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Original citation:
Weigel, M. , Elci, E. M. and Fytas, N. (2016) Connectivity properties of the random-cluster model. Journal of Physics: Conference Series, volume 681 (1): 012014

URL: http://dx.doi.org/10.1088/1742-6596/681/1/012014
DOI: 10.1088/1742-6596/681/1/012014

Publisher: IOP Publishing Ltd

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2016 J. Phys.: Conf. Ser. 681 012014

(http://iopscience.iop.org/1742-6596/681/1/012014)

Journal of Physics: Conference Series **681** (2016) 012014 doi:10.1088/1742-6596/681/1/012014

Connectivity properties of the random-cluster model

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Abstract.

We investigate the connectivity properties of the random-cluster model mediated by bridge bonds that, if removed, lead to the generation of new connected components. We study numerically the density of bridges and the fragmentation kernel, i.e., the relative sizes of the generated fragments, and find that these quantities follow a scaling description. The corresponding scaling exponents are related to well known equilibrium critical exponents of the model. Using the Russo-Margulis formalism, we derive an exact relation between the expected density of bridges and the number of active edges. The same approach allows us to study the fluctuations in the numbers of bridges, thereby uncovering a new singularity in the random-cluster model as $q < 4\cos^2(\pi/\sqrt{3})$ in two dimensions. For numerical simulations of the model directly in the language of individual bonds, known as Sweeny's algorithm, the prevalence of bridges and the scaling of the sizes of clusters connected by bridges and candidate-bridges play a pivotal role. We discuss several different implementations of the necessary connectivity algorithms and assess their relative performance.

1. Introduction

The random-cluster model was introduced by Fortuin and Kasteleyn as a generalization of the much studied Ising model of ferromagnetism and the percolation model introduced in the 1950s [1]. Through its relation to the Potts model and, ultimately, as a subject of study in its own right it developed into an important generalized model particularly for the study of continuous and first-order phase transitions [2, 3]. While a significant number of exact results are available in two dimensions, and in particular for the special cases $q \to 0$, $q \to 1$ and $q = 2$, the main approach for studying the random-cluster model away from such islands of exact solutions are Markov chain Monte Carlo simulations [4]. While these are quite efficient in general, studies of systems close to points of phase transitions are affected by a dynamical loss of ergodicity: for second-order transitions this effect is known as critical slowing down. It is caused by the diverging length scale at criticality that, in turn, leads to a divergence of time scales for the decorrelation of the system in the Monte Carlo dynamics. For this case, *cluster updates* turn out to be extremely efficient at speeding up the simulation. The algorithms of Swendsen and Wang [5] (and its generalization by Chayes and Machta [6]) and the single-cluster variant by Wolff [7] are rather well-known and frequently applied, but they are not applicable for $q < 1$. On the other hand, an algorithm due to Sweeny [8] to simulate the system directly in the random-cluster language is significantly less well-known, although a potentially superior performance has been demonstrated previously [8, 9] and it is applicable for any $q > 0$. This is due to the algorithmic complication that for this

single-bond update one requires information about the large-scale connectivity structure of the current cluster configuration at each time step. Efficient implementations of such algorithms, as the ones developed here, have only recently become available.

The performance of such connectivity algorithms crucially depends on the cluster structure at criticality where such algorithms are typically applied. How strongly connected are such clusters? What happens if a random bond is removed? These questions are relevant for the understanding of the fragmentation properties of the random-cluster model. While fragmentation has been studied for bond percolation, the additional variability in the connectivity structure implied by the tunable cluster weight q allows for the modeling of a wider variety of different random structures. Using the highly efficient simulation algorithms developed in the first part, we will present a detailed study of the fragmentation properties of critical RC clusters below. The fragmentation scaling exponents λ and ϕ are determined and derived from the set of equilibrium critical exponents. Bridges are bonds that upon removal lead to a fragmentation event. Using a probabilistic approach known as Russo-Margulis formalism [3], we find that the expected density of such bonds can be rigorously related to the mean number of active (or open) edges, revealing that the number of bridge bonds is essentially an energetic quantity. The same approach also allows us study the fluctuations in the number of bridges, thereby uncovering a new singularity in the random-cluster model occurring (in two dimensions) for $q < \tilde{q} = 4\cos^2(\pi/\sqrt{3}) = 0.2315891\cdots$.

