian coordinates on \mathbb{M}^2 , then the point (x, t) is mapped to (X, T) where

$$\begin{aligned} X &= \frac{f(x+t) + g(x+t)}{2} - \frac{g(x-t) - f(x-t)}{2} \\ T &= \frac{f(x+t) + g(x+t)}{2} + \frac{g(x-t) - f(x-t)}{2} \end{aligned} \tag{1}$$

The purpose of this article is to show how the standard null coordinates on two-dimensional Minkowski space provide a more perspicuous characterization of the causal automorphisms on Minkowski space, and to investigate the relationship between the two characterizations.

2 Characterization

First, let (x, t) be the usual Minkowskian coordinates on two-dimensional Minkowski space, \mathbb{M}^2 , and define null coordinates (u, v) by $u = t + x$, $v = t - x$.

We will show that $\phi : \mathbb{M}^2 \rightarrow \mathbb{M}^2$ is a causal automorphism if and only if it is of one of the forms

$$(u, v) \mapsto (U(u), V(v)) \text{ or } (u, v) \mapsto (U(v), V(u))$$

where U and V are both order-preserving bijections.

First, we observe that an order-preserving bijection $\mathbb{R} \rightarrow \mathbb{R}$ is necessarily a homeomorphism. For it is easy to see that an order-preserving bijection is continuous, and then (by invariance of domain [5]) it is a homeomorphism.

So now we consider the case of a mapping of the first form above,

$$(u, v) \mapsto (U(u), V(v)),$$

which is the case of an orientation preserving causal automorphism of \mathbb{M}^2 . If we denote $U(u)$ and $V(v)$ by U and V respectively, then (abusing notation by identifying points in

\mathbb{M}^2 with their coordinates) we immediately see that

$$\begin{aligned}(U_1, V_1) \in I^+(U_2, V_2) &\Leftrightarrow U_1 > U_2 \wedge V_1 > V_2 \\ &\Leftrightarrow u_1 > u_2 \wedge v_1 > v_2 \\ &\Leftrightarrow (u_1, v_1) \in I^+(u_2, v_2)\end{aligned}$$

and hence the mapping is a causal automorphism.

An analogous argument establishes that mappings of the second type, which reverse the orientation of \mathbb{M}^2 , are also causal automorphisms.

Next, suppose that $\phi : \mathbb{M}^2 \rightarrow \mathbb{M}^2$ is a causal automorphism. Then ϕ is a bijection which maps Alexandrov open sets to Alexandrov open sets. Since \mathbb{M}^2 is strongly causal, the Alexandrov topology coincides with the manifold topology [6]. Hence ϕ is continuous and open, and so is a homeomorphism.

Now, since the null geodesics of \mathbb{M}^2 bound the future and past sets, the mapping ϕ must preserve them. It follows that ϕ must be of one of the forms

$$(u, v) \mapsto (U(u), V(v)) \text{ or } (u, v) \mapsto (U(v), V(u)).$$

Since ϕ is a bijection, each of U and V must be a bijection, and since ϕ preserves the causal ordering, each of U and V must be order preserving, and so must be continuous.

Thus we have a simple characterization of a causal automorphism of \mathbb{M}^2 : it is specified by a pair of order-preserving bijections on \mathbb{R} .

3 Correspondence

So we must ask, how do this U and V relate to the functions f and g specified by Kim?

We must re-express in terms of (x, t) the causal automorphism in terms of (u, v) . Again, we consider explicitly the case where orientation is preserved, which is the case where f is increasing. This yields

$$\begin{aligned}X &= \frac{U - V}{2} = \frac{1}{2} [U(t + x) - V(t - x)] \\ T &= \frac{U + V}{2} = \frac{1}{2} [U(t + x) + V(t - x)]\end{aligned}\tag{2}$$

Now recall equation (1), which tells us that the functions f and g give the causal automorphism via

$$\begin{aligned}X &= \frac{f(x + t) + g(x + t)}{2} - \frac{g(x - t) - f(x - t)}{2} \\ T &= \frac{f(x + t) + g(x + t)}{2} + \frac{g(x - t) - f(x - t)}{2}\end{aligned}$$

Comparing (1) and (2) we see that

$$\begin{aligned}g(z) + f(z) &= U(z) \\ g(z) - f(z) &= V(-z).\end{aligned}$$

so that

$$\begin{aligned}g(z) &= U(z) + V(-z) \\f(z) &= U(z) - V(-z).\end{aligned}$$

This yields

$$\frac{g(z + \delta z) - g(z)}{f(z + \delta z) - f(z)} = \frac{[U(z + \delta z) - U(z)] - [V(-z) - V(-z - \delta z)]}{[U(z + \delta z) - U(z)] + [V(-z) - V(-z - \delta z)]}.$$

We therefore see that the conditions Kim finds on f and g are equivalent to the conditions that U and V be increasing, continuous, and onto.

The case where f is decreasing is entirely analogous.

4 Conclusions

1. There is a simple characterization of causal automorphisms in terms of null coordinates.
2. Kim's conditions have a natural interpretation when the mapping they represent on \mathbb{M}^2 is represented in null coordinates, and in that framework it is easy to see why they correspond to a causal automorphism.

Comment: It is also worth observing that by considering the situation in terms of Cartesian coordinates, we can see that X and T are both given by solutions of the wave equation on \mathbb{M}^2 (at least in the case where they are sufficiently differentiable). It would be interesting to know whether there is a useful characterization of just which solutions of the wave equation give rise to causal automorphisms of \mathbb{M}^2 .

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