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Krylov Subspace Model Order Reduction for Nonlinear and Bilinear Control Systems

Agbaje, Oluwaleke

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Krylov Subspace Model Order Reduction for Nonlinear and Bilinear Control Systems

By Oluwaleke Abimbola Agbaje

BEng-Industrial-and-Production-Engineering-MSc-Control-Engineering-

September-2016-



A-thesis-submitted-in-partial-fulfilment-of-the-University'srequirements-for-the-degree-of-Doctor-of-Philosophyii

Abstract

The use of Krylov-subspace-model-order-reduction-for-nonlinear/bilinear-systems,over-the-past-few-years,-has-become-an-increasingly-researched-area-of-study.-Theneed-for-model-order-reduction-has-never-been-higher,-as-faster-computations-forcontrol,-diagnosis-and-prognosis-have-never-been-higher-to-achieve-better-systemperformance.- Krylov-subspace-model-order-reduction-techniques-enable-this-tobe-done-more-quickly-and-efficiently-than-what-can-be-achieved-at-present.-

The-most-recent-advances-in-the-use-of-Krylov-subspaces-for-reducing-bilinearmodels-match-moments-and-multimoments-at-some-expansion-points-which-haveto- be- obtained- through- an- optimisation- scheme. This- therefore- removes- thecomputational-advantage-of-the-Krylov-subspace-techniques-implemented-at-anexpansion-point-zero.-

This-thesis-demonstrates-two-improved-approaches-for-the-use-of-one-sided-Krylov-subspace-projection-for-reducing-bilinear-models-at-the-expansion-pointzero.- This-work-proposes-that-an-alternate-linear-approximation-can-be-usedfor-model-order-reduction.- The-advantages-of-using-this-approach-are-improvedinput-output-preservation-at-a-simulation-cost-similar-to-some-earlier-worksand-reduction-of-bilinear-systems-models-which-have-singular-state-transitionmatrices.-

The comparison of the proposed methods and other original works done in this area of research is illustrated using various examples of single input single output (SISO) and multi-input multi-output (MIMO) models.

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Firstly, I-would-like-to-express-my-gratitude-towards-my-former-director-of-studiesand-supervisor-Prof-Keith-J.-Burnham.-Also-I-would-like-to-thanks-my-supervisors-and-current-director-of-studies-Dr-Olivier-Haas, Dr-Malgorzata-Sumisławska,-Dr-Othman-Maganga-and-Dr-Dina-Laila-for-their-support-over-all-the-years-ofmy-study.-

I-would-also-like-to-acknowledge-all-the-staff-of-the-Control-Theory-and-Applications-Centre-(CTAC)-who-have-taught-me-over-the-years-and-have-inspired-me-to-work-hard-to-achieve-my-goals.-

Dedication

This-work-is-dedicated-to-my-family.- My-dad-Mr-Festus-Oni-Agbaje,-my-mum-Deaconess-Oluyemi-Agbaje.- Also-to-my-siblings-Mrs-Opeyemi-Ovabore,-Olumide-Babafemi-Agbaje-and-Eyitayo-Omoniyi-Agbaje.-

 $\label{eq:I-would-also-like-to-dedicate-this-work-to-God-Almighty, who-is-a-constant-in-my-life.$

Glossary of nomenclature

Abbreviations

BIBO Bounded Input Bounded Output BIRKA-.... Bilinear-Iterative-Rational-Krylov-Algorithm-BT- Balanced-Truncation-FA-.... Firefly-Algorithm-FB- Feng-and-Benner-HOBM- Higher-Order-Bilinear-Model-IAE- Integral-of-Absolute-Error-IP-.... Improved-Phillips-IRKA-.... Iterative-Rational-Krylov-Algorithm-KSBA-.... Krylov-Subspace-Bilinear-Approximation-MIMO Multiple Input Multiple Output MOR-Model-Order-Reduction-MSE-.... Mean-Square-Error-NIAE-.... Integral-of-Absolute-Error-Divided-by-Number-of-Samples-P- Phillips-PID- Proportional, Integral, Differential-PLA- Parametrised-Linear-Approximation-POD- Proper-Orthogonal-Decomposition-PSO- Particle-Swarm-Optimisation-

- QA-.... Quadratic-Approximation-
- RC-....Resistance/Capacitance-
- ROM Reduced Order Model
- RSS- Residual-Sum-of-Squares-
- SE-.... Squared-Error-
- SPARK Stability Preserving Adaptive Rational Krylov
- SISO- Single-Input-Single-Output-
- SPA- Singular-Perturbation-Approximation-
- SPM-..... Solar-Panel-Model-
- SQP Sequential-Quadratic Optimisation-
- SSE- Sum-of-Square-of-Error-
- SSR-..... Sum-of-Squared-Residuals-
- ST-.... Simulation-Time-
- SVD-..... Singular-Value-Decomposition-
- TBIRKA-.... Truncated-Bilinear-Iterative-Rational-Krylov-Algorithm-
- TBR-.... Truncated-Balanced-Realisation-
- TPWL-.... Trajectory-Piece-wise-Linear-

Notation

a_i	i^{th} coefficient of a polynomial series
c_i	i^{th} member of a set of constants
f(x)/f	Nonlinear function
g/g(v)	Nonlinear resistor
g(x)	Nonlinear input function
h	Nonzero members of a Hessenberg matrix which is formed during the
	Arnoldi process
$h_n(\sigma_1,\ldots,\sigma_n)$.	Impulse response also known as a kernel of a Volterra series
<i>m</i>	Number of columns of input matrix/Number of columns of a second starting $% \mathcal{N} = $
	matrix/Number of bilinear state matrices (bilinear terms)
m(l)	Moment
$m(l_1, l_2, \ldots l_i)$.	Multimoment
$\hat{m}(l)$	Moment of a reduced order model
$\hat{m}(l_1, l_2, \dots l_i)$.	Multimoment of a reduced order model
<i>n</i>	State space dimension/Number of states
<i>ns</i>	Number of simulations
<i>ns</i>	Number of samples
<i>p</i>	Number of columns of output matrix
p_2	Parameter which determines the amount of columns of $V^{\{1\}}$ are used for
	computing $V^{\{2\}}$
$p_{(i)}$	Appropriate parameters for achieving orthogonality
q_i/q	Dimension of a Krylov subspace/Projection matrix algorithm parameter for $% \mathcal{A}$
	computing $V^{\{i\}}$ /Columns of the matrix Q
r/r_i	Real number/Columns of the matrix ${\cal R}$
$r_{(i)}$	Appropriate parameters for achieving orthogonality
r_{ij}	Members of the matrix ${\cal R}$ when performing the QR factorisation
<i>s</i>	Continuous time variable
<i>s</i> _{<i>i</i>}	i^{th} continuous time variable
\bar{s}_i	Member of a subspace $\mathbb S$
<i>t</i>	Time
<i>u</i>	System input
u_i	i^{th} system input

Voltage
column vectors which are member of $\mathbb V$ or $V/\mathrm{Voltage}$ at node i
Weights for weighted combination using Trajectory piece-wise linear model
order reduction
State vector
Reduced order state vector
State derivative
Reduced order derivative of system state
Expansion point for trajectory piecewise approximation
Kronecker product of the state up to the i^{th} term
Derivative of $x^{(i)}$
State vector of the Carleman bilinearisation process
State vector derivative of the Carleman bilinearisation process
System output
i^{th} System output
System output of reduced order model
System matrix
Reduced order system matrix
Reduced order linear approximations of a nonlinear function at multiple
points
i^{th} Derivative of a nonlinear function
Resulting system matrix of Carleman bilinearisation process
Member of A
State transition matrix of a linear approximation of bilinear system for a
constant input
Input matrix
Input matrix for parametrised linear approximation of a bilinear model
Reduced order input matrix
Resulting input matrix of Carleman bilinearisation process
i^{th} derivative of the nonlinear function $B(x)$
Member of N
output matrix
Reduced order output matrix
Resulting output matrix of carleman bilinearisation process

G_i	i^{th} derivative of $g(x)$
H(s)	Transfer function
$\hat{H}(s)$	Transfer function of reduced order model
<i>I</i>	Identity matrix of appropriate dimensions
I_m	Identity matrix of dimensions
K_q	q^{th} Krylov subspace
₪	Second starting matrix of a Krylov subspace
<i>N</i>	Bilinear state matrix for SISO bilinear system/model
ℕ	First starting matrix of a Krylov subspace
\hat{N}_i	i^{th} reduced order bilinear state matrix
N_i	i^{th} bilinear state matrix
<i>N</i>	Resulting bilinear system state matrix of carleman bilinearisation process
<i>P</i>	Observability Gramian
\bar{Q}	Controllability Gramian
<i>Q</i>	Set of vectors which form part of the QR factorrisation process
<i>R</i>	A matrix with nonzero members which form part of the QR factorisation /
	Residual
\mathbb{R}	Set of real numbers
S	Subspace
U(s)	Laplace transform of the input of a linear system
<i>V</i>	Right projection matrix
$V^{\{k\}}$	Basis of the kth Krylov subspace used to compute the projection matrix V
	for MOR of a bilinear model
\mathbb{V}	A set of linearly independent vectors in a subspace $\mathbb S$
W	Projection matrix formed using second Krylov subspace for two sided pro-
	jection
\bar{W}	Quadratic function of states
X(s)	Laplace transform of the state of a linear system
Y(s)	Laplace transform of the output of a linear system
\mathbb{Z}	Set of positive integers
∞	Infinity
η/η_i	Linear approximation parameter/Set of linear approximation parameters
τ	A small interval of time
η	Linear approximation parameter

 $\begin{array}{lll} \delta_i & \dots & \text{Expansion points} \\ \pi & \dots & p \mathbf{i} \\ \lambda_i(A) & \dots & i^{th} \text{ eigenvalue of } A \end{array}$

 $The above notation stands except when stated otherwise within the text of this thesis. \label{eq:constant}$

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	imation

Chapter 1

Introduction

Models- in-their-most-fundamental-form-are-considered-approximations-of-thereal-world.- These-can-be-abstract-or-physical-in-form.- Models-can-be-elementsor-amalgamations-of-elements-to-describe-a-system-or-a-process-describing-aphenomenon-under-scrutiny.- Models-can-be-classified-broadly-as-linear-andnonlinear-but-the-vast-majority-of-systems-are-nonlinear-in-nature.-

The increasing complexity of systems in automotive/aeronautic vehicles, manufacturing and energy installations has led to the requirement of sophisticated management systems for control, condition monitoring, diagnostic and prognostic purposes especially for systems which require safety and economic viability. Not only is it important to accurately control these systems to deliver the expected performance but also identify and diagnose system faults to prevent avoidable breakdown scenarios. In addition, the need for increased reliability has put more emphasis on estimation of failure modes, state of health and remaining useful life. Therefore models for control, diagnostics and prognostics are now more than ever essential in order to meet these requirements. However, models for control diagnostics and prognosis are required to be computation ally efficient for online implementation whilst retaining enough characteristics to describe the systems behaviour. This has led to the need for model order reduction-techniques-that-allow-the-resulting-models-to-run-online-and-also-containsufficient-accuracy-for-the-purpose-of-control,-diagnostics-and-prognostics.

There-is-no-fixed-method-or-rule-for-model-order-reduction-for-a-specific-system-but-rather-several-options-or-paths-that-can-be-taken.- The-methodologiesused-often-depend-on-the-particular-application-and-availability-of-a-priori-information.- For-instance, one-can-derive-a-reduced-order-model-from-an-existing-highfidelity/high-order-model-using-mathematical-manipulation-based-approaches-orin-the-case-where-there-is-no-first-principle-model-available, one-can-identify-amodel-from-data-driven-approaches.- In-addition-to-the-possibility-of-differentapproaches, there-are-also-the-questions-of-how-accurate-the-model-needs-to-be,how-it-will-be-implemented-online-and-what-structure-it-needs-to-have.-

Classical-methods-for-model-order-reduction-(MOR)-are-based-on-mathematical-manipulation-of-the-higher-order-model.- These-methods-are-used-to-project-ahigh-dimensional-(high-fidelity)-model-to-a-low-dimensional-(low-fidelity)-model,while-preserving-necessary-dynamics-and-reasonable-accuracy-of-the-original-system.-There-are-different-approaches-of-obtaining-reduced-order-model-(ROM)-viamathematical-manipulation-such-as-Krylov-subspace-based,-truncation-based-andmethods-based-on-proper-orthogonal-decomposition-(POD).-MOR-techniques-by-Arnoldi (Arnoldi 1951), Lanczos (Lanczos 1950) and Moore (Moore 1981) are some of the original methods proposed for linear time-invariant models. Modelorder-reduction-techniques-for-linear-systems-can-be-classified-into-two-categories:moment-matching-and-Singular-Value-Decomposition-(SVD)-based-approaches. They-include-Balanced-Truncation, Krylov-subspace-moment-matching-methods, H2-norm-MOR-(Gugercin, Antoulas & Beattie 2008) and singular perturbationapproximation (SPA). These methods have been well researched, with variousextensions-as-documented-in-(Tan-&-He-2007,-Liu-&-Anderson-1989,-Kumar,-Tiwari-&-Nagar-2011,-Lohmann-&-Salimbahrami-2000).-MOR-for-nonlinear-systems-is-still-a-relatively-open-area-of-research.- Most-of-the-methods-developedfor-linear-systems-have-been-extended-to-nonlinear-systems. The approaches-most-popular-today-have-been-proposed-for-weakly-nonlinear-systems. These-approaches-include-quadratic (Chen, White-et-al. 2000), piecewise-linear-MOR-(Bond-&-Daniel-2007, Rewieński-&-White-2003)- and Krylov-subspace-based-MOR-for-nonlinear-systems-via-Carleman-bilinerarisation (Phillips-2000). The advantages-of-energy-function-based-approaches-for-reduction-of-linear-models, such-as-Balanced-Truncation-and H_2 -norm-approaches, are-not-easily-transferred-to-nonlinear-cases.

In-systems-engineering, -control-engineering-and-automotive-applications,-research-into-reduced-order-models-and-methods-for-achieving-them-are-increasingly-popular. The-main-objective-for-reduced-order-modelling-is-to-preserve-theinput-output-characteristics-of-a-higher-order-model. There-has-not-been-muchemphasis-placed-on-the-amount-of-effort-required-to-achieve-this. At-the-endof-the-day,-in-most-cases-it-is-not-of-much-concern. However,-this-is-a-problemwhich-has-been-raised-in-certain-literature-(Aizad,-Sumisławska,-Maganga,-Agbaje,-Phillip-&-Burnham-2014,-Baur,-Benner-&-Feng-2014). In-an-ideal-case,-itwill-be-useful-to-achieve-reduced-order-models-at-minimum-cost-whilst-achievingan-optimum-input-output-criteria.-

Most-of-the-linear-MOR-methods-discussed-so-far-are-readily-applicable-tobilinear-models.-Unfortunately-the-peculiar-disadvantages-which-apply-to-linearcases-are-also-carried-over-to-bilinear-cases.-Krylov-subspace-techniques-are-often-preferred-for-computational-efficiency-and-their-ability-to-compute-very-largematrices.-Balanced-Truncation,-unlike-Krylov-subspace-MOR-is-not-suitable-formodels-with-very-large-system-matrices.-It-is-therefore-intuitive-to-find-a-way-ofexploiting-the-advantages.- The-most-recent-advances-in-the-use-of-Krylov-subspaces-for-reducing-bilinear-models,-match-moments-and-multimoments-at-somefrequencies-which-have-to-be-obtained-iteratively-(Choudhary-&-Ahuja-2016,-Breiten-&-Damm-2010,-Benner-&-Breiten-2012*a*,-Benner-&-Breiten-2012*a*).-However,- in- (Breiten-&- Damm-2010), this-approach-has-been-said-to-require-improvement. Hence-the-iterative-nature-of-finding-the-expansion-points. In- (Benner-&-Breiten-2012*a*), the Bilinear-Iterative-Rational-Krylov-Algorithm-(BIRKA)-hasbeen-proposed. The algorithm-solves-the-interpolation-problem-in-an-optimalway-i.e. a-search-of-expansion-points-that-match-a-suitable-tolerance. This-therefore-removes-the-computational-advantage-of-the-Krylov-subspace-techniques-asimplemented-by-(Phillips-2000, Feng-&-Benner-2007, Condon-&-Ivanov-2007, Bai-&-Skoogh-2006), especially, when-solving-very-large-matrices. Other-variants-ofthe-BIRKA-such-as-the-Truncated-Bilinear-Iterative-Rational-Krylov-Algorithm-(TBIRKA)-(Benner-&-Breiten-2012*a*)-also-have-these-limitations.

However, -it-is-interesting-to-note-that-a-combination-of-methods-as-has-beendiscussed-in-(Tan-&-He-2007)-brings-about-more-possibilities-for-reduced-ordermodelling-practitioners.-This-has-also-been-discussed-in-(Benner-&-Damm-2011)and-provide-good-prospects-for-the-future.- For-most-work-done-on-hybrid-approaches,-the-TBR-approaches-are-combined-with-Krylov-subspace-MOR-techniques.- Other-aspects-of-hybrid-approaches-are-the-combination-of-data-basedapproaches-with-classical-approaches-(Saragih-2014).-

In-this-thesis, the focus-is-on-the-reduced-order-modelling-for-bilinear-systemsutilizing-Krylov-subspace-multimoment-matching-methods-which-match-multimoments-at-an-expansion-point-of-zero.- Bilinear-form-is-a-subset-of-nonlinearmodels-and-is-a-good-approximation-of-nonlinear-behaviour-(Rugh-1981,-Phillips-2000).-

1.1 Motivation and problem statement

The motivation of this work is inspired by a series of issues which have been raised over the years. In the work done by (Phillips 2000) and (Feng & Benner 2007), two one-sided projection techniques have been proposed to match the moments and multimoments bilinear models. Other authors (Breiten & Damm-

1. Introduction

2010, Wang & Jiang 2013) have proposed a two-sided technique for improving the input-output relationship of reduced models. This implies that two Krylov subspaces are used therefore utilising twice the effort for a one-sided approach. In (Bai & Skoogh 2006), a matrix inversion approach was proposed to match more moments of a bilinear model. This was shown to produce a better input-output relationship when compared to the approach presented in (Phillips 2000). However, the matrix inversion produces an awkward projection which cannot be regarded as one-sided approach and adds the need for more computational effort. Also, due to the computation of Krylov subspaces in (Feng & Benner 2007), there is often a multiplication of system matrices with singular bilinear state matrices and this could lead to loss of information and effectiveness of these techniques.

Matrix-inversions-are-not-always-possible-and-additionally,-it-is-done-at-anextra-computational-effort.- Also,-projection-of-matrix-dimensions-is-done-insuch-a-way-that-each-moment-and-multimoment-is-matched-at-a-point-in-theprojection-subspace-and-because-this-needs-to-match-multimoments,-it-presentsa-lack- of-flexibilty- in- the-application- and- therefore- quality- of- reduced- ordermodels.- The-aim-of-this-work- is- therefore- to- propose- two- techniques- whichare-computationally-efficient,-promote-flexibility- and-also-enable-the-reductionof- models- with- noninvertible-system- matrices.- This- will-improve- the-inputoutput-characteristics-of-the-reduced-order-models-and-expand-the-scope-of-itsimplementation.-

1.2 Methodology and thesis outline

1.2.1 Methodology

The thesis will be based on the original works done in this field and a mathematical analysis of those original works. Based on the analysis of these methods, new approaches are proposed. Also, the algorithms to be used will be similar in order



Figure-1.1:-Schematic-flow-of-thesis-progression-

to-highlight-the-salient-differences-for-comparison. An experiment-which-definesa-problem, in-this-case-a-higher-order-model, is-to-be-used-for-each-method-reviewed-or-proposed. Input-and-output-data-are-collected-for-each-simulation-andeach-method-is-analysed-with-a-predefined-performance-criteria. Also-simulationtime-is-of-interest-in-each-case-study.

The chapters progress in such a way that each method is first applied to a single input single output (SISO) model, followed by their extension to multiple input multiple output (MIMO) models. The proposed methods are then applied in special cases which show the significance of the work done in this thesis.

1.2.2 Thesis outline

The-thesis-outline-is-as-follows:-

Chapter 2: This-chapter forms-the-background-from-which-all-the-conceptsproposed-in-this-study-are-based.- A-literature-review-is-presented-to-provide-a-summary-to-the-state-of-the-art-research-in-this-field,-and-cover-thebasic-ideas-of-model-order-reduction-using-Krylov-subspaces-consideringa-linear-system-model-structure-for-illustration.- The-chapter-reviews-moment-matching-and-the-definition-of-moments-and-Krylov-subspaces.- Alsothe-chapter-revises-the-linear-algebra-concepts-of-subspaces,-linear-dependence-and-orthogonalisation.- The-Arnoldi-process-which-is-quite-useful-forcomputing-projection-matrices-utilised-in-one-sided-Krylov-subspace-modelorder-reduction-is-reported-and-the-stability-of-Krylov-subspace-techniquesis-discussed.-

A-literature-review-of-the-utilization-of-Krylov-subspaces-for-linear-and-nonlinear-MOR-is-presented. An-overview-of-the-use-of-Krylov-subspace-techniques-for-the-reduction-of-nonlinear-models-via-bilinearisation,-quadraticapproximation-and-piecewise-linear-approximation-is-provided.

Chapter 3: An-introduction-to-bilinear-models-is-presented-in-this-chapter. Thechapter-narrows-this-discussion-into-the-approximation-of-nonlinear-models-through-a-bilinearisation-process-called-Carleman-bilinearisation. Thisprocess-makes-it-possible-to-extend-Krylov-subspace-model-order-reduction-to-nonlinear-systems-via-its-bilinear-approximation. The Carlemanbilinearisation-process-is-discussed-and-this-is-followed-by-the-review-ofmodel-order-reduction-through-projection-for-bilinear-models-as-proposedin-literature.

The various Krylov subspaces proposed over the years are discussed and this chapter also features the Taylor series expansion, uses of bilinear models and an algorithm for computing projection matrices. A diterature review into the extension of Krylov subspace model order reduction techniques to MIMO bilinear models is presented. The chapter concludes with a discussion.

Chapter 4: This-chapter-presents- and-highlights- the-original-contribution-ofthis-thesis.- It-contains- an-analysis-of-the-one-sided-projection-techniquesproposed-in-(Phillips-2000,-Feng-&-Benner-2007)-via-their-moment-matching- capability- and- the- multimoments- matched- are- analysed- mathematically.- This- mathematical- analysis- forms- part- of- the- novelty- of- this- thesis.- This-chapter-improves-on-the-work-done-in-literature-which-has-beenreported in Chapter 3. A new method called an Improved Phillip typeprojection is presented followed by a mathematical analysis. Also a second approach for the application of Krylov subspaces to the model or der reduction problem of bilinear models is presented. This is called the parametrised linear approximation (PLA) method.

A-numerical-simulation-analysis-of-an-alternate-linear-approximation-forbilinear-models-is-done-and-the-results-are-discussed.- This-has-been-compared-to-the-traditional-linear-approximation-as-used-by-the-other-Krylovsubspace-methods-described-in-Chapter-3.- This-alternate-linear-approximation-forms-the-foundation-for-the-parametrised-linear-approximationapproach-which-is-the-second-proposal-of-this-chapter.-

This-chapter-also-presents-a-simulation-based-study-which-helps-to-identifythe-parameters-of-an-algorithm-for-computing-Krylov-subspace-projectionmatrices. These-parameters-are-then-used-to-compute-reduced-order-models-from-all-the-methods-described-in-Chapters-3-and-4. Two-examples-fromliterature-have-been-used-to-demonstrate-these-Krylov-subspace-model-order-reduction-methods.-

Chapter 5: More-original-contributions, focusing-on-bilinear-systems, are-presented-in-this-chapter. The extension-of-the methods-proposed-in-Chapter-4- are-applied-to-MIMO-case-studies. The presentation-starts-with-a-literature-review-into-the extension-of-Krylov-subspace-MOR-techniques-to-MIMO-bilinear-models. Also, a-mathematical-analysis-which-is-an-extension-of-the-mathematical-analysis-done-in-Chapter-4-are-given-for-MIMObilinear-models.

This-chapter-contributes-to-the-extension-of-the-Feng-and-Benner-(Feng-&-Benner-2007),-Improved-Phillips-type-projection-and-the-parametrisedlinear-approximation-(PLA)-approaches-to-the-MIMO-cases.-Two-arbitrarybilinear-models-have-been-used-to-illustrate-the-application-of-the-proposedresults. These are to be compared with the other reviewed methods and the advantages of the newly proposed methods are analysed.

Chapter 6 This-chapter-provides-two-applications-of-the-techniques-proposedin-Chapters-4-and-5.- A-hybrid-MOR-technique-for-SISO-and-MIMO-bilinear-systems/models-is-introduced.- This-combines-the-techniques-forparameter-estimation,-artificial-intelligence-and-optimisation-to-optimisethe-parameters-used-for-computing-the-parametrised-linear-approximationfor-reduced-order-modelling-via-Krylov-subspace-MOR.-The-second-application-is-the-use-of-PLA-for-MOR-of-a-pseudo-singular-bilinear-system.-Pseudo-singular-bilinear-models-have-been-defined-therein.-

Using-these-applications,-two-case-studies-have-been-presented-to-show-theunique-implications-of-the-techniques-proposed-in-this-thesis.- The-numerical-simulations-have-been-analysed-using-plots-and-a-set-of-performancecriteria.-

Chapter 7: This-chapter-provides-an-overall-conclusion-of-the-works-reportedin-the-thesis.-The-avenues-for-further-work-are-also-presented.-

1.3 Contributions

In-summary, the contributions of this research are given as follows:-

- 1.- The-matching-of-a-higher-number-of-multimoments-whilst-avoiding-themultiplication-of-nonsingular-matrices.- This-has-been-called-the-Improved-Phillip-type-projection-(Chapter-3).-
- 2.- The-proposal-of-a-reduced-order-modelling-approach-using-Krylov-subspaces-by-applying-a-so-called-better-linear-approximation.- This-approachis-called-the-Parametrised-Linear-Approximation-(PLA).-

- 3.- The-analysis-of-multimoment-matching-for-the-Feng-and-Benner-type-projection-(Feng-&-Benner-2007),-J.-R.-Phillip-type-projection-(Phillips-2000)and-the-Improved-Phillip-type-projection.-
- 4.- The extension of the Improved Phillip type projection, Parametrised Linear-Approximation projection and the Feng and Benner type projection (Feng-& Benner 2007) to MIMO cases.
- 5.- The analysis of multimoment matching for MIMO bilinear model reduction using Krylov subspaces.
- 6.- The use of PLA for the reduced order modelling of pseudo-singular bilinear systems to enable the reduction of systems with nonsingular system matrices.-
- 7.- The use of an optimisation scheme for finding parameters which form an alternate linear approximation of a bilinear system/model and the use of these parameters for model order reduction.