The rest of the paper is organized as follows. In Sec. 2 we introduce the random-cluster model and discuss its relation to other models of statistical mechanics. In Sec. 3 we discuss Sweeny's algorithm for simulations of the random-cluster model and compare a number of algorithmic approaches for the required connectivity checks. Section 4 is devoted to the behavior of clusters of the random-cluster under the random removal of bonds, i.e., a fragmentation process. In Sec. 5 we outline some of the analytical results on the role of bridge bonds. Finally, Sec. 6 contains our conclusions.

2. The random-cluster model

The random-cluster (RC) model [3], introduced by Fortuin and Kasteleyn in a series of seminal papers [1, 10, 11], serves as a generalization of percolation and the Ising model to a continuum of models parameterized by the cluster weight $q \geq 0$. Its partition function is

$$Z_{\mathrm{RC}} = \sum_{A \subseteq G} v^{|A|} q^{K(A)}, \tag{1}$$

where $A \subseteq E$ denotes the set activated edges on a graph $G = (V, E)$ with vertex set V and edge set E, resulting in $K(A)$ connected components, and the bond weight $v = p/(1-p)$. Hence the RC model assigns weights to (spanning) sub-graph configurations A, i.e., subsets of activated edges and the complete set of vertices, of the underlying graph G. A special limit of Eq. (1) is $q \to 1$ in which case the model reduces to uncorrelated percolation.

As shown by Fortuin and Kasteleyn [1] for integer values of q the model is equivalent to the q-state Potts model with Hamiltonian

$$\mathcal{H} = -J \sum_{e \in E} \delta_{\sigma_i, \sigma_j}, \tag{2}$$

where $e = (i, j)$ is an edge in the graph G, and $\sigma_i \in \{1, \ldots, q\}$. This, in turn, up to a constant reduces to the Ising model for the case of only two possible states, i.e., $q = 2$. For the purposes of the present discussion, we restrict ourselves to systems on finite patches of the square lattice with dimensions $L \times L$, applying periodic boundary conditions. For this case, the ordering transition of the Potts model occurs at the coupling $J/k_B T = \ln(1 + \sqrt{q})$, corresponding to the

critical bond weight $v_c = \sqrt{q}$ in (1). This transition is continuous for $q \leq 4$ and first-order for $q > 4$ [2].

3. Simulation using Sweeny's algorithm

While the construction of Swendsen and Wang [5], published in 1987, of a rejection-free update of spin clusters in connection with bond variables implied by the random-cluster representation is well known and widely used to simulate Ising, Potts and (through the cluster embedding trick) also continuous spin models [7], a simpler and more direct approach was proposed by Sweeny [8] in 1983 already. His approach is rooted in the results of Fortuin and Kasteleyn [1] in directly attempting to sample bond configurations of the RC model according to the weight

$$w_{\mathrm{RC}}(A) = q^{K(A)} v^{|A|}, \tag{3}$$

implied by the partition sum (1). For a given sub-graph or bond configuration A, the basic update operation is then given by the deletion of an occupied edge or the insertion of an unoccupied edge. As is seen from Eq. (3), the resulting transition probabilities are determined by the change $\Delta|A| = \pm 1$ in open edges and the change ΔK in connected components. While the value of $\Delta|A|$ is trivially clear depending on whether one is attempting to open or to close a given edge, the change in cluster number depends on whether the chosen edge is *pivotal* [12], i.e., whether it is a *bridge* bond that increases the number of connected components on its closing or a *candidate bridge* that amalgamates two components upon its opening. For such pivotal edges, one has $\Delta K = \pm 1$, while for *non-pivotal* edges the cluster number remains unchanged, $\Delta K = 0$. The construction and implementation of data structures supporting the efficient calculation of ΔK constitutes the intricacy of Sweeny's algorithm.

3.1. Connectivity algorithms

We implemented three main types of connectivity algorithms to facilitate the calculation of ΔK for edge moves: breadth-first searches (BFS), a union-and-find algorithm (UF), and a dynamic connectivity algorithm (DC).

3.1.1. Breadth-first search In the simplest case, one does not keep any state information relating to the connectivity of points in the graph, but checks for the pivotality of a given edge $e = (i, j)$ each time an opening or closing of e is proposed. This entails a traversal of the components connected to i and j, respectively, to decide whether these are connected through paths not crossing e. We implement such traversals in a breadth-first (BFS) manner [13], but depth-first variants lead to essentially identical runtime scaling. Regarding the typical runtime for such searches for the RC model at criticality, arguments relating to the scaling of the shortest path ℓ between points i and j and the so-called *spreading dimension* $\hat{d}$ [14] imply that the average number of sites touched by the BFS for a non-pivotal edge is [15]

$$\langle \ell^{\hat{d}} \rangle \sim L^{d_F - x_2}. \tag{4}$$

Here, d_F is the cluster fractal dimension [16], while x_2 corresponds to the two-arm exponent [17, 18]. For pivotal edges, on the other hand, a complete traversal of the encountered clusters is required to terminate, and hence the expected number of operations scales as the typical cluster size $M_2 \sim L^{\gamma/\nu}$. As $\gamma/\nu > d_F - x_2$ everywhere for the RC model, the operations on pivotal edges dominate the asymptotic runtime, see the scaling results compiled in Table 1.

An improvement of the BFS approach can be achieved through the *interleaving* of searches in the two cluster arms joined by $e = (i, j)$ [19, 20]. In this case cluster traversal for a pivotal edge can stop once the smaller cluster has been exhausted. The size of this smaller cluster is

Table 1. Asymptotic runtime scaling at criticality of the elementary operations of opening or closing of pivotal and non-pivotal edges, respectively, using sequential breadth-first search (SBFS), interleaved BFS (IBFS), union-and-find (UF) or the fully dynamic connectivity algorithm (DC) as a function of the linear system size L.

move	SBFS	IBFS	UF	DC
non-bridge insertion	$L^{d_F-x_2}$	$L^{d_F-x_2}$	const.	$\log^2 L$
bridge insertion	$L^{\gamma/\nu}$	$L^{d_F-x_2}$	const.	$\log^2 L$
non-bridge deletion	$L^{d_F-x_2}$	$L^{d_F-x_2}$	$L^{d_F-x_2}$	$\log^2 L$
bridge deletion	$L^{\gamma/\nu}$	$L^{d_F-x_2}$	$L^{\gamma/\nu}$	$\log^2 L$
dominant scaling	$L^{\gamma/\nu}$	$L^{d_F-x_2}$	$L^{\gamma/\nu}$	$\log^2 L$

argued in Ref. [20] to scale $\sim L^{d_F-x_2}$, thus leading to a uniform $L^{d_F-x_2}$ scaling of all operations for this variant of the algorithm. This behavior is indeed found in the numerical simulations [20, 15, 21].

3.2. Union-and-find

The use of suitably designed data structures can reduce the runtime of at least some of the required operations by storing and updating connectivity information continuously. If one restricts the move set to the successive opening of edges, this leads to a continuous amalgamation of clusters if pivotal edges are selected. A suitable data structure are balanced trees using path compression that have previously been used for highly efficient simulations of the uncorrelated bond percolation problem [22]. Using these, opening moves can be performed in a runtime that is practically independent of the size of the system. The closing of edges, however, requires more expensive updates to the data structure [15]. In the case of closing a non-pivotal edge, we can use interleaved BFS to traverse both clusters, resulting in a runtime scaling proportional to $L^{d_F-x_2}$. For the case of a pivotal edge, leading to fragmentation of the original cluster, a complete re-labeling of both new clusters is required, however, resulting in a total scaling of $\sim L^{\gamma/\nu}$ for this step. As Table 1 illustrates, the asymptotically dominant runtime scaling for this approach is hence $L^{\gamma/\nu}$, provided the different edge types appear all with (asymptotically) constant probability (which is indeed the case as we show below). We note that a further improvement of this scaling can be achieved by augmenting the union-and-find data structure by cluster identifier labels, thus reducing the runtime for the closing of a bridge bond to the form $L^{d_F-x_2}$ as well. Details will be presented elsewhere [23].