This-thesis-also-served-as-a-resource-for-understanding-and-practical-implementation-of-reduced-order-modelling-using-Krylov-subspaces.-

1.4 List of publications

During-the-period-of-study, some-publications-have-been-made-under-the-guidance-of-my-supervisors- and- collaboration- with-other-researchers. The-publications-cover-a-wide-range-of-techniques-for-producing-reduced-order-modelsvia-data-based-approaches-and-classical-methods. These-publications-are-listedbelow.

Agbaje, O, Kavanagh, D., Sumisławska, M., Howey, D., McCulloch, M. & Burnham, K., Estimation of temperature dependent equivalent circuit

parameters for traction-based electric machines, in Hybrid and Electric Vehicles Conference 2013 (HEVC 2013), IET, pages 1-6, 2013.

- 2.- Aizad, T., Sumisławska, M., Maganga, O., Agbaje, O., Phillip, N. & Burnham, K. J., Investigation of model order reduction techniques: A supercapacitor case study, in 'Advances in Systems Science', Springer, pages 795–804, 2014.
- 3.- Sumisławska,-M.,-Agbaje,-O.,-Kavanagh,-D.-F.-&-Bumham,-K.-J.,-Equivalent-circuit-model-estimation-of-induction-machines-under-elevated-temperature-conditions,-in-'UKACC-International-Conference-on-Control-(CON-TROL2014)',-pages-413-418,-2014.-

The knowledge-gained-from-linear-projection-technique-which-has-been-investigated-in-(Aizad-et-al.-2014)-has-been-expanded-on-for-nonlinear-systems-andforms-the-focus-of-the-research-and-original-results-presented-in-this-thesis.- Databased-techniques-have-been-used-in-(Agbaje,-Kavanagh,-Sumislawska,-Howey,-McCulloch-&-Burnham-2013,-Sumislawska,-Agbaje,-Kavanagh-&-Burnham-2014)to-estimate-the-parameters-of-a-low-order-equivalent-circuit-model-at-extremeconditions.- Some-of-these-techniques-such-as-the-use-of-identifiability-analysis,parameter-estimation-and-weighted-optimisation-have-been-used-to-optimise-theresults-presented-in-this-thesis.-

Chapter 2

Mathematical Background Preliminaries

2.1 Definition of terms

Definition 2.1.1 (Moment) Moments have been defined as the coefficients of a Taylor series expansion (Tan & He 2007). Consider a continuous-time system with input u and output y. The transfer function H(s)- describing the system behaviour is represented as

$$H(s) = \frac{y(s)}{u(s)}.$$
(2.1)

Then the Taylor series expansion of the transfer function at the expansion point s = -0-is defined as

$$H(s) = \sum_{l=0}^{\infty} m(l) s^l \tag{2.2}$$

where the moments, m(l), are defined as

$$m(l) = \frac{1}{l!} \times \frac{d^l H(s)}{ds^l}|_{s=0}$$

$$(2.3)$$

i.e. the moments are defined from the corresponding derivatives of the transfer function with respect to s.

Note-that-the-moments-can-be-defined-for-expansion-points-other-than-zero-butin-this-thesis-the-methods-described-are-only-for-moments-for-s = -0. Also-in-subsequent-chapters, this-same-expansion-point-is-considered-when-multimomentsare-discussed. However, moment-matching-for-expansion-point-other-than-zerohave-been-considered-in-literature-and-are-quite-easily-derived-(Salimbahrami-&-Lohmann-2002).

Definition 2.1.2 (Multimoments) Multimoments are the coefficients of the Taylor series expansion for a multivariable transfer function.

Definition 2.1.3 (Krylov subspace) The q^{th} Krylov subspace is defined as

$$K_q(\mathbb{N}, \mathbb{M}) = span\{\mathbb{N}^0 \mathbb{M}, \mathbb{N}^1 \mathbb{M}, ..., \mathbb{N}^{q-1} \mathbb{M}\}$$
(2.4)

where $\mathbb{N} \in \mathbb{R}^{n \times n}$, $\mathbb{M} \in \mathbb{R}^{n \times m}$ and $q, n, m \in \mathbb{Z}$. \mathbb{N} and \mathbb{M} are known as the starting matrices and they form the basis of the Krylov subspace. When considering a single input single output (SISO) system, \mathbb{M} would be a vector and consequently, m = 1. Moreover, for multi input multi output (MIMO) systems, m > 1.

Definition 2.1.4 (Subspace) Given that an n-vector is an $n \times 1$ -matrix with real numbers as components and all n-vectors belong to the subset of vectors defined by \mathbb{R}^n called the n-space. A subspace can then be defined as a set of \mathbb{R}^n within which the following properties are inherent

1. If \bar{s}_1 and \bar{s}_2 are in \mathbb{S} , then $\bar{s}_1 + \bar{s}_2$ is in \mathbb{S}

$$\bar{s}_1 \in \mathbb{S}, \bar{s}_2 \in \mathbb{S}, \bar{s}_1 + \bar{s}_2 \in \mathbb{S} \tag{2.5}$$

2. If r is any real number, and \bar{s}_i is any vector in \mathbb{S} , then $r \times \bar{s}_i$ is in \mathbb{S}

$$r \times \bar{s}_i \in \mathbb{S} \tag{2.6}$$

For these conditions to hold, it is expedient that S is nonempty, i.e. a set that contains at least one component.

Definition 2.1.5 (Linear Independence) If \mathbb{V} is a set of m vectors in the subspace \mathbb{S} , \mathbb{V} can be said to be linearly independent if it is not possible to find constants, c_1, c_2, \ldots, c_m such that

$$c_1v_1 + c_2v_2 + \dots + c_nv_m = 0$$

where v_1, v_2, \ldots, v_m are the column vectors in \mathbb{V} . This means that the vectors are only linearly independent when all the constants are zero i.e.

$$c_1 = c_2 = \dots = c_n = 0$$
 (2.8)

Definition 2.1.6 (Span) The vectors in \mathbb{V} can be said to span \mathbb{S} or \mathbb{S} is said to be spanned by \mathbb{V} if every vector in \mathbb{S} is a linear combination of the vectors in \mathbb{V} .

$$\mathbb{S} = span\{\mathbb{V}\} \tag{2.9}$$

This makes \mathbb{V} a unique subset of \mathbb{S} . The vectors of \mathbb{V} are called the basis of \mathbb{S} .

Definition 2.1.7 (Bounded input bounded input (BIBO) stability) A system is BIBO stable iff, for any bounded input, the output is bounded at all times given zero initial conditions.

2.2 Model order reduction (MOR)

Given-a-model-(2.1)-of-any-structure-which-describes-the-input-output-behaviourof-a-system. The-model-order-reduction-problem-is-to-find-another-model-whichcan-be-used-to-replace-the-former, where-this-new-model-is-of-a-lower-dimensional-space- (vectors, matrices, equations), less-storage-requirements- and-lowevaluation/simulation-time, it-is-said-to-be-a-reduced-order-model. The-processfor-getting-this-reduced-order-model-is-called-model-order-reduction-(MOR).

Some-MOR-techniques-preserve-the-structure-of-the-higher-order-model-andsome-do-not.-MOR-techniques-can-broadly-be-divided-into-data-based-techniques-
and classical techniques which are based on mathematical manipulation. The later takes advantage of the mathematical properties of the models to derive reduced order models. Data based techniques (Sumisławska et al. 2014, Agbaje et al. 2013) will require taking input-output data, structure selection, data manipulation and processing techniques to achieve a reduced order model.

The approach taken for MOR will depend on the type of system considered. Classical model order reduction techniques have been used for both linear and nonlinear models. MOR for linear models forms a background for extending MOR to other model structures.

2.2.1 MOR for linear systems

Linear systems have been defined in general as systems that obey the laws of superposition (Nise 2007) and have simple structures. This simple structure lends itself for implementation of control and diagnostic algorithms. To introduce Krylov subspace MOR, consider a state space form of linear model,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
 (2.10)

$$y(t) = Cx(t)$$
 (2.11)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are called the system matrix, input matrix and output matrix respectively and $n, p, m \in \mathbb{Z}$. For a single input single output (SISO) system model, p and m are equal to one. Several electrical circuits and some classes of mechanical systems can be represented in this form as described in (Feng & Benner 2007, Silveira, Kamon, Elfadel & White 1997, Freund 2000). The solutions derived here are easily realizable in other-linear system formations.

Moment-matching-and-Gramian-based-model-order-reduction-methods-arethe-most-popular-for-reducing-the-order-of-large-linear-systems/models.-Gramianbased-approaches-such-as-Balanced-Truncation-(BT)-guarantee-stability-and-provide-an-error-bound-for-the-higher-order-model-and-reduced-order-model.-OtherMOR-techniques-which-are-regarded-as-Gramian-based-are-singular-perturbation-approximation-(SPA)-(Benner,-Quintana-Orti-&-Quintana-Ortí-2000,-Liu-&-Anderson-1986,-Varga-1991,-Aizad-et-al.-2014),-Balanced-stochastic-truncation-(Benner,-Quintana-Ortí-&-Quintana-Ortí-2001),-Frequency-weighted-balanced-truncation-(Gawronski-&-Juang-1990)-and-Hankel-norm-approximation-(Glover-1984,-Benner,-Quintana-Ortí-&-Quintana-Ortí-2004).-These-approachesare-referred-to-because-they-involve-the-balancing-of-the-higher-order-model-bycomputing-the-observability-and-controllability-Gramians,-*P* and-*Q*,-respectively.-For-a-linear-system,-they-are-defined-mathematically-as-

$$P = \int_0^\infty e^{At} B B^T e^{A^T t} dt \qquad (2.12)$$

$$\bar{Q} = \int_0^\infty e^{A^T t} C C^T e^{At} dt.$$
(2.13)-

The observability and controllability Gramians can be computed by solving two Lyapunov equations

$$AP + PA^T + BB^T = 0$$

$$A^T \bar{Q} + \bar{Q}A + C^T C = 0. \tag{2.15}$$

This forms the first step for the Gramian based approaches. Using the Gramians, a reduced order model can be obtained by using a balancing procedure followed by truncation. This is referred to as BT. Other variations of BT can be found in (Phillips, Daniel & Silveira 2003, Phillips & Silveira 2005, Reis & Stykel 2010, Benner 2010). As part of the procedure within this method, a set of linear equations need to be solved. The size of this set is the same as the dimension of the higher order model, thus the computational complexity of Gramian based approaches is $O(n^3)$, whilst the required storage is of order $O(n^2)$. Due to this disadvantage of BT, new and more efficient methods for solving large Lyapunov equations have been proposed (Jaimoukha, Kasenally & Limebeer 1992, Ahmad, Jaimoukha & Frangos 2010). These approaches use Krylov-subspaces-to-overcome-the-computational-complexity-of-Gramian-basedapproaches.-

Moment-matching-methods-refer-to-MOR-approches-which-match-momentsof-the-higher-order-model-and-reduced-order-model-such-that-the-reduced-ordermodel-is-accurate-up-to-a-certain-degree-depending-on-the-amount-of-momentsmatched.- These-are-also-known-as-Krylov-subspace-based-approaches.- Togetherwith-Gramian-based-approches- (Antoulas-&-Sorensen-2001,-Tan-&-He-2007),they-have-been-classified-as-projection-based-MOR-for-linear-systems-in-(Tan-&-He-2007).-

2.3 Krylov subspace MOR methods

In- cases- where- very- large-system- matrices- which- are- not- suitable- for- certainapplications- are- considered,- it- is- necessary- to- reduce- the- order- of- the-systemstates- using- techniques- that- are- appropriate- to- meet- a- predetermined- criteria.-Krylov- subspace- algorithms- introduced- in- (Arnoldi- 1951)- and- (Lanczos- 1950)are- some- of- the- original- results- proposed- in- this- area.- Improvements- to- thesetechniques- and- associated- difficulties- in- their- implementation- are- discussed- indetail- in- (Lohmann- &- Salimbahrami- 2000,- Silveira- et- al.- 1997,- Odabasioglu,-Celik- &- Pileggi- 1997,- Kerns,- Wemple- &- Yang- 1995).- Krylov- subspace- basedapproaches-have-been-described-as-being-quite-closely-related-to-other-projectiontechniques- (Tan-&-He-2007).- In-order-to-introduce-MOR-via-Krylov-subspaces,it-is-useful-to-consider-linear-systems-as-a-case-study.- Linear-systems-have-beenrepresented-using-the-transfer-function-in-the-s-domain-where-a-transfer-functionis-defined-as-the-ratio-of-the-input-to-the-output,-H(s) = -Y(s)/U(s).- This-can-bederived-from-obtaining-the-Laplace-transform-of-the-state-space-representation-(2.10)-- (2.11).- Using-the-relations- $\dot{x}(t) \rightarrow sX(s), -y(t) \rightarrow Y(s)$ -and- $u(t) \rightarrow U(s), -$ we-can-write-

$$sX(s) = AX(s) + BU(s)$$
 (2.16)

$$(sI - A)X(s) = BU(s)$$

$$(2.17)$$

$$X(s) = (sI - A)^{-1}BU(s).$$
(2.18)-

Therefore, the output Y(s) can be expressed as

$$Y(s) = CX(s)$$
 (2.19)

$$Y(s) = C(sI - A)^{-1}BU(s).$$
(2.20)-

Furthermore, for-SISO-systems-we-can-write-

$$Y(s) = [C(sI - A)^{-1}B]U(s)$$
(2.21)

$$Y(s)/U(s) = C(sI - A)^{-1}B$$
 (2.22)-

$$H(s) = C(sI - A)^{-1}B.$$
 (2.23)

By-using-the-Taylor-series-expansion, the transfer-function-can-be expressed-asa-polynomial-function. The coefficients of this expansion at zero are called the moments of the transfer function-

$$H(s) = \sum_{l=1}^{\infty} m(l) s^{l-1}$$
(2.24)

where m(l)-are-the-moments. When the expansion point is at infinity, the moments are called Markov parameters (Salimbahrami & Lohmann 2002). It is possible to find moments at different values of s. For the case described here, the moments are

$$m(l) = -CA^{-l}B. (2.25)^{-l}$$

Krylov-subspace-methods-aim-to-obtain-reduced-order-models-in-such-a-way-thatmakes-the-moments-of-the-reduced-and-higher-order-models-equivalent.-

2.3.1 One-sided projection

There-are-other-projection-methods. But-this-literature-review-focuses-on-Krylovsubspace-MOR-which-is-closely-related-to-other-projection-techniques. Krylovsubspace-MOR- is- known- for- its- fast- computational- time- when- compared- toother- methods. Krylov-subspace- methods- are- well-reported-in- (Celik, Pileggi-&-Odabasioglu-2002, Tan-&-He-2007).

In one-sided-projection, two-techniques-are-generally-used. In (Feng-&-Benner-2007)-they-have-been-refered-to-as-the-first-projection-technique-and-the-second-projection-technique. They-will-be-discussed-here-in-that-order. In the first-projection-technique as-discussed-in (Phillips-2000), the system-matrix-is-not-inverted-prior-to-projection-and-the-approximation- $x = V\hat{x}$ is-used, where V is-an-orthonormal-matrix-of-dimension- $n \times q$ such-that $V^T V = I$, where \hat{x} is-the-state-of-the-reduced-order-model, and premultiplying (2.10)-by- V^T , results-in-areduced-order-system-with-matrices-of-the-form-

$$\hat{A} = V^T A V, \hat{B} = V^T B, \hat{C} = C V.$$

$$(2.26)$$

The alternative method-i.e. the second-projection technique premultiplies bothsides of the equation (2.10) with A^{-1} prior to approximating the states x

$$A^{-1}\dot{x} = x + A^{-1}Bu. \tag{2.27}$$

Applying-projection-matrix-V to-(2.27)-and-(2.11)-by-substituting-the-approximation- $x \approx V\hat{x}$ and-premultiplying-(2.27)-by- V^T gives-

$$V^T A^{-1} V \dot{\hat{x}} = \hat{x} + V^T A^{-1} B u \tag{2.28}$$

$$\hat{y} = CV\hat{x} \tag{2.29}$$

and-premultiplying-both-sides-of-(2.28)-by-the-inverse-of- $V^T A^{-1}V$,-the-resultant-reduced-order-matrices- \hat{A} ,- \hat{B} and- \hat{C} are-obtained-as-

$$\hat{A} = (V^T A^{-1} V)^{-1}, \hat{B} = (V^T A^{-1} V)^{-1} V^T A^{-1} B, \hat{C} = CV.$$
(2.30)

The first-projection-technique has also been used in (Odabasioglu et al. 1997) to preserve the stability and passivity of a system. It also matches as many moments as the second projection technique (Odabasioglu et al. 1997). However, the second technique cannot be said to be one-sided due to the awkward definition of the projection matrices when compared to the first projection technique. Notice that the right projection matrix is not actually a transpose of the left projection matrix.

2.3.2 Moment matching

The moment matching properties of the one-sided projection has been discussed in detail in (Tan & He 2007, Lohmann & Salimbahrami 2000). (Tan & He 2007) state that the one-sided projection methods match q moments whilst two sided methods match 2q moments. From Taylor series expansion of the reduced order model transfer function given below,

$$\hat{H}(s) = \sum_{l=1}^{\infty} \hat{m}(l) s^{l-1}$$
(2.31)-

the-moments-of-the-reduced-order-model-can-be-defined-as-

$$\hat{m}(l) = -\hat{C}\hat{A}^{-l}\hat{B}.$$
 (2.32)

For the first moment of the reduced order model $\hat{m}(1) = -\hat{C}\hat{A}^{-1}\hat{B}$, note that $A^{-1}B$ belongs to the Krylov subspace $K_q(A^{-1}, A^{-1}B)$ and therefore can be written as $A^{-1}B = Vr_{(1)}$ and $B = AVr_{(1)}$ with $r_{(i)} \in \mathbb{R}^{q \times 1}$, $i = 1 \dots q$, being a vector with parameters which make the statement true. From the definition of the moments for the higher and lower order models, moment matching can be proved as follows. Substituting the matrices (2.26) into (2.32), for l = 1, results in

$$\hat{m}(1) = -\hat{C}\hat{A}^{-1}\hat{B}$$
 (2.33)

$$\hat{m}(1) = -CV(V^T A V)^{-1} V^T B.$$
 (2.34)

Since $A^{-1}B = Vr_{(1)}$ and $B = AVr_{(1)}$. Substituting this into (2.34) yields

$$\hat{m}(1) = -CV(V^T A V)^{-1} V^T A V r_{(1)}.$$
(2.35)

Also, $(V^T A V)^{-1} V^T A V = I$ and $Ir_{(1)} = r_{(1)}$ therefore

$$\hat{m}(1) = -CVr_{(1)}$$

= $-CA^{-1}B$ (2.36)
 $\hat{m}(1) = -m(1).$

Likewise-for-the-second moment, m(2), the moment-of-the-reduced-order-model-can-be-defines-as-

$$\hat{m}(2) = -\hat{C}(\hat{A}^{-1})\hat{A}^{-1}\hat{B}$$

$$= -CV(V^{T}AV)^{-1}(V^{T}AV)^{-1}V^{T}B$$

$$= -CV(V^{T}AV)^{-1}(V^{T}AV)^{-1}V^{T}AVr_{(1)}$$

$$= -CV(V^{T}AV)^{-1}r_{(1)}.$$
(2.37)

Using-the-orthogonality-of-the-matrix-V-and-some-manipulations-of-matrix-algebra,-we-write:-

$$\hat{m}(2) = -CV(V^T A V)^{-1} V^T V r_{(1)}$$

$$= -CV(V^T A V)^{-1} V^T A (A^{-1}) A^{-1} B.$$
(2.38)

Defining $Vr_{(2)} := (A^{-1})A^{-1}B$, we obtain

$$\hat{m}(2) = -CV(V^{T}AV)^{-1}V^{T}AVr_{(2)}$$

$$= -CVr_{(2)}$$

$$= -C(A^{-1})A^{-1}B$$

$$\hat{m}(2) = -m(2).$$
(2.39)

Continuing-further-for-the-**third moment**, m(3). The-third-moment-of-the-reduced-order-model-is-

$$\hat{m}(3) = -\hat{C}(\hat{A}^{-2})\hat{A}^{-1}\hat{B}$$

$$\hat{m}(3) = -CV(V^{T}AV)^{-2}(V^{T}AV)^{-1}V^{T}B.$$
(2.40)

Since $B = AVr_{(1)}$,

$$\hat{m}(3) = -CV(V^{T}AV)^{-2}(V^{T}AV)^{-1}V^{T}AVr_{(1)}$$

$$= -CV(V^{T}AV)^{-2}r_{(1)}$$

$$= -CV(V^{T}AV)^{-2}V^{T}Vr_{(1)}$$

$$\hat{m}(3) = -CV(V^{T}AV)^{-2}V^{T}A(A^{-1})A^{-1}B.$$
(2.41)

Also $A^{-2}B = Vr_{(2)}$, therefore,

$$\hat{m}(3) = -CV(V^{T}AV)^{-2}V^{T}AVr_{(2)}$$

$$= -CV(V^{T}AV)^{-1}r_{(2)}$$

$$= -CV(V^{T}AV)^{-1}V^{T}Vr_{(2)}$$

$$\hat{m}(3) = -CV(V^{T}AV)^{-1}V^{T}AA^{-1}Vr_{(2)}.$$
(2.42)

Since $Vr_{(2)} = A^{-2}B$ and $A^{-3}B = Vr_{(3)}$, therefore,

$$\hat{m}(3) = -CV(V^{T}AV)^{-1}V^{T}A(A^{-1})(A^{-1})A^{-1}B$$

$$= -CV(V^{T}AV)^{-1}V^{T}AVr_{(3)}$$

$$= -CVr_{(3)}$$

$$= -C(A^{-2})A^{-1}B$$

$$\hat{m}(3) = -m(3).$$
(2.43)

As can be observed, the crucial step in proving the matching of moments is that the vector $(A^{-1})^q B = Vr_{(q)}$ belongs to the Krylov subspace $K_q(A^{-1}, A^{-1}B)$. From the proof of moment matching shown above, the following theorem suffices.

Theorem 2.3.1 Given a projection matrix, V, for the Krylov subspace span $(V) = K_q(A^{-1}, A^{-1}B)$, where A is the system matrix and B is the input vector, q moments of the higher order model are matched by the reduced order model if the reduced order model matrices are formed such that $\hat{A} = V^T A V$, $\hat{B} = V^T B$, $\hat{C} = CV$ (Tan & He 2007).

Note that the case described here is for moments at an expansion point zero where the matrix V spans the Krylov subspace and can be computed using algorithms proposed by (Arnoldi 1951) and (Lanczos 1950). When V is computed using one Krylov subspace and $V^T V = I$, this is referred to as a one-sided projection. There exists/are methods which use two Krylov subspaces such that $W^T V = I$ where W spans the Krylov subspace, $K_q(A^{-T}, A^{-T}C)$. This implies that more computational effort is needed for computing the projection matrices.

Moments-can-also-be-matched-at- $\delta = -\infty$. In this-case, they are referred-toas-Markov-parameters. The work-done-in-(Grimme-1997)-matches-the-first-twomoments-using-a-two-sided-approach-at-multiple-expansion-points, $\delta_1, \delta_2, \ldots, \delta_k, k \in \mathbb{Z}$, by-using-a-rational-Krylov-algorithm. Improvements-to-this-approach-forselecting-expansion-points-in-an-optimal-manner-have-been-reported-in-(Frangos-&-Jaimoukha-2007*a*).

The-most-recent-work-done-in-this-area-is-inspired-by-(Grimme-1997)-where-rational-Krylov-algorithms-are-used-for-computing-the-projection-matrices-V and W where-

$$Span(V) = K_q((A - \delta_1 I)^{-1}B, (A - \delta_2 I)^{-1}B, \dots, (A - \delta_n I)^{-1}B)$$
(2.44)
$$Span(W) = K_q((A^T - \delta_1 I)^{-1}C^T, (A^T - \delta_2 I)^{-1}C^T, \dots, (A^T - \delta_n I)^{-1}C^T)$$
(2.45)

In-(Gugercin-et-al.-2008), an-iterative-rational-Krylov-algorithm-(IRKA)-for-anoptimal-selection-of-the-expansion-points-was-proposed. The-method-is-reportedto-be- H_2 optimal-as-it-is-based-on-satisfying-the- H_2 error-of-the-higher-andreduced-order-model-given-by-

$$||H - \hat{H}|| = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(jw) - \hat{H}(jw)|^2 dw}.$$
 (2.46)-

However, - the- preservation- of- stability- is- not- guaranteed- as- the- initial- selection- of- the- interpolation- points- is- not- clearly- defined. - The- IRKA- has- beenextended- to- several- directions- (Flagg, - Beattie- &- Gugercin- 2013, - Druskin- &- Simoncini-2011, Panzer, Jaensch, Wolf-&-Lohmann-2013). In (Flagg-et-al.-2013), a nearly-optimal-approach-is-developed. It-uses-Krylov-subspaces-for-solving-linear-system-of-equations-which-are-required-for-the-IRKA-and-therefore-is-more efficient. An adaptive-shift-computation-for-selecting-the-expansion-points-has-been-developed-in-(Druskin-&-Simoncini-2011). This-approach-has-been-reported-to-be-less-accurate-when-compared-to-the-IRKA-but-has-less-computational-cost. In (Panzer-et-al.-2013), a-stability-preserving, adaptive-rational-Krylov-(SPARK)-algorithm-was-developed-to-guarantee-stability-and-is- H_2 optimal-at-the-cost-of-being-more-expensive-computationally-when-compared-to-the-IRKA. Generally, methods-based-on-the-Krylov-subspaces-(2.44)- (2.45), match-only-one-moment-at-each-expansion-point- and-lack-flexibility-for-matching-more-moments. The- H_2 optimal-approach-has-been-extended-to-apply-to-MIMO-linear-systems-in-(Van-Dooren,-Gallivan-&-Absil-2008, Van-Dooren-et-al.-2008).