3.2.1. Dynamic connectivity algorithm The goal of maintaining connectivity information for a dynamically changing graph in constant or at most (poly-)logarithmic time per operation is known as *dynamic connectivity* problem in computer science [24, 25]. We use the approach suggested in Ref. [25] which is deterministic and features amortized runtimes of $\mathcal{O}(\log N)$ for connectivity queries and $\mathcal{O}(\log^2 N)$ for deletions and insertions on graphs of N nodes. The time complexity for connectivity queries depends on the underlying binary search tree used to encode the graphs. In our case we used splay trees [26] which result in the amortized bound. An identical worst-case bound holds for balanced binary search trees [25]. The algorithm represents the edge set in form of Euler tours, stored in balanced trees, thus allowing for the efficient implementation of the necessary tree manipulations. To facilitate the search for replacement edges for the case of the closing of edges, a level hierarchy of a depth that is logarithmic in the size of the graph is introduced. The edge hierarchy is dynamically adapted as edges are opened

and closed to ensure that replacement edges can always be found efficiently. A more in-depth discussion of the algorithm can be found in Ref. [21]. For simulations of the RC model, we can therefore perform each operation in a runtime asymptotically proportional to $\log^2 L$, hence clearly outperforming the other approaches at criticality, cf. Tab. 1.

3.3. Simulation results

The runtime scaling of the connectivity algorithms discussed above essentially determine the asymptotic performance of the corresponding implementations of Sweeny's algorithm[1]. The actual efficiency of a simulation algorithm, however, also depends on its abilities to efficiently decorrelate the configuration of the system. This latter property is encoded in the autocorrelation times or, more generally, the mixing time of the algorithm [27]. We considered the *integrated autocorrelation times* [28],

$$\tau_{\mathrm{int},\mathcal{O}} = \frac{1}{2} + \sum_{t=1}^{M} \rho_{\mathcal{O}}(t), \tag{5}$$

where M is the length of the time series and $\rho_{\mathcal{O}}(t)$ is the normalized autocorrelation function for the observable $\mathcal{O}$. $\tau_{\mathrm{int},\mathcal{O}}$ controls the efficiency of an algorithm in determining the number of steps it takes to create a statistically independent sample. Close to a critical point, we expect scaling of the form $\tau_{\mathrm{int},\mathcal{O}} \sim L^{z_{\mathrm{int},\mathcal{O}}}$, thus defining the dynamical critical exponent $z_{\mathrm{int},\mathcal{O}}$. The ultimate efficiency of our implementation is then gauged by studying the quantity

$$T_{\mathcal{O}} \equiv \tau_{\mathrm{int},\mathcal{O}} \bar{t}, \tag{6}$$

where $\bar{t}$ is the runtime per operation and hence $T_{\mathcal{O}}$ measures the effective time, per edge, it takes to generate a statistically independent sample. While this is clearly hardware-dependent, we expect the relative performance of different implementations to be largely unaffected by these details [15, 21].

The behavior of the autocorrelation times for the Sweeny update alone has a number of very interesting features, most prominently that of *critical speeding up*, i.e., a decrease of autocorrelation times with system size (in natural *sweep* time units), for certain magnetic quantities, such that $z_{\mathrm{int},\mathcal{O}} < 0$ there. These effects are discussed in detail in Refs. [9, 15, 21, 27]. Regarding the performance of the slowest mode which is given by the number of open edges (the energy) [27], we find slightly smaller values for the dynamical critical exponent than for the Swendsen-Wang-Chayes-Machta algorithm [5, 6] where it is applicable. The question is whether taking into account the *algorithmic* slowing down induced by the expense of frequent connectivity queries destroys this asymptotic advantage, which will be the case if $\bar{t}$ strongly increases with the linear system size L. As a look at Table 1 indicates, for our four different implementations of connectivity algorithms — sequential breadth-first search (SBFS), interleaved breadth-first search (IBFS), union-and-find (UF) and the dynamic connectivity algorithm based on Euler tours and an edge level hierarchy (DC) — the advantage is always preserved asymptotically for the DC approach with poly-logarithmic scaling, whereas for the other three approaches the answer depends on the values of $z_{\mathrm{int},\mathcal{O}}$ and γ/ν resp. $d_F - x_2$ for the specific value of q under consideration. In two dimensions, we find these values to be such that the advantage of the single-bond dynamics is destroyed for the SBFS, IBFS and UF approaches apart from, maybe, in the vicinity of the tricritical point $q = 4$.

For $q < 1$, on the other hand, Sweeny's algorithm is the only available method. In Fig. 1 we compare the runtime $\bar{t}$ of operations per edge for systems with very small $q = 0.005$ between

[1] Additional optimizations concern cases where connectivity queries are not necessary and move proposals can be accepted unconditionally, but these do not affect the asymptotic scaling [15].

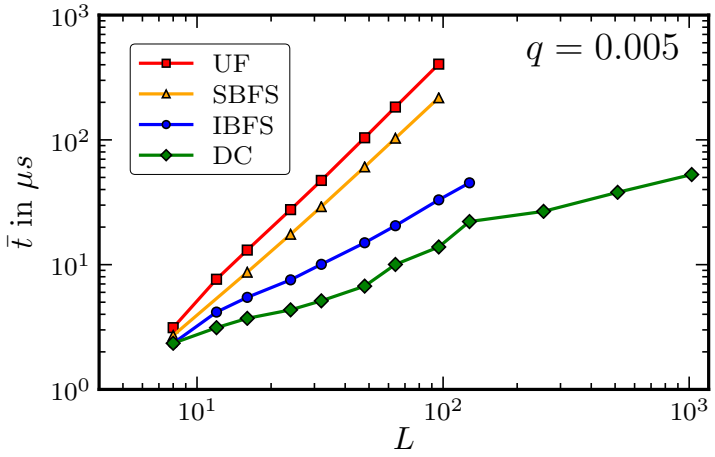

Figure 1. (Color online) Run-time per edge operation of simulations of the $q = 0.005$ square-lattice RCM and the bond algorithm employing the SBFS, IBFS, UF and DC connectivity implementations, respectively.

the four different implementations of connectivity algorithms. While the first three approaches clearly show power-law scaling (see also Table 1), we expect poly-logarithmic behavior for DC. Within the range of studied system sizes, however, this is not clearly visible. Still, the new implementation outperforms the other approaches. Further optimizations for the DC approach, in particular a reduction of the level hierarchy to only one level, further increase this advantage [27, 23].

4. Fragmentation properties

The distinction between pivotal and non-pivotal edges was found to be crucial for the asymptotic runtime scaling of most of the connectivity algorithms discussed above. The question of whether the closing (or removal) of a certain edge from the subgraph (or bond configuration) leads to the creation of a new connected component also is pivotal for the understanding of the fragmentation properties of clusters in the RC model. If one removes a single bond at the critical point $p = p_c$ this might or might not lead to a fragmentation event. The number of such *fragmenting edges* or *bridges* might depend on the cluster size and we assume the scaling form [29]

$$a(s) \sim s^{\lambda}.$$

To determine λ for the critical RC model, consider the total number of bridges,

$$\frac{\sum_s s\, n_c(s) a(s)}{\sum_s s\, n_c(s)} \sim \int s^{-\tau+1+\lambda} e^{-cs}\, \mathrm{d}s \sim L^{(\tau-\lambda)/(\sigma\nu)}, \tag{7}$$

where we have used the scaling form of the critical cluster size distribution, $n_c(s) \sim s^{-\tau} e^{-cs}$ as well as the relations $c \sim |p - p_c|^{\frac{1}{\sigma}}$ and $|p - p_c| \sim L^{-\frac{1}{\nu}}$ [16], where L is the linear dimension. Using simulations with the Sweeny and Swendsen-Wang-Chayes-Machta algorithms, we studied the density of bridges of the RC model on the square lattice numerically [30]. From the results, summarized in Fig. 2, it is clear that the critical density of bridges is asymptotically non-vanishing. Hence the average number of bridges in (7) must grow as L^d, implying $d = (\tau-\lambda)/\sigma\nu$.

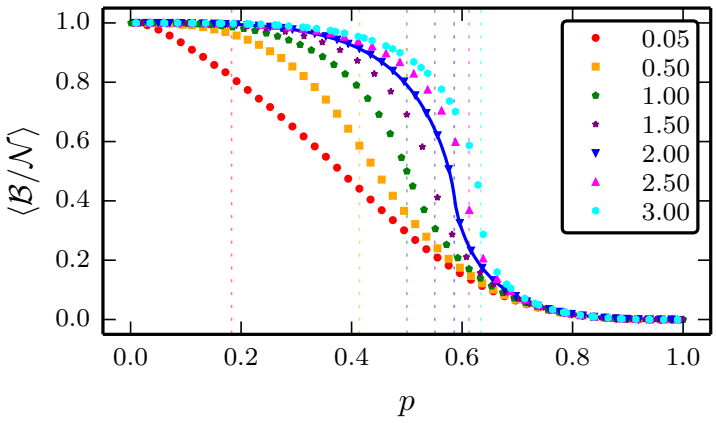

Figure 2. Proportion of bridges among active edges in the random-cluster model for different values of q. Simulation data are for systems sizes $L = 64$ ($q \neq 1$) and $L = 2048$ ($q = 1$), respectively. The solid line denotes the exact result for $q = 2$ and $L \to \infty$. The vertical dashed lines specify the location of the critical point.