2.4 Algorithms

In this section, some algorithms that enable the computation of the projection matrices are presented. It is advantageous to build an orthogonal basis for the Krylov subspace by using numerically stable processes as the computation of V can be unstable as the dimension of the Krylov subspace q gets large (Tan & He 2007). Here the processes for orthogonalisation and the Arnoldi process are discussed. The variations and genealogy of these algorithms are well reported (see for instance (Saad 2003)). Note that in this section, q_i , $i = 1, 2, \ldots$, denotes the elements of the matrix Q in the QR factorisation.

2.4.1 Orthogonalisation

Two-vectors-are-said-to-be-orthogonal-if-the-result-of-their-product-is-zero.-i.e.-

$$v_1^T v_2 = 0. (2.47)$$

For-a-set-of-vectors, this-same-rule-is-applied. The-set-is-said-to-be-orthogonalif-all-the-component-vectors-follow-this-same-rule-

$$v_i^T v_j = 0, \tag{2.48}$$

where $i \neq j$ and $i, j \in \mathbb{Z}$. The set of vectors is orthonormal if, whilst being orthogonal, each vector component has a 2-norm of 1.

An-orthonormal-basis-can-be-obtained-by-taking-the-basis-of-a-subspace-andorthonormalising-them.- This-process-of-orthonormalisation-and-orthogonalisation-is-called-the-Gram-Schmidt-process-e.g.- (Tan-&-He-2007,-Saad-2003,-Salimbahrami-&-Lohmann-2002,-Daniel,-Gragg,-Kaufman-&-Stewart-1976).-

For a set of *n*-vectors which are linearly independent, (v_1, v_2, \ldots, v_n) , the initial step of the Gram-Schmidt process is to normalize the first vector of this set by dividing it by its 2-norm. The resulting vector, q_1 is of norm 1. The following-linearly independent vector, v_2 is then orthogonalised against q_1 . This can be achieved by subtracting a multiple of q_1 from v_2 which makes the resulting vector orthogonal to q_1

$$v_2 \leftarrow v_2 - (v_2^T q_1) q_1$$
 (2.49)

The resulting vector is then normalised to obtain a vector q_2 . This means that the Gram-Shmidt process orthonormalises any vector v_j in the set against any previous vector q_{j-1} .

The Gram-Schmidt algorithm is presented below. In the algorithm, it is necessary that the matrices are linearly independent to prevent a break down of the process.

Algorithm 2.1 (Gram-Schmidt process)

1. Compute: $r_{11} = -||v_1||_2$, if $r_{11} = 0$. Stop, else Compute $q_1 = v_1/r_{11}$

2. for j = 2 :: n3. $r_{ij} = v_j^T q_i$ for i = 1, 2, ..., j - 14. $\hat{q} = x_j - \sum_{i=1}^{j-1} r_{ij} q_i$ 5. $r_{jj} = ||\hat{q}||_2$ 6. if $r_{jj} = 0$ then **Stop**, else $q_j = \hat{q}/r_{jj}$ 7. end

There exist other versions of this algorithm which have been proposed by other authors to deal with loss of orthogonality in the process (Daniel et al. 1976). The Modified Gram-Schmidt algorithm is one of them and has been reported to have better numerical properties (Saad 2003) and is used in all the algorithms in this work.

Algorithm 2.2 (Modified Gram-Schmidt process) 1. Compute: $r_{11} = ||v_1||_2$, if $r_{11} = 0$ Stop, else Compute $q_1 = v_1/r_{11}$ 2. for j = 2 : n3. $\hat{q} = x_j$ 4. for $i = 1, \dots, j - 1$ 5. $r_{ij} = \hat{q}^T q_i$ 6. $\hat{q} - r_{ij}q_i$ 7. end 8. Compute: $r_{jj} = -||\hat{q}||_2$ 9. if $r_{jj} = -0$ then Stop, else $q_j = -\hat{q}/r_{jj}$ 10. end

An-alternative-orthogonalisation-method-which-uses-a-factorisation-approachis-the-Householder's-method-(Golub-&-Van-Loan-2012).-

2.4.2 QR factorisation

As-can-be-observed-from-Algorithm-2.1-steps-4-and-5,-the-relationship-betweenthe-normalised-and-orthogonalised-vectors-at-every-step-of-the-algorithm-is-

$$v_j = \sum_{i=1}^{j} r_{ij} q_i, \tag{2.52}$$

or-in-the-matrix-form:-

$$\mathbb{V} = QR. \tag{2.53}$$

This is known as the QR decomposition of the matrix \mathbb{V} , where \mathbb{V} is a set of linearly independent vectors $[v_1, v_2, \ldots, v_n]$, $Q = [q_1, q_2, \ldots, q_n]$ and R are non-zero-elements, $r_{ij} \in \mathbb{R}$. The QR factorisation is an inbuilt function in MATLAB and can be accessed by using the command **orth**. This command has been used in Krylov subspace algorithms proposed in (Bai & Skoogh 2006, Lin, Bao & Wei 2009) and will also be used in this thesis.

2.4.3 Arnoldi process

To compute projection matrix V where $V^T V = I$, Krylov subspace methods are utilised. The starting vectors of the Krylov subspace correspond to the system matrices and input vectors of the linear system. This is the case for one sided-projection.- In-two-sided-projection,-two-Krylov-subspaces-are-utilised.-In-(Lohmann-&-Salimbahrami-2000,-Arnoldi-1951),-the-Arnoldi-and-Lanczosalgorithms-have-been-discussed-in-details.-

Using-the-starting-vectors-of-the-Krylov-subspace, the original-Arnoldi-algorithm-(Arnoldi-1951)-iteratively-formulates-a-set-of-vectors-with-norm-1-whichare-orthogonal-to-each-other. The result-of-the-algorithm-is-the-matrix-V. Whichis-orthonormal. The algorithm-as-presented-in-(Tan-&-He-2007)-is-as-follows:-

Algorithm 2.3 (Arnoldi algorithm)
1. Input: A, B, C, q
2. Compute: $r = A^{-1}B$
3. Compute: $v_1 = r/ r _2$
4. for $i = 1 : q - 1$
$5. r = A^{-1}v_i$
$6. \qquad h = (V_{[i]})^T r$
$7. r = r - V_{[i]}h$
8. if $ r _2 = 0$, end
9. $v_{i+1} = r/ r _2$
10. end
11. return V

The-algorithm-is-easily-modified-to-a-multiple-input-multiple-output-case-asdocumented-in-(Tan-&-He-2007).- The-outcome-of-the-algorithm-is-the-projectionmatrix-V where the vectors v_i are the columns of V. A two sided Arnoldialgorithm has been developed in (Lohmann & Salimbahrami 2000) to match 2q moments at zero. Rational Arnoldi algorithms which match moments at multiple expansion points have been presented in (Frangos & Jaimoukha 2007 b, Ruhe 1994).

2.5 Stability of Krylov subspace techniques

As mentioned earlier, MOR can sometimes results in unstable reduced models. While it is possible to simply discard unstable poles (Odabasioglu et al. 1997), several algorithms have been proposed to guarantee stability of the resulting models (Silveira et al. 1997, Kerns et al. 1995). In (Silveira et al. 1997), the system is said to be stable if all its eigenvalues have nonpositive real parts. Also given that the system matrix $A \in \mathbb{R}^{n \times n}$ is negative semidefinite, i.e.

$$p^T A p \le 0, \tag{2.55}$$

where in this section, p is an arbitrary non-zero vector of appropriate dimensions, it can be shown that the Arnoldi algorithm produces stable reduced order systems as follows:

$$p^T \hat{A} p \le 0$$

$$p^T V^T A V p \le 0$$

$$(Vp)^T A Vp \le 0. \tag{2.58}$$

In (Bond & Daniel 2007), an algorithm that utilises a Lyapunov function was proposed to ensure stable reduced order models for linear piecewise MOR approaches. For linear systems, a natural choice of a Lyapunov function is a quadratic function of the states

$$\bar{W} = x^T P x, \tag{2.59}$$

where $P \in \mathbb{R}^{n \times n}$ is some symmetric positive definite matrix that solves the algebraic equation

$$PA + A^T P = -Q \tag{2.60}$$

where A is the state matrix of the system and $Q \in \mathbb{R}^{n \times n}$ is a positive definite matrix, which often is chosen as the identity matrix. In order to achieve this, the left-projection matrix and the matrices of the reduced linear model are defined as

$$U^{T} = (V^{T} P V)^{-1} V^{T} P (2.61)^{-1} V^{T} P$$

$$\hat{A} = -U^T A V, \hat{B} = -U^T B, \hat{C} = -C V.$$

$$(2.62)$$

For the reduced order model (2.62), there exists a Lyapunov function, $\hat{W} = x^T \hat{P} x$ satisfying

$$\hat{P}\hat{A} + \hat{A}^T\hat{P} = -\hat{Q} \tag{2.63}$$

$$\hat{P} = V^T P V \tag{2.64}$$

which- can- be- used- to- prove- that- the- formulations- (2.61)- and- (2.62)- producestable-reduced-order-models.- From- (2.61),- (2.62)- and- (2.64),- we-can-write:-

$$\hat{P}\hat{A} = V^T P V U^T A V$$

= $V^T P V [(V^T P V)^{-1} V^T P] A V$ (2.65)
 $\hat{P}\hat{A} = V^T P A V.$

Using $\hat{A} = U^T A V$ from (2.62) and (2.64), $\hat{A}^T \hat{P}$ from (2.63) can be expressed as

$$\hat{A}^T \hat{P} = (U^T A V)^T V^T A V.$$

Based-on-the-definition-of- U^T in-(2.61),-

$$\hat{A}^{T}\hat{P} = [(V^{T}PV)^{-1}V^{T}PAV]^{T}V^{T}PV$$

$$= (V^{T}PAV)^{T}[(V^{T}PV)^{-1}]^{T}V^{T}PV$$

$$= (V^{T}PAV)^{T}$$

$$\hat{A}^{T}\hat{P} = V^{T}A^{T}PV$$
(2.67)

Thus,

$$\hat{P}\hat{A} + \hat{A}^T\hat{P} = V^T(PA + A^TP)V = -\hat{Q}$$

$$\hat{P}\hat{A} + \hat{A}^T\hat{P} = -V^TQV$$
(2.68)

Therefore, (2.63)-is-satisfied-for-a-positive-definite-matrix- $\hat{Q} = V^T Q V$.

(Silveira-et-al.-1997)-utilised-a-congruence-argument-to-guarantee-stability.-Their-work-provides-improvement-to-other-work-done-earlier-in-(Kerns-et-al.-1995).- In-(Silveira-et-al.-1997)-a-computationally-efficient-Arnoldi-algorithm-isproposed-for-arbitrary-and-stable-reduced-order-systems.-

Some-Krylov-subspace-approaches-as-proposed-in-(Bai-&-Freund-2001,-Freund-&-Feldmann-1996,-Freund-&-Feldmann-1997,-Freund-&-Feldmann-1998,-Kernset-al.-1995,-Silveira-et-al.-1997)-do-guarantee-stability-and-passivity-(Odabasiogluet-al.-1997).- However,-there-have-been-cases-where-reduced-order-models-derivedfrom-Krylov-subspaces-have-resulted-in-unstable-models-(Bai-&-Skoogh-2006)which-might-be-due-to-numerical-approximations-from-solvers-rather-than-fromthe-method-used.-

2.5.1 MOR for nonlinear systems

Most-of-the-techniques-for-reducing-linear-systems-can-be-extended-to-apply-tononlinear-systems-by-taking-advantage-of-the-Taylor-series-expansion-of-nonlinearmodels.-Examples-are-the-Carleman-bilinearisation-(Phillips-2000)-and-quadraticapproximation-(QA)-as-have-been-discussed-in-(Chen-et-al.-2000,-Chen-1999).-More-recently,-a-trajectory-piece-wise-linear-(TPWL)-method-and-all-its-variants-(Aizad-et-al.-2014,-Rewieński-&-White-2003)-have-been-proposed-for-thereduction-of-nonlinear-systems.- These-approaches-(QA-and-TPWL)-have-beenused-to-reduce-nonlinear-models-of-the-form-

$$\dot{x} = f(x) + Bu \tag{2.69}$$

$$y = Cx \tag{2.70}$$

where f is a nonlinear function such that $f := \mathbb{R}^n \to \mathbb{R}^n$, $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$. The TPWL is quite unique when compared to other approaches



Figure 2.1: Procedure for MOR of nonlinear systems using Krylov subspace approaches.

which-depend-on-the-Taylor-series-expansion-about-a-point-of-a-nonlinear-model-(Carleman-bilinearization- and-QA).-The-TPWL-method-was-first-proposed-toovercome-this-limitation-by-using-multiple-linearisation-points.- However,-ithas-been-reported-to-have-limited-small-signal-distortion-and-interpolation-fidelity-(Dong-&-Roychowdhury-2003).- In-(Dong-&-Roychowdhury-2003,-Dong-&-Roychowdhury-2008)- a-method-which-takes- advantage- of-the-TPWL- andpolynomial-representation-of-models-which-further-increases-its-complexity-wastherefore-presented.- These-nonlinear-MOR-methods-are-either-based-on-moment-matching-or-BT.-This-is-because-the-structure-provided-by-bilinearisation,quadratic-approximations-and-piece-wise-linear-approximation-of-the-nonlinearmodel-allows-linear-MOR-techniques-to-be-readily-applicable.- The-MOR-procedure-for-nonlinear-systems-using-Krylov-subspace-approaches-is-outlined-in-Figure-2.1.- Some-aspects-of-this-will-be-discussed-further-in-Chapter-3.-

2.6 Conclusion

Krylov-subspace-MOR-methods-have-been-described-as-one-of-the-most-important-algorithms-developed-in-this-century-(Dongarra-&-Sullivan-2000).- Theyhave-been-found-to-be-very-useful-when-dealing-with-systems-of-very-high-order.-

The use of Krylov subspaces has initially been proposed for linear systems, however the advantages they provide have driven research into their use for reduction of nonlinear models. In this chapter all the basic ideas which inform the reader of the hows and the whys of the use of Krylov subspaces have been discussed. The concepts such as moment matching, orthogonalisation, normalisation and the Gram-Schmidt process and their corresponding algorithms have been presented. Also, a linear model structure has been used to described projection based reduction method has been shown to produce reduced order models which match the moments of the higher order model. This is achieved by projection bases which are computed using the Arnoldi process as shown in Algorithm 2.3.

The-processes-discussed-in-this-chapter,-such-as-subspaces,-orthogonalisarionand-projection-form-a-basic-framework-for-which-Krylov-subspace-projection-canbe-extented-to-bilinear-systems-and-in-some-ways,-nonlinear-systems-which-willbe-the-focus-of-discussion-in-Chapter-3.-

Chapter 3

Bilinear Systems

Bilinear-systems- form- a-set- of- nonlinear-systems- which- are- closely-related- tolinear-systems. Bilinear-models-are-particularly-important-because-they-aresuitable-for-approximating-the-dynamics-of-nonlinear-systems-and-models-whilstretaining-a-well-structured-mathematical-framework-within-which-linear-systemsco-exist. They have been used to approximate a wide range of physical/electrical-(Bai-&-Skoogh-2006, Phillips-2003), chemical-(Espana-&-Landau-1978), biological-(Mohler-&-Barton-1978),-social-(Breiten-&-Damm-2010)-and-engineeringsystems-(Mohler-1973),-as-well-as-manufacturing-processes-(Mula,-Peidro,-Díaz-Madroñero-&-Vicens-2010).- Reduced-order-modeling-is-only-one-among-manyareas-where-bilinear-models-have-gained-interest.- Bilinear-models-have-beenutilized-for-control-system-design-(Schelfhout-1996,-Martineau,-Burnham,-Haas,-Andrews-&-Heeley-2004, Goodhart, Burnham-&-James-1994), fault-detection-(Yu-&-Shields-1996)-and-system-analysis-(Younis,-Abdel-Rahman-&-Nayfeh-2003).-In-(Martineau-et-al.-2004)-a-bilinear-PID-control-strategy-has-been-proposed.- Theirapproach-comprises-of-a-standard-linear-PID-cascaded-with-a-bilinear-compensator.-This-has-been-used-to-control-an-industrial-furnace-where-the-bilinear-PIDhas-been-observed-to-reduce-power-consumption.-In-(Goodhart-et-al.-1994)-a-bilinear-self-tuning-pole-placement-strategy-is-proposed. This-control-strategy-hasbeen-applied-to-an-industrial-heat-treatment-furnace. Comparisons-made-withan-industrial-PID-controller-show-encouraging-results-and-indicate-that-adoptingadaptive-bilinear-approaches-can-provide-significant-improvements. Also, -in-(Yu-&-Shields-1996)-a-diagnostic-observer-for-a-bilinear-system-with-unknown-inputsis-proposed. By-using-a-bilinear-fault-detection-observer, -residuals-with-a-highsensitivity-to-a-larger-class-of-faults-can-be-achieved.

In- the- following- subsections,- a- bilinear- model- structure- and- model- orderreduction-techniques-for-bilinear-models-will-be-discussed.- This-will-be-followedby- nonlinear- models- and- model- order- reduction- approaches- proposed- for- thismore-general-class-of-nonlinear-systems.-

3.1 Definition of bilinear systems

In-literature, bilinear-models-can-be-found-in-different-forms-(Zajic-2013)-but-inthis-chapter-and-subsequent-ones, we-focus-on-those-of-the-form:-

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u_i + Bu \tag{3.1}$$

$$y = Cx, \tag{3.2}$$

where $A \in \mathbb{R}^{n \times n}$, $N_i \in \mathbb{R}^{n \times n}$ for i = 1, 2, ..., m, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and zero-initial-condition, $(x_0 = 0)$, is assumed. For a single-input single-output (SISO)-model, m and p are one. Otherwise if they are both greater than one, the bilinear system is of multi-input multi-output (MIMO). Other variations of this configuration exist such as multi-input-single-output (MISO) and singleinput-multi-output (SIMO). Generally, the bilinearity is defined by a product of the system states and inputs (Mohler 1973). Therefore, for a fixed input, the bilinear model is linear in state. Also for a fixed state, it is linear in the input.

In-(Phillips-2000,-Rugh-1981,-Bai-&-Skoogh-2006),-bilinear-models-have-been-

used-to-approximate-nonlinear-models-of-the-form-

$$\dot{x} = f(x) + Bu \tag{3.3}$$

$$y = Cx \tag{3.4}$$

where f is a nonlinear function such that $f : \mathbb{R}^n \to \mathbb{R}^n$, $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$.

3.2 Volterra series representation of bilinear and nonlinear systems

Using-the-Volterra-series-functional, the input-u(t)-and-output-y(t)-relationship-of-nonlinear-systems-can-be-mapped. This-is-done-by-using-an-infinite-polynomial-sum-of-homogenous-terms-in-the-form-of-

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} h_n(\sigma_1, \dots, \sigma_n) u(t - \sigma_1) \dots u(t - \sigma_n) d\sigma_1 \dots d\sigma_n.$$
(3.5)

Some-cases-of-static-nonlinear-systems-described-by-a-polynomial-series-in-thestate-are-also-part-of-these-sets-of-nonlinear-systems-

$$\dot{x} = a_1 x + \dots + a_\infty x^\infty \tag{3.6}$$

$$\dot{x} = \sum_{n=1}^{\infty} a_n x^n. \tag{3.7}$$

For example, consider a system of differential equations which describes a nonlinear system as given in (3.3)-(3.4). The differential equations can be represented in an infinite polynomial form with convergence properties that retain the input output relationship of the nonlinear system. One useful polynomial expansion for describing the behaviour of nonlinear systems is the Taylor series expansion. The Taylor series is a representation of a nonlinear/linear function f(x) as an infinite sum of derivative terms calculated from the function at a single point.

At-a-finite-derivative-term, the Taylor-series-is-defined-mathematically-as-

$$f(x) = A_0 + A_1(x-a) + A_2((x-a)\otimes(x-a)) + A_3((x-a)\otimes(x-a)\otimes(x-a)) + \dots + A_n((x-a)\otimes(x-a)) + \dots + A_n((x-a)\otimes(x-a))), \quad (3.8)$$

where the coefficient of the i^{th} term, $A_i, i = 1, 2, 3, \dots n$ is the i^{th} derivative of the function f(x) evaluated at the point a and \otimes is the Kronecker product. Due to time and space considerations, it is quite common and efficient to use the truncated form of the Taylor series expansion. This concept was formally introduced by Brook Taylor in 1715 (Taylor 1715) although it was first discovered by James Gregory (Roy 1990). It is important to note that the form presented here is around a point zero. The Taylor series centred at zero, is also called a Maclaurin series:

$$f(x) = A_1(x) + A_2((x) \otimes (x)) + A_3((x) \otimes (x) \otimes (x)) + \dots + A_n((x) \otimes \dots \otimes (x)). \quad (3.9)$$

The Taylor series provides a framework for the reduction of nonlinear systems. Some authors have proposed direct techniques applied to the truncated expansion. The quadratic approximation and bilinear approximation are linked to the Taylor series and will be discussed in the next section.

3.2.1 Multimoment for bilinear systems

The input-output relationship of systems are often represented using the convolution theorem (Rugh 1981, Bai & Skoogh 2006). Consider a SISO bilinear model. This can be described by using an infinite sum of convolution integrals to describe the input-output characteristics of a bilinear model (3.1) - (3.2),

$$y(t) = \sum_{k=1}^{\infty} y_k(t),$$
 (3.10)-

where $y_k(t)$ is the output of the k^{th} subsystem and can be represented as

$$y_k(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} h(t_1, t_2, \dots, t_k) u(t - t_1 - t_2 - \dots - t_k) \dots \times u(t - t_k) dt_k \dots dt_1, \quad (3.11)$$

where $h(t_1, t_2, \ldots, t_k)$ -is-the-kernel-also-known-as-the-impulse-response-and-canbe-represented-as-

$$h(t_1, t_2, \dots, t_k) = Ce^{At_k - 1}N \dots e^{At_2}Ne^{At_1}B.$$
(3.12)

A-multivariable-Laplace-transform-of-the-kernels-can-be-used-to-define-a-transferfunction-for- $h(t_1, t_2, \ldots, t_k)$ -as-given-below.-

$$H(s_1, s_2, \dots, s_k) = C(s_k I - A)^{-1} N(s_{k-1} I - A)^{-1} N \dots$$
$$(s_2 I - A)^{-1} N(s_1 I - A)^{-1} B. \quad (3.13)^{-1} N(s_1 I - A)^{-1} B.$$

Also, the concept of transfer functions for time-invariant bilinear systems/modelshas been discussed in (Bai & Skoogh 2006). $H(s_1, s_2, \ldots, s_k)$ is referred to as the transfer function of the k^{th} subsystem and it can be expanded in a multivariable Maclaurin series such that

$$H(s_1, \dots, s_k) = \sum_{l_k=1}^{\infty} \dots \sum_{l_1=1}^{\infty} m(l_1, l_2, \dots, l_k) s_1^{l_1-1} s_2^{l_2-1} \dots s_k^{l_k-1}$$
(3.14)

with-

$$m(l_1, \dots, l_k) = (-1)^k C A^{-l_k} N \dots A^{-l_2} N A^{-l_1} B$$
(3.15)

 $\label{eq:consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-and-its-Maclaurin-series-expansion-consider-the-first-system-transfer-function-con$

$$H(s_1) = C(s_1 I - A)^{-1} B, \qquad (3.16)^{-1} B,$$

$$H(s_1) = \sum_{l_1=1}^{\infty} m(l_1) s_1^{l_1-1}, \qquad (3.17)^{l_1-1}$$

respectively-and-its-moments-

$$m(l_1) = -CA^{-l_1}B. (3.18)^2$$

Also-consider-the-second-subsystem-transfer-function-and-expansion-are-represented-as-

$$H(s_1, s_2) = C(s_2 I - A)^{-1} N(s_1 I - A)^{-1} B$$
(3.19)

$$H(s_1, s_2) = \sum_{l_2=1}^{\infty} \sum_{l_1=1}^{\infty} m(l_1, l_2) s_1^{l_1 - 1} s_2^{l_2 - 1}$$
(3.20)

and-the-associated-multimoments-are-

$$m(l_1, l_2) = CA^{-l_2} NA^{-l_1} B. (3.21)^{-1}$$

The aim of Krylov subspaces model order reduction for bilinear systems/modelsis-to-match-as-many-multimoments, i.e. $m(l_1) = \hat{m}(l_1)$ and $m(l_1, l_2) = \hat{m}(l_1, l_2)$ of the original model in a reduced order model. As will be shown in subsequentsections, the order of the reduced order model increases with the number of multimoments matched.