With the exponent identities $\sigma\nu = 1/d_F$ and $\tau = 1+d/d_F$, where d_F is the critical cluster fractal dimension, this shows that

$$\lambda = 1, \tag{8}$$

independent of q. This generalizes numerical results previously found for the percolation model in two and higher dimensions [31, 32].

If a fragmentation event occurs, it is natural to ask for the relative sizes of the two fragments created. This is encoded in the probability $b_{s',s}$ (the *fragmentation kernel*) of creating a fragment of size s' on fragmenting a cluster of size s. The scale-free nature of the critical RC model suggests a large-s scaling form

$$b_{s',s} \sim s^{-\phi} \mathcal{G}\left(\frac{s'}{s}, \frac{s}{L^{d_F}}\right), \tag{9}$$

which is compatible with exact results for percolation in 1D and on the Bethe lattice [33]. To connect ϕ to previously established critical exponents, we multiply Eq. (9) by s' and then integrate to find that $\mu_s \sim s^{2-\phi}\mathcal{H}(s/L^{d_F})$. From using a finite-size scaling form of the overall cluster-size distribution [20] we conclude that the scaling of the ensemble average daughter cluster size is $\langle s' \rangle \sim L^{d_F(3-d/d_F-\phi)}$. On the other hand, one can show [27] that this is proportional to the average of $C_{\min,2}$, the size of the smaller of the two clusters attached to two neighboring disconnected vertices [20, 15]. In Ref. [20] it was shown that $\langle C_{\min,2} \rangle \sim L^{d_F-x_2}$, which implies

$$\phi = 2 + (x_2 - d)/d_F = 2 - d_R/d_F, \tag{10}$$

where $d_R = d - x_2$ is the red-bond fractal dimension. This is consistent with previous results for bond percolation [31, 34, 32]. However, the scaling form suggested there, $\phi = 2 - 1/(\nu d_F)$, is only correct for percolation itself as $d_R = 1/\nu$ in this case [35]. For general values of q, on the other hand, only the form (10) gives the correct result. We also studied the fragmentation kernel using numerical simulations of the square-lattice RC model [30]. These data are summarized in Fig. 3 (left panel), demonstrating the validity of the scaling form of Eq. (9), showing an excellent collapse of data for different cluster and system sizes onto scaling functions parametrized by q. As is illustrated in Fig. 3 (right panel), our estimates for the fragmentation exponent ϕ are in

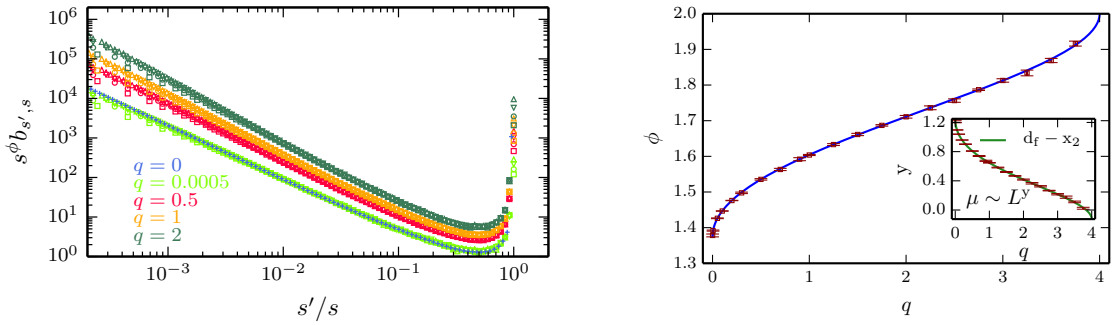

Figure 3. Left: Re-scaled conditional fragmentation probability $b_{s',s}$ according to Eq. (9) for different values of the cluster coupling q. Right: Scaling exponent ϕ of daughter clusters in the fragmentation of the square-lattice RC model as compared to the exact result (10). The inset shows the scaling exponent of the ensemble average daughter cluster size $\langle s' \rangle$.

excellent agreement with the form (10) proposed above if we substitute the exact values known from Coulomb gas arguments for d_R and d_F.