3.2.2 Multimoment for MIMO bilinear systems

As-has-been-discussed-in-Section-3.2.1, the input-output-relationship of a bilinearsystem- can- be- represented- in- a- Volterra- series. The output- $y_k(t)$ - of the k^{th} subsystem-is-given-as-

$$y_k(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} h(t_1, t_2, \dots, t_k) \cdot \left(u \left(t - \sum_{i=1}^k t_i \right) \otimes \dots \otimes u(t - t_k) \right) dt_k \dots dt_1$$
(3.22)-

In (3.22), the degree k kernel, $h(t_1, t_2, ..., t_k)$ of the MIMO bilinear model is given as

$$h(t_1, t_2, \dots, t_k) = Ce^{At_k} N(I_m \otimes e^{At_k - 1})(I_m \otimes N) \dots \underbrace{(I_m \otimes \dots \otimes I_m}_{k - 2} \otimes e^{At_2})$$
$$\underbrace{(I_m \otimes \dots \otimes I_m}_{k - 2} \otimes N) \underbrace{(I_m \otimes \dots \otimes I_m}_{k - 1} \otimes e^{At_1})(\underbrace{I_m \otimes \dots \otimes I_m}_{k - 1} \otimes B), \quad (3.23)$$

where A, B and C are the state matrix, input and output matrices respectively. N consists of the bilinear state matrices

$$N = [N_1, N_2, \dots, N_m]. \tag{3.24}$$

Consequently, the k^{th} transfer-function-can-be-derived-from-a-multivariable-Laplace-transform-and-in-given-as-

$$H(s_{1}, s_{2}, \dots, s_{k}) = C(s_{k}I - A)^{-1}N[I_{m} \otimes (s_{k-1}I - A)^{-1}](I_{m} \otimes N) \cdots$$

$$[\underbrace{I_{m} \otimes \dots \otimes I_{m}}_{k-2} \otimes (s_{2}I - A)^{-1}](\underbrace{I_{m} \otimes \dots \otimes I_{m}}_{k-2} \otimes N) \cdots$$

$$[\underbrace{I_{m} \otimes \dots \otimes I_{m}}_{k-1} \otimes (s_{1}I - A)^{-1}](\underbrace{I_{m} \otimes \dots \otimes I_{m}}_{k-1} \otimes B) \cap (3.25) \cdots$$

$$= C(s_k I - A)^{-1} N[I_m \otimes (s_{k-1} I - A)^{-1} N]) \dots$$

$$[\underbrace{I_m \otimes \dots \otimes I_m}_{k-2} \otimes (s_2 I - A)^{-1} N] . [\underbrace{I_m \otimes \dots \otimes I_m}_{k-1} \otimes (s_1 I - A)^{-1} B]. \quad (3.26)^{-1}$$

Given-(3.25),-its-Maclaurin-series-expansion-is-derived-as-

$$H(s_1, s_2, \dots, s_k) = \sum_{l_k=1}^{\infty} \dots \sum_{l_1=1}^{\infty} m(l_1, l_2, \dots, l_k) s_1^{l_{1-1}} s_2^{l_{2-1}} \dots s_k^{l_{k-1}}, \qquad (3.27)$$

where $m(l_1, l_2, \ldots, l_k)$ is the multimoments of the k^{th} subsystem such that

$$m(l_1, l_2, \dots, l_k) = (-1)^k C A^{-l_k} N(I_m \otimes A^{-l_{k-1}} N) \cdots (I_m \otimes \dots \otimes A^{-l_2} N)^{-1} (I_m \otimes \dots \otimes I_m \otimes A^{-l_1} B).$$

$$(3.28)^{-1}$$

For-example, the transfer-function and the Taylor series expansion of the second subsystem are represented respectively as

Therefore, the corresponding multimoments of the MIMO bilinear model is-

$$m(l_1, l_2) = CA^{-l_2} N(I_m \otimes A^{-l_1} B).$$
(3.29)

The first transfer function is the same as that of the SISO case except for C and B being matrices.

3.3 Bilinearization of nonlinear systems

The-bilinearization-process-described-here-has-been-described-in-(Rugh-1981)-andis-called-the-Carleman-bilinearization.- We-consider-several-classes-of-nonlinearsystems-for-which-the-Carleman-bilinearization-can-be-applied.-

3.3.1 Input affine nonlinear system with constant input matrix

Consider-the-input-affine-nonlinear-system-with-constant-input-matrix-(3.3)--(3.4).- The-resulting-bilinear-system-is-an-approximation-of-a-nonlinear-model-ofthe-form-(3.1)-- (3.2).- Using-the-Taylor-series-expansion-of-the-nonlinear-function-

$$f(x) \approx A_1 x^{(1)} + A_2 x^{(2)} + A_3 x^{(3)} \dots + A_i x^{(i)}$$
(3.30)

and-a-definition-of-new-states-x,-

$$x^{(-)} = [x^{(1)}x^{(2)}x^{(3)}\dots x^{(i)}]^T,$$
(3.31)

a-bilinear-approximation-is-achieved-where- $x^{(1)} = x, x^{(2)} = x \otimes x, x^{(3)} = x \otimes x \otimes x,$ and-so-on.- Generally, $x^{(i)} = x \otimes x \otimes \ldots \otimes x \in \mathbb{R}^{n^i}$.-

During the bilinearization process, it is crucial to note the relationship between each state element of current state in x, i.e. $x^{(i)}$, the time derivative of the original systems state, \dot{x} , and the previous state in x, $x^{(i-1)}$. The next example illustrates this relationship.

Example 3.3.1 Consider a state definition where $x = [x^{(1)}x^{(2)}]$. For $x^{(1)}$,

$$\dot{x}^{(1)} = \dot{x} = A_1 x^{(1)} + Bu. \tag{3.32}$$

For $x^{(2)}$,

$$\begin{aligned} \dot{x}^{(2)} &= -\frac{d}{dy} [x \otimes x] \\ &= \frac{d}{dt} [x^{(1)} \otimes x^{(1)}] \\ &= -\dot{x} \otimes x^{(1)} + x^{(1)} \otimes \dot{x} \\ &= (A_1 x^{(1)} + Bu) \otimes x^{(1)} + x^{(1)} \otimes (A_1 x^{(1)} + Bu) - (3.33) -$$

Since $\dot{x} = [\dot{x}^{(1)}\dot{x}^{(2)}]$, a bilinear model can be defined as

$$\dot{x} = A x + N x u(t) + B u$$
 (3.34)-

$$y = C \quad x \quad , \tag{3.35}$$

where,

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & [A_1 \otimes I + I \otimes A_1] \end{bmatrix}, N = \begin{bmatrix} 0 & 0 \\ [B \otimes I + I \otimes B] & 0 \end{bmatrix}$$
(3.36)

$$B = \begin{bmatrix} B \\ 0 \end{bmatrix}, C = \begin{bmatrix} C & 0 \end{bmatrix}$$
(3.37)

with $A \in \mathbb{R}^{(n+n^2)\times(n+n^2)}$, $N \in \mathbb{R}^{(n+n^2)\times(n+n^2)}$, $B \in \mathbb{R}^{(n+n^2)\times 1}$, $C \in \mathbb{R}^{1\times(n+n^2)}$.

For-a-state-definition-where-the-higher-order-Taylor-series-expansions-are-used, the-system-matrices-(A and-N),-output-and-input-vectors-(C , B)-as-defined-in-(Phillips-2000)-are-given-below-

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots \\ 0 & A_{21} & A_{22} & 0 & \cdots \\ \vdots & 0 & A_{31} & A_{32} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & \cdots & \\ B_{20} & 0 & \cdots & \\ B_{30} & 0 & \cdots & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$
(3.38)
$$B_{30} & 0 & \cdots & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$
$$B = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}, C = \begin{bmatrix} C & 0 & \cdots & 0 \end{bmatrix},$$
(3.39)

where $A_{ki} = A_i \otimes I \otimes \cdots \otimes I + I \otimes A_i \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes A_i$ for k > 1-and $A_{1i} = A_i$. Note that there are k terms and k - 1-Kronecker products. Likewise $B_{k0} = B \otimes I \otimes \cdots \otimes I + I \otimes B \otimes \cdots \otimes I + I \otimes I \otimes \cdots \otimes B$. The dimension of the state matrices increases exponentially A, $N \in \mathbb{R}^{(n+n^2+\cdots+n^i)\times(n+n^2+\cdots+n^i)}$.

Similarly, $B \in \mathbb{R}^{(n+n^2+\dots+n^i)\times 1}, C \in \mathbb{R}^{1\times(n+n^2+\dots+n^i)}$. In this formulation, the dimensions of zero matrices, 0, are as required in order to achieve the correct dimensions.

3.3.2 Input affine nonlinear systems

Another-class-of-nonlinear-systems-which-can-be-bilinearized-using-the-Carlemanbilinearization-procedure-is-the-general-input-affine-nonlinear-system-

$$\dot{x} = f(x) + g(x)u \tag{3.40}$$

$$y = Cx. \tag{3.41}$$

In this case, the elements of the input matrix g(x) are nonlinear functions of states. This has been discussed in (Breiten & Damm 2010, Rugh 1981). Using the same state definition as has been used in Subsection 3.3.1, the bilinearization process is possible as the nonlinear function g(x) can also be expressed as a Taylor series expansion

$$g(x) = G_0 + G_1 x + \dots + G_2(x \otimes x) + G_3(x \otimes x \otimes x) + \dots + G_n(x \otimes \dots \otimes x).$$

$$(3.42)^2$$

An-example-of-a-nonlinear-system-of-this-class-being-bilinearized-is-presented-in-Chapter-4.-

The Carleman bilinearization process is quite useful in engineering applications. Control design (Sanchez & Collado 2010), system identification (Juang & Lee 2012), filtering (Germani, Manes & Palumbo 2005*b*, Germani, Manes & Palumbo 2005*a*), motion tracking (Sayem, Braiek & Hammouri 2010, Sayem, Braiek & Hammouri 2013) are some of the applications for which it has been found to be extensively used. Its application in the use for model complexity reduction (Ghasemi, Ibrahim, Gildin et al. 2014) has been researched widely.

One-of-the-limitations-of-this-process-of-approximating-nonlinear-models-isthe-resulting-exponentially-increasing, high-dimensions-of-the-bilinear-models. This brings about the need for model order reduction. The time efficiency of Krylov subspace projection techniques makes them ideal for solving this problem. When compared to methods such as balanced truncation and H_2 model reduction for bilinear systems, where the computation of Lyapunov equations of high dimensions is necessary, the Krylov subspace projection techniques prove to be of great advantage. In some cases, the use of other model order reduction techniques are practically impossible.

3.4 Stability of bilinear models

There-exist-different-definitions-of-stability-for-bilinear-systems-as-discussed-in-(Dunoyer-1996),-some-of-which-are-related-to-stability-as-defined-for-linear-systems.-However,-in-order-to-sufficiently-guaranty-the-stability-of-bilinear-models,it-is-convenient-to-consider-a-bounded-input-bounded-output-stability-BIBO.-In-(Bose-&-Chen-1995,-Kotsios-1995,-Bibi-2004,-Siu-&-Schetzen-1991)-sufficient-conditions-have-been-given-on-the-input-of-bilinear-models-to-ensure-BIBO-stability.-The-following-definition-of-BIBO-stability-for-bilinear-models-can-be-found-in-(Zhang-&-Lam-2002)-

Definition 3.4.1 The bilinear system model of the form (3.1)-(3.2)-is said to be BIBO stable if for a bounded input, the output is bounded on $[0, \infty)$.

The-following-theorem-for-BIBO-stability-can-be-found-in-(Siu-&-Schetzen-1991,-Flagg-2012,-Zhang-&-Lam-2002)-

Theorem 3.4.1 : For a bilinear system model of the form (3.1)⁻- (3.2), suppose there exists an M > 0-so that the input $||u|| = \sqrt{\sum_{i=1}^{m} |u_i|^2}$ satisfies $||u|| \leq M$ for all t greater than zero. Let $\Gamma < \sum_{i=1}^{m} ||N_i||$. Then the output, y, given from the inputs, u_i , is bounded on $[0, \infty]$ -if there exist scalars $\beta > 0$ - and $0 < \alpha \leq$ $-max_i(Re(\lambda_i(A)))$, such that $||e^{At}|| \leq \beta e^{-\alpha t}$, $t \geq 0$ - and $\Gamma < \alpha/M\beta$ From this theorem, it-can be seen that the system represented by (3.1) - (3.2) is-BIBO stable if A is stable and N_i , i = 1, ..., m are sufficiently bounded (Zhang & Lam 2002).

3.5 MOR for bilinear models

Linear-model- order-reduction-approaches-such-as-Krylov-subspace-projection-(Lohmann-&-Salimbahrami-2000), -balanced-truncation-(Aizad-et-al.-2014)-and- H_2 model-reduction-(Gugercin-et-al.-2008)-have-been-extended-to-bilinear-models-(Phillips-2000, -Bai-&-Skoogh-2006, -Breiten-&-Damm-2010, -Condon-&-Ivanov-2007, -Hartmann, -Zueva-&-Schäfer-Bung-2010, -Benner-&-Breiten-2012*a*, -Zhang-&-Lam-2002, -Couchman, -Kerrigan-&-Böhm-2011)-with-all-their-disadvantagesand-advantages.-

Gramian-based-model-order-reduction-tecniques-have-been-proposed-for-bilinear-systems-as-described-in-(Al-Baiyat-&-Bettayeb-1993,-Benner-&-Damm-2011,-Condon- &-Ivanov-2005,-Couchman- et-al.-2011,-Hartmann- et-al.-2010).- Theuse-of-balanced-truncation-was-first-proposed-in-(Al-Baiyat-&-Bettayeb-1993).-Similar-to-MOR-for-linear-systems,-the-computation-of-observability-and-controllability-Gramians-via-two-Lyapunov-equations-of-the-bilinear-model-is-essential.-Different-notions-of-these-Lyapunov-equations-have-been-described-in-(Condon-&-Ivanov-2005).- These-Gramians-can-then-be-used-for-balancing-followed-bytruncation-of-the-system-matrices.- The-limitations-of-balanced-truncation-areeven-more-prominent-for-bilinear-models-derived-from-Carleman-bilinearizationdue-to-the-exponential-increase-in-model-dimensions-which-in-turn-increases-thecomputational-complexity-of-solving-the-two-Lyapunov-equations.-

3.6 Krylov subspace MOR for bilinear models

In-2000, Phillips proposed a method which matches the multimoments of a bilinear model (Phillips 2000). This approach has influenced most of the workdone-so-far (Bai & Skoogh 2006, Feng & Benner 2007, Breiten & Damm 2010). (Bai & Skoogh 2006) proposed a method which tries to match the moments and multimoments of the bilinear model. Feng and Benner discuss a one-sided approach which they claim is equivalent to the work done by Bai and Scoogh. Aslightly different approach proposed by (Condon & Ivanov 2007) which uses a linear approximation of the bilinear model about a small constant input over a finite time period can also be explored for one sided projection.

Comparative studies of these approaches have been done in (Baur et al. 2014, Feng & Benner 2007, Bai 2002). In this section, these methods are to be discussed in some detail with numerical simulations done to compare their inputoutput preservation qualities using predefined performance criteria. An improved approach for moment matching is also proposed. The approach proposed in (Bai & Skoogh 2006) cannot be referred to as one-sided because of the awkward formulation of the reduced system matrices, therefore it will not be considered here. Note that in this section, only the methods which have been proposed for SISO-model structures are discussed.

3.6.1 Petrov-Galerkin projection for bilinear models

Considering-a-bilinear-system-of-the-form-

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u + Bu$$
 (3.43)

$$y = Cx. \tag{3.44}$$

In this subsection we focus on the case for $m = 1, A \in \mathbb{R}^{n \times n}, N_1 = N \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1}, C \in \mathbb{R}^{1 \times n}$ and we assume zero initial condition, (x = 0), is

assumed. Approximating system states will result in a system of lower dimension. Projection methods try to achieve this using the approximation $x \approx V\hat{x}$, where \hat{x} is the new set of states. Hence, (3.43) - (3.44) can be rewritten as

$$V\dot{\hat{x}} = AV\hat{x} + NV\hat{x}u + Bu \tag{3.45}$$

$$\hat{y} = CV\hat{x}.\tag{3.46}$$

 $\label{eq:premultiplying-(3.45)-by-the-transpose-of-V, results-in-a-set-comprising-of-a-new-system-matrix, -input-and-output-vectors-$

$$V^T V \dot{\hat{x}} = V^T A V \hat{x} + V^T N V \hat{x} u + V^T B u$$
(3.47)

$$\hat{y} = CV\hat{x} \tag{3.48}$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{N}\hat{x}u + \hat{B}u \tag{3.49}$$

$$\hat{y} = \hat{C}\hat{x} \tag{3.50}$$

of-lower-dimensions. Of-particular-importance-is-the-condition-that $V^T V = -I$. This-is-because-orthogonal-matrix-computations-contain-less-numerical-noise-(Tan-&-He-2007). The-reduced-system-is-of-q states, $q << n, q \in \mathbb{Z}$, with-system-matrices, -input-and-output-vectors-labelled- $\hat{A}, \hat{N}, \hat{B}$ and \hat{C} , -with-

$$\hat{A} = V^T A V \in \mathbb{R}^{q \times q} \tag{3.51}$$

$$\hat{N} = V^T N V \in \mathbb{R}^{q \times q} \tag{3.52}$$

$$\hat{B} = V^T B \in \mathbb{R}^{q \times 1} \tag{3.53}$$

$$\hat{C} = CV \in \mathbb{R}^{1 \times q}. \tag{3.54}$$

This is often referred to as the Petrov-Galerkin projection (Flagg 2012). Inanother analogy, the Petrov-Galerkin projection is derived by defining the state approximation $x \approx V\hat{x}$ such that $\hat{x} \in \mathbb{R}^q$ and enforcing the Petrov-Galerkin condition $W^T R = 0$, i.e. requiring R to be orthogonal, where R is the residual

$$R = Ax + Nxu + Bu - \dot{x} \tag{3.55}$$

and W and V are matrices with columns that span suitable subspaces. Premultiplying (3.55) by W^T and substituting x with $V\hat{x}$ results in

$$W^T R = W^T A V \hat{x} + W^T N V \hat{x} u + W^T B u - W^T V \dot{\hat{x}}$$

$$(3.56)$$

$$0 = W^T A V \hat{x} + W^T N V \hat{x} u + W^T B u - \dot{\hat{x}}.$$
(3.57)

The reduced order model is defined as in (3.49) and (3.50) where $\hat{A} = -W^T A V \in \mathbb{R}^{q \times q}$, $\hat{N} = -W^T N V \in \mathbb{R}^{q \times q}$, $\hat{B} = -W^T B \in \mathbb{R}^{q \times 1}$, $\hat{C} = -CV \in \mathbb{R}^{1 \times q}$.

In both analogies, it is required to find appropriate matrices V and/or W. This can be achieved by using Krylov subspace techniques. For one-sided Krylov subspace projection for bilinear systems, W = V.

3.6.2 Phillips type projection

In-(Phillips-2000),- a- multimoment- matching- approach- has- been- proposed- byusing-the-Krylov-subspaces-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B)$$
(3.58)

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, NV^{\{k-1\}})$$
(3.59)

$$span\{V\} = span\{\bigcup_{k=1}^{k} span\{V^{\{k\}}\}\},$$
 (3.60)

where $V^{\{k\}}$ is the basis of the q_k^{th} Krylov subspace $K_{q_k}(\mathbb{M}, \mathbb{N})$. The Krylov subspace (3.58), as defined, matches $q_1 - 1$ moments of the first subsystem of the bilinear model.

In the numerical studies which will be presented in this thesis, only $V^{\{1\}}$ and $V^{\{2\}}$ are used for computing V, i.e. the Krylov subspaces and projection matrices are defined as

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B)$$
(3.61)

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}})$$
(3.62)

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^{\{k\}}\}\}.$$
 (3.63)

The projection matrix, V, is computed as a union of $V^{\{1\}}$ and $V^{\{2\}}$. The dimension of V is therefore $n \times (q_2q_1 + q_1)$, where n is the dimension of A, q_1 and q_2 refer to the Krylov subspaces $K_{q_1}(A^{-1}, B)$ and $K_{q_2}(A^{-1}, NV^{\{1\}})$ respectively. The formulation of V and its dimension is the same for the other types of projection types to be discussed in this section.

3.6.3 Feng and Benner type

In-the-work-influenced-by-the-approach-of-(Phillips-2000)-and-(Bai-&-Skoogh-2006),-Feng-and-Benner-have-proposed-the-Krylov-subspaces-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(3.64)

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, A^{-1}NV^{k-1})$$
(3.65)

$$span\{V\} = span\{\bigcup_{k=1}^{k} span\{V^k\}\}$$

$$(3.66)$$

for-matching-the-maximum-amount-of-moments. These-sets-of-Krylov-subspacebases-are-said-to-match-the-same-amount-of-moments-as-in-(Bai-&-Skoogh-2006).-In-this-case, the-Krylov-subspaces- $span\{V^{\{1\}}\}$ and $span\{V^{\{2\}}\}$ are-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(3.67)

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$$
(3.68)

and-the-corresponding-projection-vector-is-formed-as-

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^k\}\}.$$
 (3.69)-

In (Bai & Skoogh 2006) an approach which multiplies the system equation by the inverse of the state transition matrix is used. In (Breiten & Damm 2010) a generalization of the methods discussed in (Bai & Skoogh 2006) has been proposed. They proposed a multiplication of the system matrix by an appropriate non-singular matrix of the same dimensions. In this case, the computation of the reduced order matrices is different from the projection methods described in this section.
3.6.4 Condon type Krylov subspace projection

Using-a-slightly-different-approach-from-the-other-authors-discused-in-this-sectionis-a-method-proposed-by-(Condon-&-Ivanov-2007).- Consider-a-method-whichmatches-only-moments-of-a-single-variable-expansion-of-the-bilinear-model-abouta-small-input- $u = -\eta$ (Condon-&-Ivanov-2007).- If-a-bilinear-model-is-analysedover-a-finite-time-interval- $t \in [0, \tau]$,-then-it-is-possible-to-analyse-it-as-a-linearmodel.- The-validity-of-this-linear-approximation-has-been-discussed-extensivelyin-(Condon-&-Ivanov-2005).- The-resulting-system-is-

$$\dot{x} = Ax + Nx\eta + Bu \tag{3.70}$$

$$y = Cx. \tag{3.71}$$

Applying the definition of the Krylov subspace (2.4) (defined in Chapter 2), q moments of (3.70) - (3.71) can be matched using the Krylov subspace

$$span\{V\} = K_{q_1}(A_{\eta}^{-1}, A_{\eta}^{-1}B),$$
 (3.72)

where $A_{\eta} = [A + N\eta]^{-}$ and $H_{\eta}(s) = C(sI - A_{\eta})^{-1}B$. Making use of the Petrov-Galekin projection procedure, the reduced system with, $\hat{A} = V^{T}AV$, $\hat{N} = V^{T}NV$, $\hat{B} = V^{T}B$, $\hat{C} = CV$, is achieved. In this thesis, we will refer to this as a Condon type, as it was used in (Condon & Ivanov 2007). Condon & Ivanov have used a two-sided approach to improve the output of the reduced model, but in this thesis it will be implemented using a one-sided approach.

Note-that-the-major-difference-between-the-projection-types-discussed-in-thissection-is-the-definition-of-the-Krylov-subspaces.- These-slight-differences,-as-willbe-shown-in-subsequent-subsections-and-sections,-can-have-a-significant-effect-onthe-input-output-relationship-preservation-for-the-reduced-order-model.-

Since-the-works-proposed-by-(Phillips-2000,-Feng-&-Benner-2007,-Condon-&-Ivanov-2007),-there-has-been-a-lot-of-interest-in-the-reduction-of-bilinear-modelsusing-Krylov-subspaces.- The-works-done-in-(Benner-&-Breiten-2015, Flagg-2012, -Breiten-&-Damm-2010, Benner-& Breiten-2012a)-show-multimoment-matchingat-some-frequencies.- Using-Krylov-subspaces,- (Breiten-&-Damm-2010)-showmultimoment-matching-using-this-approach-and-demonstrated-their-work-using-two-numerical-simulations.- In-their-conclusion,-it-has-been-noted-that-thisapproach-needs-improvement.- An-extension-of-the-IRKA,-namely-the-bilinear-IRKA-was-first-proposed-in-(Benner-&-Breiten-2012a).- This-approach-appliesa-two-sided-rational-Krylov-iterative-procedure-for-computing-a-reduced-ordermodel. The reduced order model is said to satisfy the conditions for H_2 optimality-and-methods-of-this-form-are-regarded-as- H_2 model-order-reduction.- Thebilinear-IRKA (BIRKA) has been described to be very expensive in (Choudhary-& Ahuja 2016). This is because the algorithm solves the projection matrices by using the solutions of two generalised Sylvester equations. The truncated BIRKA-(TBIRKA)-(Flagg-2012)-was-proposed-for-this-purpose.- Using-Krylovsubspace-methods-for-solving-the-Sylvester-equations,-the-computational-efficiency-of-the- H_2 norm-approach-has-been-improved. However, this-is-at-the-costof-solving-the-Sylvester-equations-to-some-tolerance-which-can-deteriorate-thequality-of-the-resulting-reduced-order-model.-

3.7 Krylov subspace MOR for MIMO bilinear models

Many applications of bilinear models are MIMO systems. Similar to the case of moment matching for SISO linear models as discussed in Chapter 2, multimoment matching and other classical model order reduction techniques (Phillips 2000, Feng & Benner 2007, Hartmann et al. 2010) proposed for bilinear models can be extended to MIMO models. In (Hartmann et al. 2010), different balanced truncation approaches were proposed for MIMO bilinear models. Likewise, in $(Benner-\&-Breiten-2012a,-Zhang-\&-Lam-2002),-H_2 methods-were-proposed-and-implemented.- The-structural-differences-between-SISO-and-MIMO-models-also-pose-a-set-of-different-challenges-one-of-which-is-multiple-N matrices.- Intrinsically,-MIMO-models-have-input- and-output-matrices-as-opposed-to-row- and-column-vectors-respectively.- Also,-they-might-possess-multiple-system-matrices.- Some-of-the-problems-with-extending-multimoment-matching-to-MIMO-bilinear-models-have-been-dealt-with-in-linear-cases.- In-(Tan-&-He-2007)-a-block-Arnoldi-algorithm-which-is-capable-of-computing-orthonormal-basis-for-two-starting-matrices-is-presented.-$

Krylov-subspace-model-order-reduction-techniques-for-MIMO-bilinear-modelsof-the-form-

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u_i + Bu \tag{3.73}$$

$$y = Cx, \tag{3.74}$$

where $A \in \mathbb{R}^{n \times n}$, $N_i \in \mathbb{R}^{n \times n}$ for i = 1, 2, ..., m, and $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ with $n \in \mathbb{Z}, m \in \mathbb{Z}$, and $p \in \mathbb{Z}$ was first proposed in (Lin, Bao-& Wei-2007) wherein the Phillips (Phillips 2000) type projection was extended to bilinear MIMO models. In a subsequent paper written by the same authors (Lin et al. 2009), the Bai type projection framework and algorithm was extended to MIMO models. In their work, they compared both approaches using arbitrary bilinear models and showed superior reduced order models using the Phillips type (Phillips 2000) approach.

The application of the Petrov-Galerkin condition and state approximation $x = V\hat{x}$ to (3.74)-yields a reduced order model of the form

$$\dot{\hat{x}} = \hat{A}\hat{x} + \sum_{i=1}^{m} \hat{N}_i \hat{x} u_i + \hat{B}u$$
(3.75)-

$$\hat{y} = \hat{C}\hat{x} \tag{3.76}$$

such that if $V \in \mathbb{R}^{n \times q}$, then, $\hat{A} \in \mathbb{R}^{q \times q}$, $\hat{N}_i \in \mathbb{R}^{q \times q}$, $\hat{B} \in \mathbb{R}^{q \times m}$, $\hat{C} \in \mathbb{R}^{p \times q}$, where

 $\hat{A} = V^T A V, \hat{N}_i = V^T N_i V, \hat{B} = V^T B, \hat{C} = CV.$ Both analogies discussed in this chapter for SISO bilinear systems also apply here.

3.7.1 Krylov subspace for MIMO Phillips type projection

In-the-work-presented-by-Lin,-Bao-and-Wei-(Lin-et-al.-2007)-an-algorithm-thatcomputes-the-projection-bases-for-MIMO-bilinear-was-proposed.- The-following-Krylov-subspaces-were-said-to-match-the-multimoments-of-the-MIMO-bilinearmodel.-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B)$$
(3.77)

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, N_i V^{\{1\}})$$
(3.78)

$$i = 1, 2, \dots, m \tag{3.79}$$

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}.$$
 (3.80)

The algorithm presented was shown to be able to preserve the input output characteristics of the high order bilinear model. In (Baur et al. 2014) the work done by Lin, Bao and Wei has been identified as an extension of the Phillips type projection of SISO bilinear models.

3.7.2 Bai type projection

An alternative to the projection formulation for the reduced order models as proposed for one-sided and two-sided projections for bilinear systems has been utilised in (Bai & Skoogh 2006). This alternative formulation has been achieved by premultiplying the bilinear system equation by A^{-1}

$$A^{-1}\dot{x} = x + A^{-1}Nxu + A^{-1}Bu.$$
(3.81)-

Utilizing the change in state approximation, $x = V\hat{x}$, such that $\hat{x} \in \mathbb{R}^{q \times q}$.

$$A^{-1}\dot{x} = V\hat{x} + A^{-1}NV\hat{x}u + A^{-1}Bu \tag{3.82}$$

$$\hat{y} = CV\hat{x}.\tag{3.83}$$

Premultiplying (3.82)-with V^T results in a reduced order system.

$$V^{T}A^{-1}V\dot{\hat{x}} = V^{T}V\hat{x} + V^{T}A^{-1}NV\hat{x}u + V^{T}A^{-1}Bu$$
(3.84)-

$$\hat{y} = CV\hat{x}.\tag{3.85}$$

Note-that-due-to-the-orthogonality-of-V,-the-system-matrix-for-the-reduced-ordersystem-is-identity,- $V^T V = -I$.- However,-an-equivalent-definition-of-reduced-ordermatrices-can-be-achieved-using-the-inverse-of- $V^T A^{-1} V$.- Therefore-

$$\dot{\hat{x}} = (V^T A^{-1} V)^{-1} \hat{x} + (V^T A^{-1} V)^{-1} V^T A^{-1} N V \hat{x} u + V^T A^{-1} V^T A^{-1} B u \quad (3.86)^{-1} \hat{y} = C V \hat{x} \quad (3.87)^{-1} V \hat{y} = C V \hat{x}$$

he new definition of the reduced system matrix
$$\hat{A}$$
 is $(V^T A^{-1} V)^{-1}$. This new

The new definition of the reduced system matrix \hat{A} is $(V^T A^{-1} V)^{-1}$. This new definition can be made computationally effective by using \hat{A} for computing the other system matrices as follows:

$$\hat{N} = \hat{A}V^T A^{-1} N V \tag{3.88}$$

$$\hat{B} = \hat{A}V^T A^{-1} B \tag{3.89}$$

$$\hat{C} = CV. \tag{3.90}$$

According-to-(Bai-&-Skoogh-2006), if-it-is-assumed-that- $VV^T = I$, then-it-canbe-proved, for-the-Bai-type-projection, that-for-k = 1, 2, the q^k moments-of-thereduced-order-model- $\hat{m}(l_1, l_2)$ -matches- q^k moments-of-the-higher-order-model $m(l_1, l_2)$ -by-utilising-the-Krylov-subspaces-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(3.91)

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}N_iV^{\{1\}}) \cdot i = 1, 2, \dots, m$$
^m
(3.92)

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{N} span\{V_i^{\{2\}}\}\}\}, \qquad (3.93)$$

where $l_1, l_2 = 1, 2, ..., q$. Numerical-studies-carried-out-by-the-authors-of-(Bai-&-Skoogh-2006)-have-shown-that-using-this-method, the-input-output-relationship-of-higher-order-bilinear-models-is-preserved-in-the-reduced-order-models.

3.8 Algorithms for computing projection matrices

For computing the projection matrix, V, a modified version of the algorithm proposed in (Bai & Skoogh 2006) is described in this section. The algorithm has been shown to be useful for matching moments and multimoments (Bai & Skoogh 2006, Breiten & Damm 2010). The required inputs to the algorithm are the starting matrices for the first Krylov subspace M, \mathbb{N} and q_1 for computing $V^{\{1\}}$. Another parameter, p_2 , is required to select the number of columns of $V^{\{1\}}$ to be used for computing $V^{\{2\}}$. q_2 is used for computing the second Krylov subspace $spanV^{\{2\}}$. Note that the bases, M, \mathbb{N} will vary depending on the method used. The outline of this algorithm is as follows:

Algorithm 3.1 (Computation of projection base, V)

- 1. **Input:** $\mathbb{N}, \mathbb{M}, q_1, p_2, q_2, N$
- 2. Compute: r = M
- 3. Compute: $v_1^{\{1\}} = r/||r||_2$
- 4. for $i = 1 : q_1 1$
- 5. $r = \mathbb{N}v_i^{\{1\}}$
- 6. $h = (V_{[i]}^{\{1\}})^T r$

7.
$$r = -r - V_{[i]}^{\{1\}}h$$

8. $if \cdot ||r||_2 = 0$, end
9. $v_{i+1}^{\{1\}} = -r/||r||_2$
10. end
11. $G = -\mathbb{N}NV_{[p_2]}^{\{1\}}$
12. $V^{\{2\}} = -orth(G)$
13. $for \cdot i = 1 : \cdot p_2(q_2 - 1)$
14. $r = -\mathbb{N}v_i^{\{2\}}$
15. $h = (V_{[p_2+i-1]}^{\{2\}})^T r$
16. $r = -r - V_{[p_2+i-1]}^{\{2\}}h$
17. $if \cdot ||r||_2 = 0$, end
18. $v_{p_2+i}^{\{2\}} = -r/||r||_2$
19. end
20. $V = -orth([V^{\{1\}}V^{\{2\}}])$

The routine from step-3-to-step-10-and step-12-to-step-19-can-be-recognised-as-the-Arnoldi-process-which-has-been-used-to-compute- $V^{\{1\}}$ and $V^{\{2\}}$ respectively.-Step-11-is-used-to-define-the-base-for- $V^{\{2\}}$ which-is-dependent-on- $V^{\{1\}}$.-Using-the-in-built-MATLAB-function-**orth** in-steps-12-and-20,-a-QR-decomposition/Gram-Schmidt-process-is-carried-out-to-generate-an-orthonormal-basis-for-the-range-of-G and- $[V^{\{1\}}V^{\{2\}}]$.-The-output-of-the-algorithm-is-V which-is-to-be-used-for-

 $computing \ the \ reduced \ order \ model \ as \ discussed \ in \ Section \ 3.6.1.$

The projection matrix-computed here using Algorithm 3.1 is for the Feng and Benner (Feng & Benner 2007) type projection where $\mathbb{N} = A^{-1}$ and $\mathbb{M} = A^{-1}B$ as at step 11, $NV_{[p_2]}^{\{1\}}$ is multiplied by \mathbb{N} . This is not required in Phillips type projection (Phillips 2000), as only $NV_{[p_2]}^{\{1\}}$ is used instead. The subscript $[p_2]$ denotes the number of columns of $V^{\{1\}}$ which are used in forming $V^{\{2\}}$.

3.9 MIMO projection algorithms

In order to construct a projection matrix for matching multimoments of a MIMObilinear model, a series of algorithms need to be put in place. Firstly, a block Arnoldi process as described in (Tan & He 2007, Lin et al. 2009) is needed in order to compute an orthonormal basis for a block Krylov subspace $K_q(\mathbb{N}, \mathbb{M})$ to handle the matrix \mathbb{M} . The Block Arnoldi algorithm is presented below:

```
Algorithm 3.2 (Block Arnoldi for MIMO models)

1. Input: \mathbb{N}, \mathbb{M}, q

2. Q = orth(\mathbb{M})^{-}

3. W = Q

4. for i = 1 : q - 1^{-}

5. R = -\mathbb{N}Q

6. R = -R - W(W^{T}R)^{-}

7. Q = orth(R)^{-}

8. W = [W, Q]^{-}
```

9. end

10. Return W

At-the-end-of-the-algorithm, -a-matrix-W is-returned-where-

$$W = [W^{\{1\}}, W^{\{2\}}, \dots, W^{\{q\}}]$$
(3.96)

and the columns of W form basis for the Krylov subspace spanned by $K_q(\mathbb{N}, \mathbb{M})$.

Algorithm-3.2-can-be-used-for-computing-the-projection-matrix-V for-a-bilinear-model-using-the-procedure-in-the-following-algorithm-proposed-in-(Linet-al.-2009).- A-similar-version-of-this-algorithm-proposed-for-the-Phillips-typeprojection-method-has-been-presented-in-(Lin-et-al.-2007).-

Algorithm 3.3 (Computation of V for MIMO bilinear models)
1. Input: A, B, N₁,..., N_m, m, q₁, q₂, p₂
2. Compute an orthonormal basis, V^{1}, for the Krylov subspace: K_{q1}(A⁻¹, A⁻¹B), using Algorithm 3.2
3. for i = 1 : m, compute an orthonormal basis, V^{2}_i, for the Krylov subspace: K_{q2}(A⁻¹, A⁻¹NV^{{11}_[p2]), using Algorithm 3.2
4. end
5. V = orth([V^{1}, V^{2}₁, ..., V^{{2}_m])²
6. Return V

Observing-steps-2-and-4,-the-Krylov-subspaces-utilised-here-are-for-the-Fengand-Benner-(Feng-&-Benner-2007)-type-and-the-Bai-(Bai-&-Skoogh-2006)-typeprojections. For the other projection types, the Krylov subspace starting vectors should be changed accordingly. The output of the algorithm can be used for computing reduced order MIMO bilinear models for one-sided projection using the following algorithm.

Algorithm 3.4 (Computation of matrices for MIMO models) 1. Input: V 2. $\hat{A} = V^T A V$ 3. $\hat{N}_i = V^T N_i V$ 4. $\hat{B} = V^T B$ 5. $\hat{C} = -CV$

The-Bai-type-projection-(Bai-&-Skoogh-2006)-reduced-order-matrices-can-becomputed-by-using-the-following-algorithm.-

Algorithm 3.5 (Computation of matrices for Bai type)
1. Input: V

2.
$$\hat{A} = (V^T A^{-1} V)^{-1}$$

3.
$$\hat{N}_i = \hat{A} V^T A^{-1} N_i V$$

$$4. \quad \hat{B} = \hat{A} V^T A^{-1} B$$

5.
$$\hat{C} = CV$$

The-computation-of-the-reduced-order-matrices-for-Bai-(Bai-&-Skoogh-2006)type-projection-has-been-explained-in-Subsection-3.7.2.-

3.10 Discussion

Krylov-subspace-model-order-reduction-techniques-for-bilinear-systems-have-beenquite-useful-for-reduction-of-bilinear-and-nonlinear-models.- They-have-been-successfully-applied-to-many-nonlinear-systems.- In- (Bai-&-Skoogh-2006)-a-nonlinear-transmission-line-model-and-an-electrostatic-gap-closing-actuator-have-beenreduced-using-Krylov-subspaces.- Also-(Benner-&-Damm-2011)-used-Krylov-subspaces-to-reduce-the-order-of-a-heat-transfer-model.- In-(Breiten-&-Damm-2010)-aflow-model-which-can-be-used-for-modelling-engineering-problems-such-as-trafficflow-and-gaseous-systems-has-been-reduced-for-control-applications.-

When-compared-to-other-methods-such-as-balanced-truncation-and- H_2 modelorder-reduction-for-bilinear-systems, it-has-been-reported-that-the-Krylov-subspace-approaches-are-more-desirable-due-to-the-ease-of-implementation-(Bauret-al.-2014). This-is-because-the-computation-of-controllability-and-observability-Gramians-for-high-dimensional-systems-are-very-costly-and-in-some-cases, solving-Lyapunov-equations-is-impossible-(Damm-2008). In-(Benner-&-Damm-2011)a-hybrid-approach-which-combines-the-Gramian-computation-approaches- andthe-Krylov-subspace-MOR-has-been-proposed. This-implements-the-methodsby-reducing-the-dimensions-of-the-bilinear-model-using-Krylov-subspaces-beforethe-computation-of-the-Gramians-of-a-reduced-bilinear-model-to-a-much-smallerdimension.-

One-limitation-that-Krylov-subspace-projection-methods-have-is-the-needfor-inversion-of-the-system-matrix.- There-exist-systems-whose-matrices-are-notinvertible-and-this-poses-a-problem-for-the-discussed-methods.- In-(Mach,-Pranić-&- Vandebril-2013)- and- (Chu,- Lai-&- Feng-2008),- Krylov-subspace-approacheswhich-do-not-require-explicit-matrix-inversions-have-been-proposed.- However-in(Mach-et-al.-2013)-they-have-been-reported-not-to-deliver-good-results-for-theapplication-therein-and-further-investigation-is-needed.-

The Carleman bilinearization process often produces sparse matrices and the system matrix *N* is in some cases singular. This is likely to pose some numerical issues for the Krylov subspaces where the multiplication of *N* with a matrix occurs. This is an issue which has not been discussed in literature previously and is an interesting prospect for discussion as model order reduction using Krylov subspaces have been reported to have numerical issues (Choudhary & Ahuja 2016).

3.11 Conclusion

In this chapter, model order reduction of nonlinear systems via bilinearization has been discussed focusing on the Krylov subspace based model order reduction methods for bilinear systems. By using the Taylor series expansion, the approximation of nonlinear systems is made possible not only via bilinearization but also quadratic approximation. A second derivative truncation of the Taylor series is used to explain the bilinearization process. This is followed by examples of the different applications of bilinearization. Other model order reduction approaches for nonlinear systems have also been highlighted.

In-literature, the input-output behaviour of reduced order models is determined by the amount of multimoments matched. Within the following chapter, a detailed analysis of the multimoment matching capacity of the original works done in this field is presented. This analysis which has not been seen in this forms for multimoment matching form part of the contributions of this work. The next chapter introduces two new approaches for MOR using Krylov subspaces and solves some of the problems which arise when using Krylov subspace MOR.

Chapter 4

Improved Phillips and Parametrised Linear Approximation for MOR of Bilinear Systems

4.1 Introduction

Some-bilinear-models-result-in-non-invertible-A and/or-N matrices-(Bai-&-Skoogh-2006,-Breiten-&-Damm-2010,-Couchman-et-al.-2011).- This-means-that-the-model-order-reduction-of-these-models-is-not-possible-using-the-methods-discussed-in-the-previous-chapters-that-involve-the-use-of-matrix-inversion.- Also,-when-the-N matrix-is-singular,-there-is-likely-to-be-an-irreversible-loss-of-information-when-multiplying-matrices.- The-method-proposed-in- (Feng-&-Benner-2007)-uses-a-routine-which-multiplies-two-matrices-of-the-same-dimensions- $(A^{-1}N)$.- However,-if-one-of-these-matrices-is-singular,-then-the-loss-of-information,-such-as-loss-in-rank,-affects-the-input-output-preservation-of-the-reduced-order-model.- These-are-questions-which-have-not-been-considered-in-other-work-and-an-investigation-

of-this-effect-is-presented-in-this-chapter.-

As-mentioned-earlier, the overall desire of model-order reduction is to preserve the input-output relationship of the high-order/fidelity model. Research in this topic that exploits the use of Krylov subspace projection at the expansion point s = -0, is seemingly exhaustive. In this chapter, a new method for improving the input-output preservation is proposed based on using a so called better linear approximation of the bilinear model. Some simulation studies to illustrate the usefulness of this proposed method are provided.

In this chapter, an analysis of multimoment matching for the Phillips (Phillips 2000) type projection and the Feng and Benner (Feng & Benner 2007) type projection are presented. This forms part of the original contributions of this thesis. Based on this analysis, a new approach is presented which is called the Improved Phillips (IP) type projection. A multimoment matching analysis for this new approach is also presented. This is followed by a proposal for using alternate-linear approximations for MOR-using Krylov subspaces. This is called the parametrised-linear approximation (PLA) for Krylov subspace MOR.

4.2 Multimoment matching

4.2.1 Multimoment matching for Phillips type projection

By- using- the- Krylov- subspace- $K_{q_1}(A^{-1}, B)$ - for- computing- $V^{\{1\}}$, - as- has- beenshown- in- Section- 2.3.2, - this- only- matches- $q_1 - 1$ - moments- of- the- first- transfer- function- of- the- bilinear- model. - This- consequently- affects- the- multimomentmatching- when- $V^{\{1\}}$ is- used- for- computing- $V^{\{2\}}$. - However, - this- formulation- of-Krylov-subspaces- presented- by- Phillips- (Phillips- 2000)- can- be-shown- to- matchmultimoments- of- the- multivariable- transfer- function- $H(s_1, s_2)$ - of- the- bilinearmodel.-

Theorem 4.2.1 The Krylov subspaces $K_{q_1}(A^{-1}, B)$ and $K_{q_2}(A^{-1}, NV^{\{1\}})$ when

used to compute projection vectors, V^T and V as defined in (3.61)⁻⁻ (3.63)-match the multimoments of a bilinear model (3.1)⁻⁻ (3.2)- and a reduced order bilinear model such that $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1 - 1$, $l_2 = 1, \ldots, q_2 - 1$, if the reduced order model matrices are computed as $\hat{A} = V^T A V$, $\hat{N} = V^T N V$, $\hat{B} = V^T B$ and $\hat{C} = CV$, where V spans the Krylov subspaces $K_{q_1}(A^{-1}, B)$ - and $K_{q_2}(A^{-1}, NV^{\{1\}})$ - and $V^T V = I$.

PROOF. This-multimoment-matching-property-can-be-shown-by-first-substituting-the-reduced-order-matrices-(3.51)-- (3.54)-as-given-in-Subsection-3.6.1into-the-multimoments-of-the-reduced-order-model,- $\hat{m}(l_1, l_2)$.- From-(3.15)-themultimoment-of-the-reduced-order-bilinear-model-can-be-defined-as-

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}.$$
(4.1)-

Now, using the definition of the reduced order matrices, \hat{A} , \hat{N} , \hat{B} and \hat{C} , this multimoment equation can be rewritten as

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB.$$
(4.2)

Because $A^{-(q_1-1)}B$ belongs to the Krylov subspace K_{q_1} , it can be written that $A^{-(q_1-1)}B = V^{\{1\}}r_{(q_1)}$. Then $B = A^{(q_1-1)}V^{\{1\}}r_{(q_1)}$. Also because $V^{\{1\}} \in V$, then $V^{\{1\}}r_{(q_1)} = Vp_{(q_1)}$, where $r_{(i)}$ and $p_{(i)}$ are appropriate parameters and dimensions, where $r_{(i)} \in \mathbb{R}^{q_1}$, $p_{(i)} \in \mathbb{R}^{q_1+q_1q_2}$ for $i \leq q_1$ and $p_{(i)} \in \mathbb{R}^{(q_1+q_1q_2)\times q_1}$ for $i > q_1$. From the routine of moment matching as shown in (2.40) - (2.43), for any value of $q_1 \in \mathbb{Z}|q_1 > 0$, when $l_1 = q_1 - 1$.

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNVp_{(q_1)}.$$
(4.3)-

Since $Vp_{(q_1)} = V^{\{1\}}r_{(q_1)}$ then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)}.$$
(4.4)-

Moreover, from (3.62) - (3.63), $NV^{\{1\}} \in V$, therefore $NV^{\{1\}} = Vp_{(q_1+1)}$ and

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TVp_{(q_1+1)}r_{(q_1)}$$

$$= CV(V^TAV)^{-l_2}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}.$$
(4.5)

Further, since $A^{-1}NV^{\{1\}} \in V$ and $A^{-1}NV^{\{1\}} = A^{-1}Vp_{(q_1+1)} = Vp_{(q_1+2)}$, therefore,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TAVp_{(q_1+2)}r_{(q_1)}$$

$$= CV(V^TAV)^{-l_2+1}p_{(q_1+2)}r_{(q_1)}.$$
(4.6)

Using-this-routine-iteratively-until $q_2 = (l_2 + 1), (4.6)$ -becomes-

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CVp_{(q_1+l_2+2)}r_{(q_1)}.$$
(4.7)-

Now $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2+2)}$, so

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NV^{\{1\}}r_{(q_1)}.$$
(4.8)-

Since $V^{\{1\}}r_{(q_1)} = A^{-(q_1-1)}B$ and $l_1 = q_1 - 1$, then,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B.$$
(4.9)

Therefore,-

$$\hat{m}(l_1, l_2) = -m(l_1, l_2), \tag{4.10}$$

where $l_1 = 1, \dots, q_1 - 1$ and $l_2 = 1, \dots, q_2 - 1$.

This-proof-forms-an-extension-of-the-proof-of-moment-matching-as-done-is-(Tan-&-He-2007)-to-multimoment-matching.-A-proof-of-multimoment-matching-for-(3.61)-- (3.63)-has-been-done-in-(Phillips-2000).- However,-the-author-has-stated-that- l_1 can-be-less-than-or-equal-to- q_1 .- This-has-been-shown-not-to-be-the-case-here.- Also-unique-to-this-proof-is-the-relationship-between- l_2 and- q_2 .-

Consequently, this proof can be generalised for cases where the transfer function has more than two variables i.e. for $H(s_1, s_2, \ldots, s_k)$,

$$\hat{m}(l_1, l_2, \dots, l_k) = m(l_1, l_2, \dots, l_k),$$
(4.11)-

such that $l_1 = 1, 2, ..., q_1 - 1, l_2 = 1, 2, ..., q_2 - 1, ..., l_k = 1, 2, ..., q_k - 1$ where $V^{\{k\}}$ for k = 1, 2, ..., as defined in (3.58)-(3.59) have been used for computing the projection matrix V (3.60).

4.2.2 Multimoment matching for Feng and Benner type projection

Theorem 4.2.2 Given the Krylov subspaces as proposed by Fend and Benner (Feng & Benner 2007), $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$, multimoments of the bilinear model (3.1) - (3.2) and a reduced order model can be matched such that $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2$, when the reduced order model matrices are computed as $\hat{A} = V^T AV$, $\hat{N} = V^T NV$, $\hat{B} = V^T B$ and $\hat{C} = -CV$ with V spans the Krylov subspaces $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$ and $V^T V = -I$.

PROOF. To-match-the-multimoments-of-the-multivariable-transfer-function,the-Krylov-subspaces-proposed-by-Feng-and-Benner-(Feng-&-Benner-2007)-canbe-used-following-a-similar-procedure-as-in-Section-4.2.1.- The-multimoments-ofthe-reduced-order-model-are-defined-as-

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}.$$
(4.12)-

Using the definition of the reduced order matrices we have, $\hat{A} = V^T A V$, $\hat{N} = V^T A V$, $\hat{B} = V^T B$ and $\hat{C} = -CV$. Substituting these matrices into (4.12), gives

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB. \quad (4.13)$$

Because $A^{-q_1}B = V^{\{1\}}r_{(q_1)}$, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)}$$
(4.14)-

$$= CV(V^T A V)^{-l_2} V^T A A^{-1} N V^{\{1\}} r_{(q_1)}.$$
(4.15)-

From (3.68), $A^{-1}NV^{\{1\}} \in V$, therefore $A^{-1}NV^{\{1\}} = Vp_{(q_1+1)}$ and

$$\hat{C}\hat{A}^{-l_{2}}\hat{N}\hat{A}^{-l_{1}}\hat{B} = CV(V^{T}AV)^{-l_{2}}V^{T}AVp_{(q_{1}+1)}r_{(q_{1})}$$

$$= CV(V^{T}AV)^{-l_{2}+1}p_{(q_{1}+1)}r_{(q_{1})}$$

$$= CV(V^{T}AV)^{-l_{2}+1}V^{T}AA^{-1}Vp_{(q_{1}+1)}r_{(q_{1})}.$$
(4.16)

Moreover, $A^{-2}NV^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$ therefore,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = -CV(V^TAV)^{-l_2+1}V^TAVp_{(q_1+2)}r_{(q_1)}$$

$$= -CV(V^TAV)^{-l_2+2}p_{(q_1+2)}r_{(q_1)}.$$
(4.17)

Following-this-routine-results-in-

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = -CVp_{(q_1+l_2)}r_{(q_1)}$$

$$= -CA^{-l_2}NV^{\{1\}}r_{(q_1)}$$
(4.18)-

for any value of l_2 . Note that $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2)}$. Since $V^{\{1\}}r_{(q_1)} = A^{-q_1}B$,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B.$$
(4.19)

Therefore,-

$$\hat{m}(l_1, l_2) = m(l_1, l_2),$$
(4.20)

where $l_1 = 1, \ldots, q_1, l_2 = 1, \ldots, q_2$ $p_{(i)}, i = 1, \ldots, (q_1 + q_2)$ and $r_{(i)}, i = 1, \ldots, q_1$ are appropriate parameters for achieving orthogonality.

The same approach used in (Tan & He 2007) for linear systems momentmatching has been extended to multimoment matching for bilinear models. In the proof for multimoment matching shown in (Feng & Benner 2007), it has been assumed that $VV^T = I$. This is not the case in this thesis. The condition for computing the projection matrices, i.e. $V^TV = I$ has been used for showing multimoment matching. This proof can also be generalised for cases where the transfer function has more than two variables. For $H(s_1, s_2, \ldots, s_k)$,

$$\hat{m}(l_1, l_2, \dots, l_k) = m(l_1, l_2, \dots, l_k),$$
(4.21)

such that $l_1 = 1, 2, ..., q_1, l_2 = 1, 2, ..., q_2, ..., l_k = 1, 2, ..., q_k$ where $V^{\{k\}}$ for k = 1, 2, ..., as defined in (3.64)-(3.65) have been used for computing the projection matrix V (3.66).