5. The role of bridges

Our arguments for the value $\lambda = 1$ of the exponent governing the density of fragmenting bonds were based on the numerical observation of a value $\langle \mathcal{B}/\mathcal{N} \rangle$ of the expected relative density of bridge bonds among all open edges that is strictly greater than zero and less than one. It turns out that this property follows as a corollary from a much more general property of bridge bonds in the random-cluster model [12].

This property can be derived starting from a well-known identity in percolation theory known as Russo-Margulis formula [36],

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}_{p,G}[X] = \sum_{e \in E} \mathbb{E}_{p,G}[\delta_e X], \tag{11}$$

where $X(A) : \Omega \to \mathbb{R}$ is an arbitrary observable and the quantity $\delta_e X$ is called the *influence* of e on X and is given by

$$(\delta_e X)(A) \equiv X(A^e) - X(A_e),$$

where A is a given bond configuration and A^e refers to A modified by opening e while A_e correspond to A with edge e closed. Here, $\mathbb{E}_{p,G}[X]$ refers to an expectation with respect to the probability measure of uncorrelated bond percolation on a graph G. Applying this formalism to the cluster-number observable K, for which $K(A) - K(A_e) = -1$ if e is a bridge, $e \in B(A)$, and 0 otherwise, it is possible to show that [12]

$$\mathbb{E}_{p,q,G}[\mathcal{B}] = \frac{\mathbb{E}_{p,q,G}[\mathcal{N}] - p}{(1-p)(1-q)}, \tag{12}$$

which we refer to as *bridge-edge formula*. Here, $\mathbb{E}_{p,q,G}[X]$ refers to the expectation of X with respect to the random-cluster weight (3). This is a general relation, valid for any graph and any value of $q \neq 1$ (the case of $q \to 1$ is discussed in Ref. [12]). Since we know that for $0 < p < 1$ the edge density $\mathbb{E}_{p,q,G}[\mathcal{N}]$ satisfies $0 < \mathbb{E}_{p,q,G}[\mathcal{N}] < 1$, it follows that $\lambda = 1$ exactly for all graphs and for critical as well as off-critical conditions.

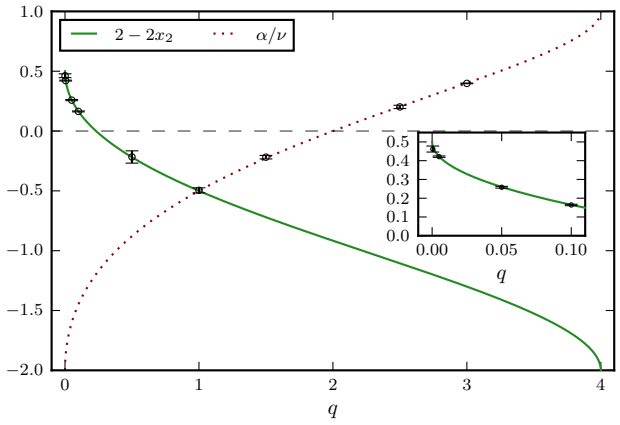

Figure 4. The exponents α/ν and $2-2x_2$ appearing in the system size scaling of $\mathrm{Var}_q[|B|]/|E|$ at criticality. The solid and dotted lines show the exact value of $2-2x_2$ and α/ν, respectively, following from the Coulomb gas mapping. The symbols denote our numerical estimates from fitting to the variance of the bridge density, cf. Ref. [12].

As the number of open edges in the RC model is an energy-like quantity, through Eq. (12) the same holds true for the density of bridges. Exact results for the energy and hence the density of open edges at the self-dual, critical point [2] allow us to exactly find the critical density of bridges,

$$\mathbb{E}_{p_{\mathrm{sd}}(q),q,\mathbb{Z}^2}[\mathcal{B}] = \frac{1}{2(1+\sqrt{q})}. \tag{13}$$

From the same connection to the internal energy of the Potts model, one may also deduce the finite-size corrections to the asymptotic result (13), and one finds that the dominant term is proportional to $L^{1/\nu-d}$ [12]. Similarly, it is possible to derive analogous relations for other types of edges, i.e., non-bridges as well as candidate bridges and candidate non-bridges. Details can be found in Ref. [12].