4.3 Improved Phillips type projection

The method proposed by Phillips (Phillips 2000) which has been discussed in Subsection 3.6.2 can be readily improved because it has been shown that the Krylov-subspace-used to compute $V^{\{1\}}$ matches only $q_1 - 1$ -moments of its linear approximation. Further observation of the Phillips (Phillips 2000) type projection shows that in order to match more moments of the linear approximation of the bilinear model, the Krylov-subspace-bases (3.58) to compute $V^{\{1\}}$ should be replaced by

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B).$$
(4.22)-

This formulation for computing $V^{\{1\}}$ would match q_1 moments, i.e. if we are only considering the linear approximation of the bilinear model. This implies that the Krylov subspace multi-moment matching of the bilinear model can be improved by using (4.22). Also, since the methods so far use the system matrix A for the linear approximation of the bilinear model, exchanging A with a so-called better linear approximation and using it for computing $V^{\{1\}}$ will result in improved input-output preservation for the reduced bilinear model. This approach can be demonstrated numerically as we will show in Section 4.4 and Section 4.5.

In-Subsection-4.3.1, an analysis of multimoment matching will be shown. The approach presented here is an extension of the proof for moment matching as shown in (Tan & He 2007). This approach has also been used to analyse matched multimoments for methods proposed by Phillips (Phillips 2000) as well as Feng and Benner (Feng & Benner 2007) in Subsections 3.6.2 and 3.6.3 respectively. An improved parametrised linear approximation for MOR of the bilinear model will also be discussed in Section 4.4.

4.3.1 Multimoment matching for Improved Phillips projection

For-matching-the-multimoments-of-the-multivariable-transfer-function,-the-following-Krylov-subspaces-is-proposed-in-this-thesis:-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(4.23)

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}})$$
(4.24)-

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^k\}\}.$$
(4.25)

Theorem 4.3.1 (Improved Phillips projection) For a bilinear system/model as defined in (3.1)-- (3.2), a reduced order model of dimensions $q_1q_2 + q_1$ can be constructed by using projection matrices, V and V^T , $V^T V = I$, If V is computed using the Krylov subspaces (4.23)- and (4.24). This formulation of Krylov subspaces matches multimoments of the multivariable transfer function $H(s_1, s_2)$ - of the bilinear model such that $\hat{m}(l_1, l_2) = m(l_1, l_2), l_1 = 1, \ldots, q_1, l_2 = 1, \ldots, q_2 - 1$. This has been called the Improved Phillips type projection projection.

PROOF. From (3.15) the multimoment of the reduced order bilinear modelcan be defined as

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}.$$
(4.26)

Also, using the definition of the reduced order matrices, $\hat{A} = V^T A V$, $\hat{N} = V^T N V$, $\hat{B} = V^T B$ and $\hat{C} = C V$, and substituting the matrices into (4.26), gives

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB.$$
(4.27)-

Because, $A^{-q_1}B = V^{\{1\}}r_{(q_1)}$, where $r_{(i)}$ appropriate parameters for achieving orthogonality, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)}$$
(4.28)

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From-(4.24), $NV^{\{1\}} \in V$, therefore $NV^{\{1\}} = Vp_{(q_1+1)}$, where $p_{(i)}$ are appropriate parameters for achieving orthogonality, thus

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TVp_{(q_1+1)}r_{(q_1)}$$

$$= CV(V^TAV)^{-l_2}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}.$$
(4.29)

Since $A^{-1}NV^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$

$$\hat{C}\hat{A}^{-l_{2}}\hat{N}\hat{A}^{-l_{1}}\hat{B} = CV(V^{T}AV)^{-l_{2}}V^{T}AVp_{(q_{1}+2)}r_{(q_{1})}$$
$$= CV(V^{T}AV)^{-l_{2}+1}p_{(q_{1}+2)}r_{(q_{1})}$$
$$= CV(V^{T}AV)^{-l_{2}+1}V^{T}AA^{-1}Vp_{(q_{1}+2)}r_{(q_{1})}.$$
(4.30)

Moreover, $as^{-2}NV^{\{1\}} = Vp_{(q_1+3)} = A^{-1}Vp_{(q_1+2)}$

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2+1}V^TAVp_{(q_1+3)}r_{(q_1)}$$

$$= CV(V^TAV)^{-l_2+2}p_{(q_1+3)}r_{(q_1)}.$$
(4.31)-

Using-this-iterative-scheme,-it-can-be-derived-that-

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CVp_{(q_1+l_2+1)}r_{(q_1)}$$

$$= CA^{-l_2}NV^{\{1\}}r_{(q_1)}$$
(4.32)-

for any value of l_2 , where it is true that $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2+1)}$. Since $V^{\{1\}}r_{(q_1)} = A^{-q_1}B$, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B.$$
(4.33)-

Note that $p_{(i)}$ are matrices for $i > q_1$ otherwise they are vectors and $r_{(i)}$ are vectors for SISO-bilinear models. Using this analysis, it can be observed that the Krylov subspaces (4.23) - (4.24) match the multi-moments, $\hat{m}(l_1, l_2)$ and $m(l_1, l_2)$, such that $l_1 = 1 \dots q_1$, and $l_2 = 1 \dots q_2 - 1$.

This-proof-of-multimoment-matching-for-the-Improved-Phillip-approach-isunique-to-this-thesis.-

4.3.2 For higher order subsystem of the bilinear model

In-order-to-compute-the-projection-matrices-for-higher-order-subsystems-of-thebilinear-model,-the-following-definition-of-Krylov-subspaces-is-then-used:-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(4.34)

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, NV^{\{k-1\}})$$

$$(4.35)$$

$$span\{V\} = span\{\bigcup_{k=1}^{k} span\{V^{\{k\}}\}\}$$
 (4.36)

where k is the number of Krylov subspaces computed. This formulation matches the multimoments with all the indices $l_1, l_2, l_3, \ldots, l_k$.

The new-results for the proof of multimoment matching using the Improved-Phillips approach can also be generalised for a higher order subsystem of the bilinear model i.e. for $H(s_1, s_2, \ldots, s_k)$,

$$\hat{m}(l_1, l_2, \dots, l_k) = m(l_1, l_2, \dots, l_k),$$
(4.37)

such that $l_1 = 1, 2, ..., q_1, l_2 = 1, 2, ..., q_2 - 1, ..., l_k = 1, 2, ..., q_k - 1$ where $V^{\{k\}}$ for k = 1, 2, ..., as defined in (4.34)-(4.35) have been used for computing the projection matrix V (4.36).

Remark 4.3.1 Note that the formulation of V and its dimension are the same for all the types of projection discussed in this section. The only difference between these projection types is the definition of the matrices configuration at the initial setting of the Krylov subspaces. These seemingly slight differences as will be shown in subsequent subsections and sections have a significant effect on the input-output relationship preservation for the reduced order model.

The advantage of using the Improved Phillips approach is that it is a compromise between the Phillips type projection (Phillips 2000) and the Feng and Benner (Feng & Benner 2007) type projection. It improves the matched linear moments when computing $V^{\{1\}}$ and reduces the loss of information which occurs

by-multiplying-the-inverse-of-the-system-matrix-A with-N.- This-is-for-the-casewhere N is a singular matrix.

Parametrised linear approximation for mul-4.4 timoment matching

The formulation of the projection matrix $V^{\{1\}}$ for the methods, i.e. Phillips-(Phillips 2000)-type-projection, Feng- and Benner (Feng- & Benner 2007)-typeprojection-and-the-newly-proposed-Krylov-subspaces-for-multimoment-matching-(4.23)-- (4.25), -aattempts-to-match-moments-of-the-linear-approximation-of-thebilinear-model.- In-this-subsection,-we-propose-a-new-and-improved-approach,allowing-any-linear-approximation-of-the-bilinear-model-to-satisfy-the-conditionfor-matching-multimoments-of-the-bilinear-system.- This-approach-promises-toachieve-'better'-preservation-of-the-input-output-propertied-of-the-bilinear-model.-This-is-demonstrated-here-by-utilising-the-linear-approximation-of-

$$\dot{x} = Ax + Nxu + Bu \tag{4.38}$$

for-a-constant-input- $u = \eta$ as given-below-

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$$\dot{x} = Ax + Nx\eta + B\eta \tag{4.39}$$

$$y = Cx, \tag{4.40}$$

where $A_{\eta} = [A + N\eta]$ and $B_{\eta} = B \times \eta$, the linear approximation of the bilinear system-can-be-written-as-

$$\dot{x} = A_{\eta}x + B_{\eta}u \tag{4.41}$$

$$y = Cx. \tag{4.42}$$

This forms a linear approximation of the bilinear model for a constant input η (Flagg-2012).- This-will-be-referred-to-as-the-parametrised-linear-approximationof-the-bilinear-model.-With-this-linear-approximation,-a-Krylov-subspace-can-bedefined-such-that-

$$span\{V^{\{1\}}\} = K_{q_1}(A_{\eta}^{-1}, A_{\eta}^{-1}B_{\eta})$$
(4.43)

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, NV^{\{k-1\}})$$
(4.44)

$$span\{V\} = span\{\bigcup_{k=1}^{\kappa} span\{V^{\{k\}}\}\}.$$
 (4.45)

This-formulation-a-parametrised linear-approximation-of the-bilinear-systemallow-the-computation-of-the-projection-matrices-and-subsequently-the-reducedorder-models-that-can-be-described-as-being-one-sided.- The-parametrised-linearapproximation-has-been-applied-to-the-Improved-Phillips-type-projection-(4.43)-- (4.45).-

All-the-methods-discussed-in-Section-3.5-use-a-conventional-linear-approximation, with-transition-matrix-A, of-the-bilinear-model-for-achieving-this. Notethat-the-linear-approximation-approach-can-be-applied-to-the-other-momentmatching-approaches-such-as-Phillips-type-projection-(Phillips-2000)-and-Fengand-Benner-type-projection-(Breiten-&-Damm-2010). A-comparison-of-thesemethods-can-be-carried-out-numerically. When-reducing-the-order-of-a-bilinearmodel-via-Krylov-subspaces, the computation-of- $V^{\{1\}}$ is-done-first. As-thereexists-a-linear-approximation-of-the-bilinear-model-for-a-constant-input-appliedto-the-bilinear-systems-over-a-finite-time, consider-using-a-bilinear-model-asdescribed-in-Example-4.4.1-where-the-outputs-of-both-linear-approximations-arecompared-using-input-output-plots-over-time.-

Example 4.4.1 Linear approximation of bilinear models

Consider the bilinear model presented in (Flagg 2012):

$$A = \begin{bmatrix} -1 & 0 & 0.1667^{-} & 0^{-} \\ 0^{-} & -2 & 0 & 0.25^{-} \\ 0 & 0^{-} & -3 & 0^{-} \\ 0 & 0^{-} & 0^{-} & -4^{-} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 0^{-} \\ 2 & 0 & 0 & 0^{-} \\ 0 & 3 & 0 & 0^{-} \\ 0 & 0 & 4 & 0^{-} \end{bmatrix}, \quad (4.46)^{+}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$
(4.47)

where $A \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ and n = 4.

A-constant-input-of- $\eta = 0.5$ -is-chosen-to-achieve-a-linear-approximation-of-thesystem. This-is-compared-to-a-linear-approximation-of-the-bilinear-model-wherethe-linear-approximation-uses-only-the-system-matrix-A. All-three-models-areexcited-using-a-sinusoidal-input- $u = -\sin(t)$. The-plotted-output-of-the-models-canbe-observed-in-Figure-4.1, where-the-parametrised-linear-approximation, tends-



Figure 4.1: Comparison of linear approximation using the conventional state transition matrix and the parametrised linear approximation

to-provide-a-better-approximation-of-the-bilinear-model.-

Similar-linear-approximations-have-been-used-in-various-applications.- In-(Juang-&-Lee-2012)-a-bilinear-system-identification-algorithm-was-developed-byintroducing-constant-inputs-at-designated-sampling-points. A-linear-approximation-for-bilinear-systems-of-this-kind-was-also-used-to-compute-observability-andcontrollability-Gramians-for-bilinear-systems-in-(Condon-&-Ivanov-2005).

4.5 Case studies

4.5.1 Simulation based study

A-form-of-this-simulation-based-study-as-described-in-this-subsection-has-beenused-in- (Baur-et-al.-2014)-to-understand-the-effect-of-the-parameters-on-thereduction-process.-Here,-using-the-input-defined-as-

$$u = (\cos(2\pi t/10) + 1)/2, \tag{4.48}$$

the effect of the parameters q_1 , q_2 and p_2 on the reduced order model are investigated. It is expected that the accuracy of reduced order model will improve when the order increases and vice versa as this corresponds to the multimoments matched. However, this is not always the case as it has been reported that Krylov subspace methods lose accuracy as the subspace dimension increases. The parameters, q_1 , q_2 and p_2 affect the dimensions of the reduced order model and will also determine the input-output preservation of the reduced order model. However, it is more desirable to achieve a much lower order at a reasonably high accuracy.

Using- $V^{\{1\}}$ and $V^{\{2\}}$ to compute the projection bases, V, an experiment is carried out by defining four cases. Each case possesses different values of q_1, q_2 and p_2 for the Algorithm 3.1.

- Case 1: $q_1 = 20, q_2 = 1$ and $p_2 = 1$
- Case 2: $q_1 = 11, q_2 = 2$ and $p_2 = 5$
- Case 3: $q_1 = 5, q_2 = 16$ and $p_2 = 1$

• Case 4: $q_1 = 17, q_2 = 2$ and $p_2 = 2$

This-is-done-for-all-the-multimoment-matching-methods-discussed-in-Chapter-3-(Phillip-type-projection-(Phillips-2000)-and-Feng-and-Benner-type-projection-(Breiten-&-Damm-2010)),-and-the-Improved-Phillip-type-projection-discussed-in-Section-4.3.-

All-the-cases-presented-here-are-expected-to-achieve-the-same-reduced-orderdimensions.- The-results-are-to-be-assessed-using-performance-criteria-which-willbe-predefined,-graphic-output-and-absolute-error-plots.-

4.5.2 Projection procedure

After selecting the parameters, q_1 , q_2 and p_2 , The algorithm used to compute the projection matrices V and V^T is given in Algorithm 3.1 which requires a slight modification for the Phillips type projection. (Phillips 2000) and Improved Phillips Type projection. For the Phillips type projection, the input-vectors are $\mathbb{N} = A^{-1}$ and $\mathbb{M} = B$. For the Improved Phillips type projection, the inputvectors are $\mathbb{N} = A^{-1}$ and $\mathbb{M} = A^{-1}B$. Also in step 11, $G = NV_{[p_2]}^{\{1\}}$ for the Phillips (Phillips 2000) and Improved Phillips type projection. Note that for the Condon type projection, only $V^{\{1\}}$ has been used as the right projection base, i.e. $V = V^{\{1\}}$. This means that an algorithm which matches only moments is sufficient. An algorithm for matching moments has been presented in (Tan & He 2007). This process for constructing $V^{\{1\}}$ is know as the Arnoldi process and has been discussed in Chapter 2. After getting the projection bases, the reduced order models are computed as described in Subsection 3.6.1. An algorithm for this is given below:

Algorithm 4.1 (Reduced order matrix computation)

1. Input: V

2. $\hat{A} = -V^T A V$ 3. $\hat{N} = -V^T N V$ 4. $\hat{B} = -V^T B$ 5. $\hat{C} = -CV$

The implementation of the Algorithm and simulation of models has been carried out using Matlab and Simulink. Two examples of bilinear models are presented as discussed next. The first example demonstrates the use of case studies for determining parameters q_1, q_2 and p_2 . This is followed by the MOR of a flow model. These two examples have been used by (Bai & Skoogh 2006) and (Breiten & Damm 2010) respectively to demonstrate Carleman bilinearization and Krylov subspace MOR. In order to compare the different reduced order models, some performance criteria have been used as discussed next.

4.5.3 Performance criteria

The quality of a reduced model using Krylov subspaces is determined by how many moments are matched. Reduced models are only useful when pre-set criteria are reached.

C.1- Integral-of-absolute-error:- In-order-to-assess-the-goodness-of-fit-of-thereduced-order-model-in-the-time-domain,-a-quantitative-efficacy-index-isrequired.-In-this-thesis-the-integral-of-absolute-error-(IAE)-of-the-differencein-the-output-responses-between-the-high-order-bilinear-model-and-thereduced-order-bilinear-model-is-used,-which-is-calculated-as-

IAE =
$$\int_{0}^{n_s} |y(t) - \hat{y}(t)| dt$$
 (4.50)

where y and \hat{y} are the outputs of the higher order bilinear model and the reduced-order-bilinear-model-respectively.- Terms- n_s refer-to-the-numberof the samples collected in the time sequence and t is the time sequence. These-notations-are-also-used-for-the-remaining-performance-criteria-to-bediscussed.

C.2- Coefficient- of- determination: The- coefficient- of- determination- (RT^2) ismathematically-defined-as-

$$RT^{2} = 100 \times \frac{||\hat{y} - y||_{2}^{2}}{||y - y_{mean}||_{2}^{2}}, \qquad (4.51)^{2}$$

where y_{mean} is the mean of bilinear/nonlinear system output. The desire is-to-keep-the-coefficient-of-determination-as-high-as-possible. Generally, amodel-with-an- $RT^2 = -90$ -is-regarded-as-highly-acceptable.-

C.3- Mean-square-error: The-mean-square-error-(MSE)-computes-the-averageof-the-squared-deviations-of-the-reduced-order-model-from-the-high-ordermodel-

$$MSE = \frac{1}{n_s} \sum_{i=1}^{n_s} (\hat{y} - y)^2$$
(4.52)

and is analogous to mean squared deviation (MSD). It is widely used instatistics, regression-analysis-and-parameter-estimation.-

C.4- Sum-of-square-of-error: The-sum-of-square-of-error-(SSE)-is-a-mathematicalfunction-which-computes-the-sum-of-the-squared-errors-of-the-reduced-ordermodel-and-is-mathematically-defined-as:-

SSE =
$$\sum_{i=1}^{n_s} (\hat{y} - y)^2$$
 (4.53)

The-SSE-is-also-known-as-the-residual-sum-of-squares-(RSS)-and-the-sum-ofsquared-residuals-(SSR).-The-desire-is-to-keep-the-SSE-as-small-as-possible.-This-criterion-is-widely-used-in-parameter-and-model-selection.-

C.5- IAE-divided-by-number-of-samples-(NIAE):-The-NIAE-is-the-IAE-dividedby-the-number-of-samples-

NIAE
$$= \frac{1}{n_s} \int_0^{n_s} |y(t) - \hat{y}(t)| dt.$$
 (4.54)

C.6- Simulation-time-in-seconds-(ST)-During-the-simulations,-the-time-for-eachalgorithm-for-computing-the-projection-matrices-is-measured,-

$$ST = \frac{1}{ns} \sum_{i=1}^{ns} ST_i \tag{4.55}$$

where-ns is-the-number-of-simulations-carried-out.-

These criteria have been used to determine the accuracy of the reduced ordermodels.- Also,-the-magnitude-error-of-each-reduced-order-model-with-respect-tothe input-u and the absolute error with respect to time depicted in graphic plots have-been-used-to-analyse-the-reduced-order-models-in-Subsections-4.5.4-and-4.5.5.- Note-that-the-time-series-used-in-computations-used-in-this-thesis-havenot-been-equally-spaced-as-a-variable-time-step-has-been-used-for-simulations.-However, both-high-order-and-low-order-model-output-values-which-have-beenanalysed-using-the-performance-criteria-are-identical-in-size-and-sampling-points.-

Case study 1: A nonlinear RC circuit 4.5.4

The nonlinear model-used in this paper to compare the methods discussed is a transmission-line-model-of- 20^{th} order, i.e. n = -20-as-illustrated-in-Figure-4.2.

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Figure-4.2:- Nonlinear-circuit-

The nonlinear circuit model is of the form (3.3) - (3.4) discussed in Chapter 3, where f(x) = f(v), input B and output C matrices are given as

$$f(v) = f_k v = \begin{bmatrix} -g(v_1) - g(v_1 - v_2)^{-} \\ g(v_1 - v_2) - g(v_2 - v_3) \\ \vdots \\ g(v) - g(v_{k-1} - v_k)^{-} \\ \vdots \\ g(v) - g(v_{k-1} - v_n)^{-} \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$
(4.56)

The model is composed of linear capacitors which are assumed to have capacitance values of unity, i.e. C = 1, and nonlinear resistors where the resistance g(v)-is-a-function-of-voltage:-

$$g(v) = \exp(40v) + v - 1$$
 (4.57)

As-the-definition-of-the-output-vector-indicates, the-output-of-the-nonlinearcircuit-is-the-voltage-between-node-1-and-the-ground.

A-state-vector- $x = [x^{(1)}x^{(2)}x^{(3)}]$ -has-been-defined-for-the-Carleman-bilinearization-of-the-nonlinear-model. This-results-in-system-matrices, input-andouput-vectors-with-the-following-dimensions, $A \in \mathbb{R}^{8420 \times 8420}$, $N \in \mathbb{R}^{8420 \times 8420}$, $B \in \mathbb{R}^{8420 \times 1}$, $C \in \mathbb{R}^{1 \times 8420}$.

Results:

First-of-all,-the-results-of-the-Phillips-(Phillips-2000)-type-projection-for-the-casestudies-is-presented.- The-outputs-y of-the-different-cases-are-shown-in-the-firstrow-of-Figure-4.3.- The-middle-row-of-this-figure-shows-the-input-u whilst-theabsolute-error-values-are-shown-in-the-bottom-row.- In-Table-4.1,-the-performancecriteria-values-of-the-different-cases-are-shown.-

Table 4.1: Performance criteria for different experimental cases of the Phillips (Phillips 2000) type method.

Phillip-type-	RT^2	MSE-	IAE-	NIAE-	SSE-	NSSE-
Case-1-	96.98-	9.6002e-07-	0.4470-	7.4617e-04-	5.7505e-04-	9.6002e-07-
Case-2-	99.87-	4.0522e-08-	0.0799-	1.3338e-04-	2.4273e-05-	4.0522e-08-
Case-3-	99.56-	1.3929e-07-	0.1528-	2.5504e-04-	8.3435e-05-	1.3929e-07-
Case-4-	99.90-	3.0749e-08-	0.0697-	1.1632e-04-	1.8419e-05-	3.0749e-08-

As-presented, the results suggest that Case 4-with parameter values $q_1 = 17$, $q_2 = 2$, and $p_2 = 2$, produce the best results. Next, the experimental results of the Feng and Benner (Feng & Benner 2007) type projection are presented. Figure 4.4 compares the output of the reduced order model to that of the nonlinear model in the top figure. In the middle is the input *u* and the absolute error values



Figure 4.3: Time response y of bilinear model and reduced order of different cases, input u and absolute error values for all the cases using the Phillips (Phillips 2000) type projection.

are-shown-in-the-bottom-figure. In-Table 4.2, the performance-criteria-values of the different-cases 1-to-4-are-shown. For the Feng-and-Benner-projection, the, Case 2- $(q_1 = -11, q_2 = -2, p_2 = -5)$ -and-Case 4- $(q_1 = -17, q_2 = -2, p_2 = -2)$ -show-similar-results. However, Case 2-is-slightly-better-with $RT^2 = -99.86, -0.11$ -more-than-Case 4. The experimental results of the Improved Phillips type projection are presented in Figure 4.5. This compares the output of the reduced order-model-to-that-of-the-nonlinear-model-in-the-first-row. In the second-row-is the input (u)- and the absolute error values are shown-in-the third-row. In Table 4.3, the performance-criteria-values of the different-cases 1-to-4-are shown. Also-in-the experimental-values for the Improved Phillips type projection show-improved-



Figure 4.4: Time response y of bilinear model and reduced order of different cases, input (u) and absolute error values for all the cases using the Feng and Benner (Feng & Benner 2007) type projection.

values for the Cases 2 and 4 when compared to the other cases. The results for the experiment suggest that choosing q_1 too high does not necessarily improve the output of the reduced order model. Likewise, selecting a high value of q_2 does not improve the results. However, values of p_2 higher than 1 is likely to increase the input output preservation capacity of the reduced order model. These experimental results suggest that the computation of a reduced order model that preserves input output relationship which satisfies a set of performance criteria is highly dependent on $V^{\{1\}}$ and therefore the correct selection of the parameters q_1 and p_2 is critical. This knowledge has been used to manually derive a much lower reduced order model for the Phillips type projection (Phillips 2000), Feng

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Table-4.2:- Performance-criteria-for-different-experimental-cases-of-the-Feng-and-Benner-type-method.-

Feng-and-Benner-	RT^2	MSE-	IAE-	NIAE-	SSE-
Case-1-	94.51-	1.7247e-06-	0.4939-	9.3022e-04-	9.1583e-04-
Case-2-	99.86-	4.4082e-08-	0.0764-	1.4385e-04-	2.3408e-05-
Case-3-	96.46-	1.1107e-06-	0.3932-	7.4053e-04-	5.8980e-04-
Case-4-	99.75-	7.7999e-08-	0.0964-	1.8163e-04-	4.1418e-05-



Figure 4.5: Time response y of bilinear model and reduced order of different cases, input (u) and absolute error values for all the cases using the Improved Phillips type projection.

and Benner (Feng & Benner 2007), Improved Phillips and Parametrised Linear Approximation projection (PLA) techniques whose results will be shown next. The results of using a one-sided Condon (Condon & Ivanov 2007) approach are also included. The computation of $V^{\{1\}}$ for the PLA-type projection uses the

Table-4.3:- Performance-criteria-for-different-experimental-cases-of-the-Improved-Phillips-type-method.-

Improved-Phillips-	RT^2	MSE-	IAE-	NIAE-	SSE-
Case-1-	96.38-	1.1453e-06-	0.4118-	7.5701e-04-	6.2306e-04-
Case-2-	99.86-	4.3794e-08-	0.0770-	1.4160e-04-	2.3864e-05-
Case-3-	96.53-	1.0976e-06-	0.4001-	7.3544e-04-	5.9710e-04-
Case-4-	99.82-	5.5529e-08-	0.0874-	1.6058e-04-	3.0208e-05-

parameter $\eta = 0.522$. The results presented in here are for reduced order models of 7th order-where-the-parameters $q_1 = 5, p_2 = 2$ and $q_2 = 1.5$





(a) Plot of reduced order bilinear models via Condon (Condon & Ivanov 2007), Feng and Phillips (IP) and parametrise linear approxi-Benner (Feng & Benner 2007) (FB), Phillips mation (PLA) reduced order bilinear models. (Phillips 2000) (IP) and nonlinear model.

(b) Plot of nonlinear model, Improved

Figure 4.6: Plots of outputs \hat{y} for reduced order models and nonlinear model output-y.-

Figure 4.6(a)-shows-a-graphic-comparison-of-the-simulated-outputs-for-the-Condon-type-projection-(Condon-&-Ivanov-2007), Phillips-type-projection-(Phillips-2000), Feng- and Benner-type-projection (Feng-& Benner-2007), the Improved-Phillips type projection and the nonlinear circuit model. As can be observed, the approach proposed by Condon et. al. (Condon & Ivanov 2007), when applied-to-a-one-sided-projection,-is-not-as-effective-as-the-other-methods.- This-isbecause-only-moments-are-matched.-Figure-4.6(b)-compares-the-output-for-the-Improved-Phillips-type-projection-with-that-of-the-PLA-output.- The-effect-of-
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the so-called better-linear approximation of the bilinear model can be observed. This trend can also be observed in Table 4.4 which shows the RT^2 , SSE, IAE, NIAE and MSE values of the reduced order models. Comparing the results presented in Table 4.4, IP improves, in terms of RT^2 , the results found using Phillips (Phillips 2000) and Fend and Benner (Feng & Benner 2007) by 1.4% and 0.28% respectively. Applying PLA for MOR further improves on the results from IP in terms of RT^2 by 0.02% and IAE by 20%. These results confirm the effectiveness of using an alternate linear approximation for computing a reduced order system model, see Section 4.4.

The simulation time values for computing the projection matrices are 271seconds, 292-seconds, 424-seconds and 293-seconds for Phillips (Phillips 2000), Improved Phillips, Feng and Benner (Feng & Benner 2007) and PLA respectively. These values were computed using the average of 6-simulation runs. This shows similar simulation times for IP, Phillips (Phillips 2000) and PLA, with Phillips being the fastest, as expected. Feng and Benner (Feng & Benner 2007) is about 1.5-times slower than IP-due to the additional matrix inversion required.

	RT^2	MSE-	IAE-	NIAE-	SSE-
Condon-	65.01-	1.0982e-05-	1.3207-	0.0025-	00.58-
Phillips-	98.50-	4.6991e-07-	0.2846-	5.3504e-04-	2.4999e-04-
Feng-and-Benner-	99.66-	1.0718e-07-	0.1121-	2.1063e-04-	5.7020e-05-
Improved-Phillips-	99.94-	1.9686e-08-	0.0482-	9.0529e-05-	1.0473e-05-
PLA-	99.96-	1.1669e-08-	0.0385-	7.2371e-05-	6.2077e-06-

Table-4.4:- Performance-criteria-for-7th-order-reduced-order-models.-

According-to-(Rugh-1981)-the-error, $y - \hat{y}$, of-bilinear-approximations-is-a-function-of-the-input-and-number-of-Taylor-series-expansion-terms. In-Figures-4.7, 4.8-and-4.9-the-reduced-order-model-errors-corresponding-to-the-increasing-input-is-shown. Comparing-the-reduced-order-model-of-Feng-and-Benner-type-projection-with-the-Phillips-type-projection-in-Figure-4.7, the-Feng-and-Benner-type-projection-shows-less-error-for-most-of-the-input-values. This-can-also-be-

observed-in-the- RT^2 values.- However, this-is-not-the-case-when-it-is-compared-the-Improved-Phillips-type-projection-in-Figure-4.8.-



Figure 4.7: Comparison of Phillips (Phillips 2000) type projection with Fengand Benner (Feng & Benner 2007) (FB) type projection using plot of magnitude error against corresponding ascending input values.



Figure 4.8: Comparison of Improved Phillips type projection with Feng and Benner (Feng & Benner 2007) (FB) using plot of magnitude error against corresponding ascending input values.

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Figure 4.9: Comparison of Improved Phillips (IP) type projection with Parametrised Linear Approximation (PLA) approach using plot of magnitude error against corresponding ascending input values.

As-can-be-observed-in-Figure-4.9, the Improved-Phillips-type-projection-showsless-errors-for-smaller-inputs-but-as-the-input-gets-bigger, the PLA-maintains-itserror-range-whilst-the-first-method-becomes-worse. Overall, the parametrisedlinear-approximation-approach-shows-a-better-input-output-preservation-for-thenonlinear-model.

4.5.5 Case study 2: Flow model

Another-model-which-can-be-used-to-demonstrate-bilinearization-and-Krylovsubspace-model-order-reduction-for-nonlinear-models-is-the-flow-model-whichhas-been-used-in-(Breiten-&-Damm-2010).- The-system-presented-therein-is-aone-dimensional-Burgers-equation:-

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = \frac{\partial}{\partial t} \left(v \frac{\partial w}{\partial x} \right), \text{ for } (x, t) \in (0, L) \times (0, T), \tag{4.58}$$

with-

$$w(x,0) = p(x) \text{ for } x \in (0,L)$$
 (4.59)

$$w(0,t) = -u(x) \text{-for-} t \in (0,T) \text{-}$$

$$(4.60) \text{-}$$

$$w(L,t) = q(x) \text{-for-} t \in (0,T),$$
 (4.61)-

where x is a point at time, w(x, t) is the velocity, and v is the viscosity coefficient which-also-depends-on-space-and-time.- In-order-to-reduce-the-model-order,-(Breiten-&-Damm-2010)-assumed-a-constant-viscosity-coefficient.-A-zero-initialcondition-is-also-imposed-on-the-system. Only-the-left-boundary-condition-iscontrolled-while-the-right-boundary-condition-is-0.- A-spacial-discretization-of-(4.58)-results-in-a-nonlinear-control-system-with-nonlinear-functions-of-systemstates.



This-nonlinear-control-system-is-of-the-form-

$$\dot{w} = f(w) + g(w)u \tag{4.63}$$

$$y = Cw, \tag{4.64}$$

where f(w)-and g(w)-are-nonlinear-functions-with-Taylor-series-expansions-

$$f(w) = A_1 w + A_2 (w \otimes w)$$

$$(4.65)$$

and-

$$B(w) = G_0 + G_1 w, (4.66)^2$$

where $A_1 \in \mathbb{R}^{n \times n}$ and $A_2 \in \mathbb{R}^{n \times n^2}$ are the first and second derivatives of f(w). $G_0 \in \mathbb{R}^n$ and $G_1 \in \mathbb{R}^{n \times n}$ are the solution of g(w) at w = 0 and the first derivative of g(w) respectively.

A-Carleman-bilinearization-is-then-carried-out-on-the-nonlinear-control-systemby-introducing-a-state-vector-

$$x = \begin{bmatrix} w \\ w \otimes w \end{bmatrix}. \tag{4.67}$$

This-results-in-the-following-bilinear-system-matrices,-

$$A = \begin{bmatrix} A_1 & \frac{1}{2}A_2 \\ 0 & A_1 \otimes I + I \otimes A_1 \end{bmatrix}, \quad N = \begin{bmatrix} G_1 & 0 \\ G_0 \otimes I + I \otimes G_0 & 0 \end{bmatrix}$$
(4.68)
$$B = \begin{bmatrix} G_0 \\ 0 \end{bmatrix}, \quad C = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$
(4.69)

The simulations have been done with the parameters of the nonlinear model-L = 1, v = 0.1 and with state space order, n = 50.5

Using-Algorithms-3.1-and-4.1, the high-order bilinear model-is reduced to a 3rd-order-model-using-the-Phillips-(Phillips-2000)-type, Improved-Phillips-typeand-Feng- and-Benner- (Feng- & Benner-2007)- type-projection-methods. The parameters-of-Algorithms-3.1-have-been-set-as- $q_1 = -1, q_2 = -1$ -and- $p_2 = -1$. Theseparameters-have-been-obtained-experimentally. Simulation-results-using-inputfunction,-

$$u = \sin((2\pi/10) \times 50t + 50)$$
(4.70)

are-presented.-

Results:

The model-reduction-outcomes-of-this-case-study-uses-performance-criteria- RT^2 ,-MSE,-SSE,-IAE- and-NIAE- to-show- the-goodness-of-fit-for- each-reduced-order-model-which-have-been-derived-by- the-Phillips- (Phillips-2000)-type-projection,-Feng- and-Benner- (Feng-&-Benner-2007),-Improved-Phillips- and-PLA-type-projections.- Graphical-plots-have-also-been-used-to-show-the-system-outputs-and-error-values.-

Table 4.5 shows the performance criteria values of the reduced order models. The numerical figures for Phillips (Phillips 2000) type projection and the Improved Phillips type projection are identical. This is because they are quite similar in the computation of their subspaces. Whilst the figures for Feng and Benner show better results. For all the three methods, $RT^2 = -100$.

Table-4.5:- Performance-criteria-values-of-MSE,-IAE,-NIAE-and-SSE-for-reducedorder-models-via-Phillips-(Phillips-2000)-type,-Feng-and-Benner-(Feng-&-Benner-2007)-and-Improved-Phillips-type-projections.-

Methods-	MSE-	IAE-	NIAE-	SSE-
Phillips-	$5.4760e - 07^{-1}$	5.0134-	$5.3019e - 04^{-1}$	0.0052-
Fend-and-Benner-	$1.2352e - 11^{-1}$	0.0114-	$1.2074e - 06^{-1}$	$1.168e - 07^{-1}$
Improved-Phillips-	$5.4760e - 07^{-1}$	5.0134-	$5.3019e - 04^{-1}$	0.0052-

Figure 4.10 shows the output (y, average speed) plots of the reduced ordermodels and that of the high order bilinear model against time (t) in the first row. The second row shows the input plot against time (t). Figure 4.11 shows the plot absolute error of the model outputs against input in ascending orderin the first row and shows the plot of absolute error values in time. In Figure 4.11, it can be observed, as it was for Table 4.5 with MSE = 5.3019e - 04, SSE = 1.168e - 07, IAE = 1.2074e - 06 and NIAE = 1.168e - 07 that the Feng and Benner (Feng & Benner 2007) type projection produces reduced order models of-higher-accuracy-when-compared-to-Phillips-(Phillips-2000)-and-the-Improved-Phillips-type-projection-with-MSE-=-5.4760e - 07,-SSE-=-0.0052,-IAE-=-5.0134-and-NIAE-=-5.3019e - 04-when-the-flow-model-is-considered.-



Figure 4.10: Time response y (average speed) of high order bilinear model-(HOBM) and reduced order models via Phillips (Phillips 2000) (P), Feng and Benner (Feng & Benner 2007) (FB) and Improved Phillips (IP) type projections for input u.

The-application-of-PLA-via-Feng-and-Benner-(Feng-&-Benner-2007)-can-bedone-by-using-the-Krylov-subspaces-as-defined-below:-

$$span\{V^{\{1\}}\} = K_{q_1}(A_{\eta}^{-1}, A_{\eta}^{-1}B_{\eta})$$
(4.71)

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$$
(4.72)

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^{\{k\}}\}\}.$$
 (4.73)-

Computing-the-reduced-order-model-using-PLA-produces-the-following-resultsfor-simulations-carried-out-using-the-input,-

$$u = \sin((2\pi/10) \times 50t) + 500. \tag{4.74}$$





Figure 4.11: Time response y of bilinear model and reduced order of different cases, input u and absolute error values for all the cases using the Feng and Benner type projection.

Table 4.6 shows the performance criteria values for the reduced order models using Feng and Benner and PLA via Feng and Benner. The amplitude of the input has been increased in this case to highlight the advantage of the PLA approach. The linear approximation parameter has been chosen to be $\eta = 10$.

4.6 Discussion

Considering-Case-study-1,-the-proposed-Improved-Phillips-(Phillips-2000)-typeapproach-tends-to-be-as-effective-as-the-Feng-and-Benner-(Feng-&-Benner-2007)approach-for-model-order-reduction.- However,-it-seems-to-be-more-promisingwhen-computing-models-of-very-low-order-in-light-of-the-case-study.-

In-the-different-cases-of-parameters-presented-for-Case-study-1,-there-is-adeflation-in-Case-3-which-results-in-an-order-of-17.- Deflation-does-not-occur-

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Table-4.6:- Performance-criteria-values-of-IAE-and-SSE-for-reduced-order-models-via-Feng-and-Benner-(Feng-&-Benner-2007)-and-PLA-via-Feng-and-Benner-type-projection.-

Methods-	IAE	SSE
Fend-and-Benner-	73.8428-	12.5305-
PLA-	73.6805-	12.5249-

when the operating parameters of the algorithm $(q_1 = 5, q_2 = 1, p_2 = 2)$ have been reduced. This results in reduced order models of 7^{th} order.

In-Case-study-2, the-Feng-and-Benner-(Feng-&-Benner-2007)-type-projectionproduces-a-reduced-order-model-with-better-performance-criteria-when-comparedto-the-Improved-Phillips-type-and-the-Phillips-(Phillips-2000)-type-projection.-When-the-PLA-approach-is-then-applied-via-L.-Feng-and-P.-Benner-(Feng-&-Benner-2007), the-performance-criteria-values-of-the-PLA-were-improved-compared-to-those-for-the-Feng-and-Benner-(Feng-&-Benner-2007)-approach.-

Whilst-it-is-suspected-that-the-best-reduced-order-model-derived-from-all-the-simulated-methods-will-depend-on-the-type-of-matrices-possessed-by-the-high-order-system, the-parametrised-linear-approach-promises-to-reduce-the-error-when-applied-to-each-of-these-approaches. This-improved-input-output-preservation-has-been-achieved-at-very-low-cost-of-computing- A_{η} and B_{η} . When-compared-to-the-simulation-time-of-the-other-approaches, the-parametrised-linear-approximation-(PLA)-is-quite-efficient.

The implications of using an alternate linear approximation for reducing bilinear models are not only significant for preserving the input-output behaviour of the bilinear model. Their application suggests that they can be used for reducing systems with non-invertible matrices. This is possible if the alternate linear approximate of the bilinear model is non-singular. This will be discussed further in Chapters 5 and 6.

4.7 Conclusion

Two-novel-results-have-been-proposed-in-this-chapter;-the-Improved-Phillips-and-the-parametrization-of-the-linear-model.-

The present-study-showcases-the-use-of-different-Krylov-subspace-projectionmethods-which-have-been-proposed-by-other-authors-for-matching-the-momentsand-multimoments-of-a-bilinear-model.-With-the-use-of-Carleman-bilinearizationthe-approximation-of-nonlinear-models-has-been-shown.-An-improved-approachwhich-matches-multimoments-of-the-multivariable-transfer-function-of-the-resulting-bilinear-model-has-been-implemented.-In-addition,-it-has-been-demonstratedthat-the-moments-of-the-first-transfer-function-of-a-bilinear-model-can-be-improved-by-using-an-alternative-linear-approximation-of-the-bilinear-system.-Thesefindings-have-been-illustrated-by-reducing-a-bilinearised-nonlinear-circuit-model.-With-the-use-of-coefficient-of-determination,-mean-square-error-and-magnitudeerror-and-graphic-plots-a-comparison-of-the-different-Krylov-subspace-reducedmodels-has-been-carried-out.-

An experimental procedure has been done to identify the effect of the parameters of the Algorithm 3.8 on the reduced order model. The results of the experiment agree with other authors as according to (Baur et al. 2014) the input-output presevation of the reduced order model is highly dependent on the computation of $V^{\{1\}}$.

The results-obtained-here-suggest-that-the-use-of-Krylov-subspaces-for-matching-multimoments-is-quite-subjective-as-the-quality-of-the-reduced-or-model-isdependent-on-the-nature-of-system-matrices-being-used-for-computing-the-Krylovsubspace.- But-the-high-dependence-of-the-reduced-order-model-on-the-linearapproximation-of-the-bilinear-model-can-be-used-to-improve-its-input-outputpreservation-via-the-PLA-approach.-

Chapter 5

IP and PLA for MIMO Bilinear Models

The results presented in (Lin et al. 2007) and (Lin et al. 2009) propose Krylov subspaces for matching multimoments and moments for MIMO bilinear models. These are basically extensions of the Phillips type projection (Phillips 2000) and Bai-type projection (Bai 2002) which have been proposed for SISO bilinear models.

As-has-been-discussed-in-Chapter-4,-the-Krylov-subspaces-proposed-in-(Feng-&-Benner-2007)- match- the- multimoments- $m(q_1, q_2)$ - and- $\hat{m}(q_1, q_2)$. This- has-been-shown- to- be-less- effective- in- some- cases- for- preserving- the- input-output-relationship- of- the- high- order- models- when- compared- to- the- following- Krylov-subspaces-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(5.1)

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}})$$
(5.2)-

$$span\{V\} = span\{\bigcup_{k=1}^{2} span\{V^{\{k\}}\}\},$$
 (5.3)-

proposed-in-this-thesis-for-the-SISO-case-studies.-It-has-also-been-shown-that-the-PLA-approach-for-MOR-can-be-used-to-improve-input-output-preservation-in-allthe cases discussed. While the extension of Phillips (Phillips 2000) projection to MIMO Bilinear model already exists, improvement can be achieved using other methods.

In-this-chapter, the-Improved-Phillips, Feng- and Benner- (Feng- & Benner-2007)- and the parametrized linear approximation approaches for model orderreduction-are extended to MIMO-bilinear models. An analysis and multimomentmatching for bilinear models are also illustrated. The newly proposed approaches are compared to the work done in (Lin et al. 2007, Lin et al. 2009).

5.1 IP type projection for MIMO bilinear models

In order-to-extend-the-Improved-Phillips-type-projection-to-MIMO-bilinear-models,-the-multiple-bilinear-state-matrices-will-have-to-be-taken-into-consideration.- $V^{\{1\}}$ remains- the-same- with-the-SISO-case.- The-Krylov-subspace-which-contains- the-bilinear-state-matrices-is- $V^{\{2\}}$.- The-result-of-this-for-MIMO-bilinearprojection-is-that-there-are-multiple-matrices-which-are-members-of- $V^{\{2\}}$.- Thiscan-be-represented-by-utilising-indices,-i.e- $V_i^{\{2\}} \in V^{\{2\}}$,- where $i = 1, 2, \ldots, m$.-Therefore-the-Krylov-subspace- $V^{\{2\}}$ is-defined-below:-

$$span\{V^{\{2\}}\} = span\{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}$$
(5.4)

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, N_i V^{\{1\}}), \ i = 1, 2, \dots, m.$$
(5.5)

Utilizing- $V^{\{1\}}$ and $V^{\{2\}}_i,$ the projection-matrix-V is then computed as given

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}.$$
 (5.6)

The extension of Krylov subspace techniques for MOR of MIMO bilinear models follows this pattern. For completeness, the Krylov subspaces for higher k^{th}

subsystem-of-the-bilinear-model-is-given-as-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(5.7)-

$$span\{V_i^{\{k\}}\} = K_{q_k}(A^{-1}, N_i V^{\{k-1\}}), \ i = 1, 2, \dots, m$$
(5.8)

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}.$$
 (5.9)-

5.2 Feng and Benner type projection for MIMO bilinear models

The-Feng-and-Benner-(Feng-&-Benner-2007)-type-projection-for-the-model-orderreduction-of-MIMO-bilinear-models,-to-the-best-of-the-author's-knowledge,-isbeing-proposed-first-in-this-thesis.-

As- in- the- case- with- the- other- methods,- the- Krylov- subspaces- proposed- in-(Feng-&-Benner-2007)-can-be-extended-to-MIMO-bilinear-models.- This-can-beachieved- for- the- second- subsystem- of- the- bilinear- model- by- using- the- Krylovsubspaces-given-below-

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B)$$
(5.10)-

$$span\{V_i^{\{2\}}\} = K_{q_2}(A^{-1}, A^{-1}N_iV^{\{1\}}), \ i = 1, 2, \dots, m$$
(5.11)

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}.$$
 (5.12)

5.3 Multimoment matching for MIMO bilinear models

Theorem 5.3.1 For a MIMO bilinear model, the Krylov subspaces $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, N_i V^{\{1\}})$ -match the multimoments, $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2$, of the higher order and reduced order bilinear models if the reduced order model is computed such that $\hat{A} = V^T A V$, $\hat{N}_i = V^T N_i V$, $\hat{B} = V^T B$ and $\hat{C} = CV$, where $V^{\{1\}} \in V$, $V_i^{\{2\}} \in V$ and $V^T V = I$.

PROOF. If the Krylov subspaces in (5.10) - (5.12) as proposed in (Feng & Benner 2007) for projection are considered, multimoment matching for MIMObilinear systems can be established. It has been shown previously that $A^{-l_1}B \in span\{V^{\{1\}}\}, \text{ so } A^{-l_1}B = V^{\{1\}}r_{(q_1)}$ and because $V^{\{1\}} \in V, A^{-l_1}B = Vp_{(q_1)}$ for $l_1 = 1, \ldots, q_1$,

$$A^{-l_1}B = Vp_{(q_1)}. (5.13)$$

Therefore, for the multimoments of the reduced order model and using the definitions of the reduced order matrices

$$\hat{C}\hat{A}^{-l_{2}}\hat{N}(\hat{I}_{m}\otimes\hat{A}^{-l_{1}}\hat{B}) = CV[V^{T}AV]^{-l_{2}}V^{T}NV(I_{m}\otimes[V^{T}AV]^{-l_{1}}V^{T}B) = CV[V^{T}AV]^{-l_{2}}V^{T}NV(I_{m}\otimes V^{T}Vp_{(q_{1})}) = CV[V^{T}AV]^{-l_{2}}V^{T}NV(I_{m}\otimes p_{(q_{1})}).$$

$$(5.14)$$

Note-that-due-to-the-definition-of-N and the Kronecker-product, we have

$$\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} = CV[V^TAV]^{-l_2}V^TN_iVp_{(q_1)}, \text{ for } i = 1, \dots, m$$

$$= CV[V^TAV]^{-l_2}V^TN_iV^{\{1\}}r_{(q_1)}, \text{ for } i = 1, \dots, m$$

$$= CV[V^TAV]^{-l_2}V^TAA^{-1}N_iV^{\{1\}}r_{(q_1)}, \text{ for } i = 1, \dots, m.$$
(5.15)

Since $A^{-1}N_iV^{\{1\}} \in V$, therefore $A^{-1}N_iV^{\{1\}} = Vp_{(q_1+1)}$ and

$$\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} = -CV[V^TAV]^{-l_2}V^TAVp_{(q_1+1)}r_{(q_1)}$$

= $-CV[V^TAV]^{-l_2+1}p_{(q_1+1)}r_{(q_1)}$
= $-CV[V^TAV]^{-l_2+1}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}, \text{ for } i = 1, \dots, m.$
(5.16)-

Also, $A^{-2}N_iV^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} = CV[V^TAV]^{-l_2+1}V^TAVp_{(q_1+2)}r_{(q_1)}$$

$$= CV[V^TAV]^{-l_2+2}p_{(q_1+2)}r_{(q_1)}$$

$$= CV[V^TAV]^{-l_2+2}V^TAA^{-1}Vp_{(q_1+2)}r_{(q_1)}, \text{ for } i = 1, \dots, m.$$
(5.17)-

Continuing-this-routine,-it-can-be-shown-that-

$$\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} = CVp_{(q_1+l_2)}r_{(q_1)}$$

$$= CA^{-l_2}N_iV^{\{1\}}r_{(q_1)}, \text{ for } i = 1, \dots, m.$$
(5.18)-

Note-that- $A^{-l_2}N_iV^{\{1\}} = Vp_{q_1+l_2}$. Also, since $V^{\{1\}}r_{q_1} = A^{-q_1}B$,

$$\hat{C}\hat{A}^{-l_2}\hat{N}_i\hat{A}^{-l_1}\hat{B} = CA^{-l_2}N_iA^{-l_1}B, \text{ for } i = 1, \dots, m.$$
(5.19)-

Considering the definition of N and the use of a Kronecker product, we have

$$\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}^{-l_1}\hat{B}) = CA^{-l_2}N(I_m \otimes A^{-l_1}B), \qquad (5.20)$$

where $r_{(q_1)}$ and $p_{(i)}$ are appropriate parameters for achieving orthogonality and $r_{(q_1)} \in \mathbb{R}^{q_1}, p_{(i)} \in \mathbb{R}^{q_1+q_1q_2m \times 1}$ for $i \leq q_1$ and $p_{(i)} \in \mathbb{R}^{(q_1+q_1q_2m \times q_1) \times q_1}$ for $i > q_1$.

Using this analysis, it can be derived that for a one-sided projection of a MIMO-bilinear-model, the Krylov subspaces proposed in (Lin-et-al. 2007) match the multimoments, $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1 - 1$, $l_2 = , \ldots, q_2 - 1$. Also, it can be derived that the Improved Phillips type projection matches the multimoments $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2 - 1$. The proof shown here differs from those done in (Lin-et-al. 2007, Lin-et-al. 2009) for MIMO-bilinear models because they (Lin-et-al. 2007, Lin-et-al. 2009) have assumed that $VV^T = I$.

5.4 PLA for MIMO bilinear models

The parametrised linear approximation approach is also proposed for MIMO cases. Considering a MIMO bilinear model of the form (3.1)- (3.2), the multiple

bilinear-state-matrices-with-an-application-of-a-constant-input- $u = [u_1 u_2 \dots u_m]^T$ over-a-short-period-of-time-results-in-the-linear-approximation-of-the-MIMObilinear-model-where-the-state-matrix-is-

$$A_{\eta} = A + N_1 \eta_1 + N_2 \eta_2 + \dots + N_m \eta_m, \qquad (5.21)^2$$

given that $u_1 = \eta_1, u_2 = \eta_2, \dots, u_m = \eta_m$ are the so-called parameters for a linear approximation of the bilinear model. The input matrix B_η can also be defined using these parameters such that

$$B_{\eta} = B \times \eta \tag{5.22}$$

where $\eta = [\eta_1 \eta_2 \dots \eta_m]^T$. Therefore, the following set of equations can be used for computing projection bases for model order reduction using the parametrised linear approximation approach:

$$span\{V^{\{1\}}\} = K_{q1}(A_{\eta}^{-1}, A_{\eta}^{-1}B_{\eta})$$
(5.23)

$$span\{V_i^{\{2\}}\} = K_{q2}(A^{-1}, N_i V^{\{1\}}), \ i = 1, 2, \dots, m$$
(5.24)

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}.$$
 (5.25)-

As-can-be-observed,-the-set-of-equations-use-the-Improved-Phillips-type-projection-for-applying-the-parametrised-linear-approximation.- However,-the-PLA-forprojection-of-bilinear-models-can-be-applied-to-the-Phillips-type-(Phillips-2000)projection,-Feng-and-Benner-(Feng-&-Benner-2007)-and-Bai-(Bai-&-Skoogh-2006),type-projections-for-MIMO-bilinear-model-reduction.-

Two-versions-of-the-PLA-are-developed-in-this-chapter. The-first-is-(5.23)--(5.25). The-second-version-is-presented-below-

$$span\{V^{\{1\}}\} = K_{q_1}(A_{\eta}^{-1}, A_{\eta}^{-1}B_{\eta})$$
(5.26)-

$$span\{V_i^{\{2\}}\} = K_{q_2}(A_{\eta}^{-1}, N_i V^{\{1\}}), \ i = 1, 2, \dots, m$$
(5.27)

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}.$$
 (5.28)-

For systems which have noninvertible state transition matrices, (5.26) - (5.28) are used. This is because (5.24) does not use the alternate state transition matrix for computing $V^{\{2\}}$ and will not be useful in this case.