It turns out that the Russo-Margulis formalism allows for even more advanced investigations. Following along similar lines as for the derivation of the bridge-edge formula (12) one can investigate the *fluctuations* in the number of bridge bonds. The variance $\mathrm{Var}_{p,q,G}[|B|]/|E|$ is found to be exactly given by

$$\frac{\mathbb{E}_{p,q,G}[\mathcal{N}]\,(2p-1)-p^2+\mathrm{Var}_{p,q,G}[N]/|E|}{(1-q)^2(1-p)^2}+\mathbb{E}_{p,q,G}[\mathcal{B}]+\frac{1}{1-q}\frac{1}{|E|}\sum_{e\neq f\in E}\mathbb{P}_{p,q,G}[e\Leftrightarrow f], \tag{14}$$

where we write $e \overset{A}{\Leftrightarrow} f$ for the event that both e and f are non-bridges in the same cycle such that removing e will cast f into a bridge. The first term in Eq. (14) is essentially the fluctuation in the number of open edges which, in turn, corresponds to the internal energy. Hence we expect asymptotic critical scaling proportional to $L^{\alpha/\nu}$. The last term, on the other hand, can be shown to lead to scaling proportional to L^{d-2x_2} [37, 12]. The values of these exponents are illustrated for the square-lattice model in Fig.4. In two dimensions we hence find that the term proportional to $L^{\alpha/\nu}$ dominates for $q>1$ and, as is well known, becomes positive for $q>2$. For $q<1$, on the other hand, the new term proportional to L^{2-2x_2} dominates and, in particular, leads to a new divergence for $2-2x_2>0$, which occurs for $q<\tilde{q}=4\cos^2(\pi/\sqrt{3})=0.2315891\cdots$.

6. Conclusions

We have presented a general implementation of Sweeny's algorithm for the random cluster model that works independent of the structure and dimensionality of the lattice and features, at least asymptotically, an improved efficiency in decorrelating the system as compared to the better known Swendsen-Wang-Chayes-Machta algorithm. In addition, it can be applied to the regime $q < 1$ which is inaccessible to all other known simulation approaches. It would be interesting to study the dynamic behavior of the algorithm in more detail, in particular for the random-cluster model in three dimensions.

In the course of developing and comparing different implementations of the required connectivity checks we discovered the crucial importance of bridge bonds or singly connected bonds for the dynamic as well as the static behavior of the RC model. Fragmentation processes can be understood from studying the bridge bonds and the properties of the clusters connected by bridges. Studying fragmentation we found that the density of bridges is asymptotically independent of cluster size, thus rendering the fragmentation probability upon random bond removal directly proportional to the size of clusters. We investigated the fragmentation kernel, i.e., the conditional probability of a fragmentation event on a cluster of size s to generate a daughter cluster of size s'. This is found to be highly asymmetric, i.e., an fragmentation event is very unlikely to yield fragments of comparable size, in contrast to recent claims in Ref. [38]. This property also explains our recent observation of the fact that, under suitable initial conditions, the equilibrium fragmentation exponent ϕ also governs the final fragment size distribution in a non-equilibrium process [30, 27].

Using the Russo-Margulis formalism of percolation theory, we were able to relate the expected density of bridges to the density of open edges. Some of the manifold consequences of this observation are discussed in Ref. [12]. Additionally, the study of fluctuations in the number of bridges allowed us to uncover a previously unknown singularity in the random-cluster model occurring, in two dimensions, for $q < 4\cos^2(\pi/\sqrt{3})$. Removing *all* bridges from critical RC model configurations leads to a new ensemble of structures and the sizes of the resulting fragments are found to scale according to the backbone critical exponent of the model [12]. This promises an efficient new approach for determining this otherwise hardly accessible critical exponent, especially for the case of the random-cluster model in three dimensions.

Acknowledgments

The authors acknowledge funding from the EC FP7 Programme (PIRSES-GA-2013-612707).

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