Remark 5.4.1 Using the same approach as in Theorem 5.3.1, it can be shown that the Krylov subspaces (5.23)- (5.24)-match the multimoments

$$\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}_{\eta}^{-l_1}\hat{B}_{\eta}) = CA^{-l_2}N(I_m \otimes A_{\eta}^{-l_1}B_{\eta}) - (5.29)$$

It can also be shown that the Krylov subspaces defined in (5.26)--(5.27)-match the multimoments

$$\hat{C}\hat{A}_{\eta}^{-l_2}\hat{N}(\hat{I}_m \otimes \hat{A}_{\eta}^{-l_1}\hat{B}_{\eta}) = CA_{\eta}^{-l_2}N(I_m \otimes A_{\eta}^{-l_1}B_{\eta})$$
(5.30)

Note that when as $\eta_1, \eta_2, \ldots, \eta_m$ becomes negligible,

$$\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m\otimes\hat{A}_\eta^{-l_1}\hat{B}_\eta)\cong\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m\otimes\hat{A}^{-l_1}\hat{B})$$
(5.31)-

$$CA^{-l_2}N(I_m \otimes A_\eta^{-l_1}B_\eta) \cong CA^{-l_2}N(I_m \otimes A^{-l_1}B)$$
(5.32)

and

$$\hat{C}\hat{A}_{\eta}^{-l_2}\hat{N}(\hat{I}_m\otimes\hat{A}_{\eta}^{-l_1}\hat{B}_{\eta})\cong\hat{C}\hat{A}^{-l_2}\hat{N}(\hat{I}_m\otimes\hat{A}^{-l_1}\hat{B})$$
(5.33)-

$$CA_{\eta}^{-l_2}N(I_m \otimes A_{\eta}^{-l_1}B_{\eta}) \cong CA^{-l_2}N(I_m \otimes A^{-l_1}B).$$

$$(5.34)$$

The following algorithm can be used for computing Krylov subspace projection matrix, V for PLA as presented in (5.23) - (5.27).

Algorithm 5.1 (Computation of V for MIMO models using PLA)

1. **Input:** $A, B, N_1, \ldots, N_m, m, q_1, q_2, p_2, \eta_1, \eta_2, \ldots, \eta_m$

- 2. Compute linear approximation: $A_{\eta} = A + N_1\eta_1 + N_2\eta_2 + \ldots + N_m\eta_m$, $B_{\eta} = B\eta$
- 3. Compute an orthonormal basis, $V^{\{1\}}$, for the Krylov subspace: $K_{q_1}(A_{\eta}^{-1}, A_{\eta}^{-1}B_{\eta})$, using Algorithm 3.2
- 4. for i = 1 :: m, Compute an orthonormal basis, $V_i^{\{2\}}$, for the Krylov subspace: $K_{q_2}(A^{-1}, A^{-1}N_iV_{[p_2]}^{\{1\}})$, using Algorithm 3.2
- 5. end
- 6. $V = orth([V^{\{1\}}, V_1^{\{2\}}, \dots, V_m^{\{2\}}])$
- 7. Return V

The algorithm is implemented with the condition that $p_2 \leq q_1$ where p_2 is as defined in Section 3.8. Algorithm 5.1 computes a projection matrix for implementing the PLA for MIMO bilinear models. This differs from Algorithm 3.3 in steps 1 and 2. In step 1, there are more input parameters as $\eta_1, \eta_2, \ldots, \eta_m$ are added to the algorithm and in step 2, these parameters are used for computing an alternate linear approximation of the bilinear model. The new linear approximation is then used to form $V^{\{1\}}$.

Implementing-PLA-for-singular-system-matrices, step-4-should-be-computedusing-the-alternate-system-matrix- A_{η} as-implemented-in-the-following-algorithm:-

Algorithm 5.2 (Computation of V for MIMO models using PLA)

1. **Input:** $A, B, N_1, \ldots, N_m, m, q_1, q_2, p_2, \eta_1, \eta_2, \ldots, \eta_m$

- Compute linear approximation: A_η = A + N₁η₁ + N₂η₂ + ... + N_mη_m, B_η = Bη
 Compute an orthonormal basis, V^{1}, for the Krylov subspace: K_{q1}(A_η⁻¹, A_η⁻¹B_η), using Algorithm 3.2
 for i = 1 : m, compute an orthonormal basis, V_i^{2}, for the Krylov subspace: K_{q2}(A_η⁻¹, A_η⁻¹N_iV_[p2]^{1}), using Algorithm 3.2
 end
 - 6. $V = orth([V^{\{1\}}, V_1^{\{2\}}, \dots, V_m^{\{2\}}])$
 - 7. Return V

Algorithm-5.2-differs-from-Algorithm-5.1-in-step-4.- Here-the-system-matrix-A has-been-replaced-with- A_{η} the-parametrised-linear-approximation-as-definedin-Equation-(5.21)-to-compensate-the-singularity-of-A at-the-expansion-point-ofzero.-

5.5 Numerical examples

In this section, two numerical examples of arbitrary bilinear models are used to compare the Phillips (Phillips 2000) type, Bai (Bai & Skoogh 2006) type, Feng and Benner (Feng & Benner 2007) type, the IP and PLA type projection techniques for MIMO bilinear models. The performance criteria used are RT^2 , MSE, IAE, NIAE and SSE.

5.5.1 Example 1

Consider a time-invariant-MIMO-bilinear-model of-state-space-dimension, n = -1400, number-of-inputs, m = -2, number-of-outputs, p = -3. The state-matrices-A and N_1 are given as

$$A = \begin{bmatrix} -10 & 2^{-} & 0^{-} & \cdots & 0 \\ 2^{-} & -10^{-} & 2^{-} & \ddots & \vdots^{\frac{1}{2}} \\ 0^{-} & \ddots & \cdot & \cdot & \cdot & 0^{-} \\ \vdots^{\frac{1}{2}} & \ddots & 2^{-} & -10^{-} & 2^{-} \\ 0^{-} & \cdots & 0 & 2^{-} & -10^{-} \end{bmatrix}, \quad N_{1} = \begin{bmatrix} 0^{-} & -1 & 0^{-} & \cdots & 0^{-} \\ 1 & 0^{-} & -1 & \ddots & \vdots^{-} \\ 0^{-} & \cdots & \cdot & \cdot & \cdot & 0 \\ \vdots^{\frac{1}{2}} & \ddots & 1 & 0^{-} & -1 \\ 0^{-} & \cdots & 0 & 1 & 0^{-} \end{bmatrix}. \quad (5.37)^{-}$$

Also, $N_2 = -N_1 \cdot B$ is an $n \times m$ matrix while C is a $p \times n$ matrix and are in the form given below

$$B = \begin{bmatrix} 1 & 1^{-} \\ 0 & 1^{-} \\ \vdots & \vdots^{-} \\ 0 & 1^{-} \\ 0 & 1^{-} \\ 0 & 1^{-} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1^{-} & \cdots & 1 & 1^{-} \\ 0.8 & 0.8^{-} & \cdots & 0.8 & 0.8^{-} \\ 0.5 & 0.5^{-} & \cdots & 0.5 & 0.5^{-} \end{bmatrix}, \quad (5.38)^{-}$$

where $A \in \mathbb{R}^{n \times n}$, $N_1 \in \mathbb{R}^{n \times n}$, $N_2 \in \mathbb{R}^{n \times n}$ and an initial state, x(0) = 0. The system is simulated with inputs, u_1 (See Table 5.1.) and u_2 which form the simulation input, u, where,

Table-5.1:- Table-of-input-values- u_1 for-Example-1.-

u_1	5.1449-	1.6550-	14.4810-	14.4897-	19.6743-
Time-range-(s):-	$t \in [0-:-0.9]^{-1}$	$t \in [1-1.9]^{-1}$	$t \in [2-2.9]^{-1}$	$t \in [3 - : -3.9]$ -	$t \in [4 - : -4.9]$ -
u_1	8.4851-	11.2442-	17.5686-	1.4018-	1.7853-
Time-range-(s):-	$t \in [5 : -5.9]$ -	$t \in [6 - : -6.9]$ -	$t \in [7 - : -7.9]$ -	$t \in [8 - : -8.9]$ -	$t \in [9 - : -9.9]$ -
u_1	11.74529-				
Time-range-(s):-	$t \in [10]^{-1}$				

$$u_2 = (\sin(t) + 1)/10^{-10}$$
(5.39)

$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T \tag{5.40}$$

Using-Algorithm-3.3-and-Algorithm-3.2-discussed-in-Chapter-3,-four-of-the-methods-proposed-in-(Lin-et-al.-2007,-Lin-et-al.-2009)-and-the-methods-proposed-inthis-thesis-were-applied-to-reduce-the-order-of-the-bilinear-model-(5.37)-(5.38)by-utilising-the-parameters,- $q_1 = -5, -q_2 = -5, -p_2 = -4$.- The-outputs-of-the-systemare- $y_1, -y_2, -$ and $-y_3$.- In-this-numerical-example,- the-Phillips,- Bai,- MIMO-Fendand-Benner- and-Improved-Phillips- approaches- to-Krylov-subspace-projectionare-compared.- This-comparison-is-done-using-the-performance-criteria-discussedin-Chapter-4-and-visual-output-plots.-

Results:

 $\label{eq:shows-the-} Table-5.2 \mbox{-shows-the-} RT^2, \mbox{-}MSE, \mbox{-}IAE, \mbox{-}NIAE \mbox{-}and \mbox{-}SSE \mbox{-}values \mbox{-}of \mbox{-}the \mbox{-}reduced \mbox{-}order-models \mbox{-}produced \mbox{-}by \mbox{-}the \mbox{-}methods \mbox{-}implemented. \mbox{-} As \mbox{-}can \mbox{-}be \mbox{-}observed, \mbox{-}the \mbox{-}values \mbox{-}reduced \mbox{-}be \mbox{-}observed, \mbox{-}the \mbox{-}values \mbox{-}reduced \mbox{-}reduced \mbox{-}red \mbo$

Table-5.2: Table-of-performance-criteria-values-for-Phillips-type,-Feng-and-Benner,-Bai-and-Improved-Phillips-projection.-

	MSE-	IAE-	NIAE-	SSE-
Phillips-	0.0037-	5.2415-	0.0336-	0.5746-
Bai-	0.0015-	3.2032-	0.0205-	0.2353-
Benner-	5.9975e-04-	2.0073-	0.0129-	0.0936-
Improved-Phillips-	5.4007e-05-	0.5221-	0.0033-	0.0084-

of-each-performance-criteria-for-the-different-methods-are-quite-similar. This-canalso-be-observed-in-Figure-5.1-which-shows-the-inputs- u_1 , u_2 and the-simulatedoutputs-of-the-reduced-order-models- and the-high-order-bilinear-model. Onlyone-of-the-outputs-from-these-models-has-been-plotted-which-has-been-denoted y_1 .

Figure 5.2-shows-a-zoomed-in-view-of-all-the-reduced-order-models-and-theoriginal-higher-order-bilinear-model-at-the-8-second-mark-of-Figure 5.1.- It-can-



Figure 5.1: Time response y_1 of high order bilinear model and reduced order models using inputs u_1 and u_2 .

be-seen-from-this-zoomed-in-plot-that-the-Improved-Phillips-produces-the-closestresult-to-the-original-model.- This-result-is-consistent-also-at-all-other-points-oftime.-

5.5.2 Example 2

In this example, the best-algorithm from the previous Example 1, i.e. the Improved Phillips has been applied, comparing the use of standard linearization and the parametrized linear approximation method. As in Subsection 5.5.1, a similar higher order bilinear model is used. Here, consider a time invariant 2-inputs 3-outputs bilinear model with matrices A and N_1 shown below



 $\label{eq:Figure-5.2:-Zoomed-in-time-response-of-higher-order-bilinear-model-(HOBM)-and-reduced-order-models.-$

$$A = \begin{bmatrix} -5 & 2 & 0 & \cdots & 0 \\ 2 & -5 & 2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & -5 & 2 \\ 0 & \cdots & 0 & 2 & -5 \end{bmatrix}, \quad N_{1} = \begin{bmatrix} 0 & -3 & 0 & \cdots & 0 \\ 3 & 0 & -3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 3 & 0 & -3 \\ 0 & \cdots & 0 & 3 & 0 \end{bmatrix}. \quad (5.41)$$

The matrix $N_2 = -N_1$, where $A \in \mathbb{R}^{n \times n}$, $N_1 \in \mathbb{R}^{n \times n}$. B is an $n \times m$ matrix while C-is a $p \times n$ where n = -1400, p = -3 and m = -2 and are given below

$$B = \begin{bmatrix} 1 & 1^{-} \\ 0 & 1^{-} \\ \vdots & \vdots^{-} \\ 0 & 1^{-} \\ 0 & 1^{-} \\ 0 & 1^{-} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1^{-} & \cdots & 1 & 1^{-} \\ 0.8 & 0.8^{-} & \cdots & 0.8 & 0.8^{-} \\ 0.5 & 0.5^{-} & \cdots & 0.5 & 0.5^{-} \end{bmatrix}$$
(5.42)

The model order reduction is done for an initial state of zero. The same parameters as in the Subsection 5.5.1 have also been used here i.e. $q_1 = 5, q_2 = 5, and p_2 = 4.$

The parameters η_1 and η_2 are equal-and have been chosen by trial-and error to be equal-to 0.753. The linear approximation approach for the Krylov subspaces defined in (5.23) - (5.25) have been implemented using Algorithm 5.1.

The bilinear models have been simulated using inputs u_1 and u_2 .

u_1	18.3745-	5.9107-	51.7177-	51.7490-	70.2652-
Time-range-(s):-	$t \in [0-:-0.9]^{-1}$	$t \in [1-:-1.9]^{-1}$	$t \in [2 - : -2.9]$ -	$t \in [3 - : -3.9]$ -	$t \in [4 - : -4.9]^-$
u_1	30.3039-	40.1577-	62.7450-	5.0065-	6.3760-
Time-range-(s):-	$t \in [5 : -5.9]$ -	$t \in [6 - : -6.9]$ -	$t \in [7-:-7.9]^-$	$t \in [8 - : -8.9]$ -	$t \in [9 - : -9.9]$ -
u_1	40.9033-				
Time-range-(s):-	$t \in [10]^{-1}$				

Table-5.3: Table-of-input-values- u_1 for Example-2.-

$$u_2 = (\sin(t) + 1)/10^{-10}$$
(5.43)

$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T \tag{5.44}$$

Results:

Table 5.4 shows the RT^2 , MSE, IAE, NIAE and SSE values of the reduced order models produced by using the Improved Phillips and parametrised linear approximation approaches. Figure 5.3 shows one of the outputs of the reduced order bilinear models compared to the higher order bilinear model. In order to highlight the differences between the IP and the PLA, the input u_1 has been amplified as can be observed in the second row of Figure 5.3. u_2 remains the same and has not been plotted. In the third row of Figure 5.3, the absolute errors of both reduced order models are plotted.



Figure 5.3: Time response y_1 of high order bilinear model (HOBM) and reduced order models using the Improved Phillips (IP) type projection and the Parametrised-Linear Approximation (PLA). Also plotted is the input u_1 and the absolute error values

As-can-be-observed-there-is-a-significant-increase-in-the-input-output-preservation-of-the-reduced-order-model-when-applying-the-parametrised-linear-approximation-method.-It-is-also-expected-that-the-parametrised-linear-approximationapproach-will-yield-better-results-when-applied-to-the-Phillips- (Phillips- 2000)type-projection,-Feng-and-Benner-type-(Feng-&-Benner-2007),-and-Bai-(Bai-&-Skoogh-2006)-type-projection-for-the-reduction-of-MIMO-bilinear-models.-

5.6 Conclusion

In-this-chapter, the use of Krylov-subspaces has been extended for the reduction of MIMO bilinear models and some new approaches have been proposed. The

Table- 5.4:- Table- showing- performance- criteria- for- Improved- Phillips- and-Parametrised-Linear-Approximation-for-MIMO-bilinear-models.-

	RT^2	MSE-	IAE-	NIAE-	SSE-
Improved-Phillips-	99.04-	1.8256-	365.0072-		622.5193-
PLA-	99.86-	0.2603-	139.6369-	0.4036-	8.9180-

Improved-Phillips, Feng-and-Benner-type (Feng-&-Benner-2007)-and-PLA-approaches-have-been-proposed. These-methods-have-been-compared-by-using-asimulation-study. These-methods-have-also-been-shown-to-match-multimomentsof-the-bilinear-model.

The simulation study done here shows that all the considered Krylov subspace methods discussed here perform very similarly, with slight improvement in the Improved Phillips, as seen in Figure 5.2. Further investigation into the linearization method shows that the introduction of a parametrised linear approximation (PLA) tends to make a significant impact on the accuracy of the reduced order model. This is even more in the case of MIMO models as multiple N matrices add to the linear dynamics of the system which are not represented when a standard linear approximation A is utilised for the model order reduction.

To-summarize, the main contributions to literature which are proposed in this chapter are the Improved Phillips type projection proposed for MIMO bilinear systems and the Feng and Benner type projection (Feng & Benner 2007) proposed for MIMO systems and the PLA proposed for MIMO bilinear systems.

Chapter 6

Applications of PLA and IP Projection

6.1 Introduction

Reduced- order- models- are- used- in- many- applications- such- as- control- design-(Schelfhout- 1996),- diagnostics,- hardware- in- the- loop- simulations- and- systemsdesign.- The- need- for- reduced- order- models- arises- for- various- reasons- such- ascost- and- practicality.- System-level- simulations,- optimisation- and-system- interaction- design- can- be- made- easier- in- terms- of- time- and- in- some- cases,- practically- impossible- without- the- use- of- reduced- order- models.- In- (Hung,- Yang- &-Senturia- 1997)- a- reduced- order- model- is- used- to- speed- up- the- simulation- timeof- a- micromechanical- device- via- a- macromodel- of- reduced- order.- In- (Nayfeh,-Younis- &- Abdel-Rahman- 2005),- a- state- of- the- art- literature- review- has- beendone- on- the- development- of- reduced- order- models- for- micro-electromechanicalsystems.- Reduced- order- models- have- also- been- used-in- (Filipi,- Fathy,- Hagena,-Knafl,-Ahlawat,-Liu,-Jung,-Assanis,-Peng-&-Stein-2006)-to-perform-engine-in-theloop-testing.- In-their-paper,-(Filipi-et-al.-2006)-(Filipi-et-al.-2006)-used-an-energybased-model- order- reduction- technique- to-reduce- a- high- mobility- multipurposewheeled-vehicle-model.- This-was-necessary-because-in-order-to-carry-out-enginein-the-loop-simulations,-a-model-needs-to-capture-all-the-important-dynamics-ofthe-system,- and-at-the-same-time-the-simulations-must-be-able-to-run-in-realtime.-

The nature of singular systems creates further difficulties for the application of some mathematical processes. Their frequent occurrence in modelling of electric circuits and power systems has brought about an interest into finding solutions for dealing with this type of systems. This has brought about the study of singular systems (Gray & Verriest 1989) and model order reduction for singular systems as discussed in (Xu, Lam, Liu & Zhang 2003, Bender 1987). A singular system has been defined in (Weisstein 2002) as a system whose condition number is infinite.

The purpose of this chapter is to apply the PLA projection and the Improved-Phillips projection techniques for the reduction of pseudo-singular bilinear models where a pseudo-singular bilinear model refers to a bilinear model with noninvertible state transition matrices. Also, an optimisation scheme will be used to estimate the parameters, $(\eta_1, \eta_2, \ldots, \eta_m, m \in \mathbb{Z})$, of a PLA-MOR approach. Hybrid approaches for MOR are becoming very popular. These approaches tryto combine the advantages of different MOR methods (data based and mathematical manipulation) to achieve higher accuracy or ease of implementation and practicality. For example, in (Saragih 2014), genetic algorithm has been usedwith balanced truncation to reduce a MIMO bilinear model.

In-Section-6.2, -a-hybrid-approach-for-model-order-reduction-is-introduced.-This-is-followed-by-the-use-of-PLA-for-MOR-of-so-the-called-pseudo-singularsystems-in-Section-6.3.- Two-case-studies-are-provided-in-Section-6.4.- A-solarpanel-model-with-singular-matrices-has-been-reduced-successfully-by-using-PLA.-Also, - an-optimisation-scheme-has-been-applied-to-the-bilinear-model-providedin-Subsection-5.5.2.- This-has-been-compared-to-PLA-without-an-optimisationscheme.-

6.2 Hybrid MOR using parameter estimation and optimisation techniques

The involvement of parameters in the computation of reduced order models means that there will be optimum parameter choices for which an optimal reduced order model can be achieved. Since the reduced order model has to reach some level of accuracy for it to be acceptable, these parameters can be said to be optimum for a given set of performance criteria.

An optimisation scheme can be taken into consideration. Tools such as genetic algorithm (Gen & Cheng 2000) and Nelder-Mead simplex (Lagarias, Reeds, Wright & Wright 1998) method for the optimisation of reduction parameters are some of the optimisation algorithms which can be used to find optimal parameters for MOR. Figure 6.1 shows a flow chart that gives a pictorial view of the proposed optimisation scheme.

The parameter initialisation stage in the flow chart (Figure 6.1) uses a best guess of the parameters, $\eta_1, \eta_2, \ldots, \eta_m$, from the user's experience with the system to be reduced. Normally, most algorithms can cope with an initialisation of zero but in this case, since the PLA achieves reduced order models with good performance criteria values without the optimisation scheme, an initialisation value can be obtained using trial and error whilst observing the accuracy of the reduced order model. The linear approximation of the bilinear model is

$$A_{\eta} = A + N_1 \eta_1 + N_2 \eta_2 + \dots + N_m \eta_m \tag{6.1}$$

$$B_{\eta} = B \times \eta. \tag{6.2}$$

After-which-the-projection-matrix, V is computed-using-Algorithm-5.1-and-Algorithm-3.2.- This-is-followed-by-computing-the-reduced-order-state-transition-



Figure-6.1:- Flow-chard-for-hybrid-model-order-reduction-using-an-optimisation-scheme-and-Krylov-subspace-projection-techniques.-

matrix, bilinear state matrices, input and output matrices using Algorithm 3.4. The scheme uses a performance criteria as a cost function. Depending on the cost, the optimisation algorithm updates the parameters $(\eta_1, \eta_2, \ldots, \eta_m)$. This loop continues until a desirable cost or number of loops is achieved.

6.3 Parametrised linear approximation projection for pseudo-singular systems

In-this-section,-a-pseudo-singular-system-model-is-defined.-It-is-a-bilinear-systemthat-has-a-noninvertible-state-transition-matrix.-Consider-a-bilinear-system-

$$\dot{x} = Ax + \sum_{i=1}^{m_1} N_i x u_i + Bu \tag{6.3}$$

$$y = Cx \tag{6.4}$$

where $A \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m_2}$, $C \in \mathbb{R}^{p \times n}$ and A is a singular matrix. $u = [u_1, u_2, \dots, u_{m_2}]$. Note that whilst it is normal for u to be composed of all u_i , it is not necessarily the case. In some cases, u_i could be noise or some external input to the system/model. The ideal case discussed in previous chapters is for when $m_1 = m_2$.

Using-an-alternate-linear-approximation-to-define-the-linear-behaviour-of-thepseudo-singular-bilinear-system-model,-the-model-can-be-reduced-by-using-the-Krylov-subspaces-as-defined-as-follows:-

$$span\{V^1\} = K_{q_1}(A_{\eta}^{-1}, A_{\eta}^{-1}B_{\eta})$$
(6.5)

$$span\{V_i^2\} = K_{q_2}(A_{\eta}^{-1}, N_i V^{\{1\}})$$
(6.6)

$$span\{V\} = span\{span\{V^{\{1\}}\} \bigcup \{\bigcup_{i=1}^{m} span\{V_i^{\{2\}}\}\}\}.$$
(6.7)

Comparing (6.5)-(6.7) to (5.8)-(5.9), the state transition matrix, A, has been replaced by A_{η} , assuming that A_{η} is nonsingular and forms a stable linear approximation of the bilinear model.

6.4 Case study on Solar Panel Model

6.4.1 Model description

For-illustration, a-solar-panel-model-shown-in-Figure-6.2 which-has-been-presentedin- (Tenny, Rawlings- &- Wright- 2004, Couchman- et- al. - 2011)- will- be- used- todemonstrate-the-use-of-parametrised-linear-approximation-for-a-pseudo-singularsystem. - The-solar-collector-plant-consists-of-a-heat-exchanger, -790-meter-pipe, -



Figure-6.2:-Schematic-diagram-of-solar-collector-

collector-and-a-pump.-A-fluid-is-used-to-collect-the-solar-energy-and-is-transportedusing-the-pipe-to-the-heat-exchanger-for-extraction-of-the-heat-energy.-The-fluidis-then-transported-back-to-the-collector.-A-pump-is-used-to-control-the-flowrate-of-the-fluid-through-the-pipes,-therefore-the-fluid-flow-is-the-control-variable.-The-operating-outlet-temperature-of-this-system-is-573-K.-

A-model-of-this-system-is-derived-by-discretizing-the-return-loop,-heat-ex-

changer-and-collector-across-a-1-dimensional-space-where-the-temperatures-ateach-node-are-states-of-the-model.-The-resulting-model-is-of-a-bilinear-structure-(Couchman-et-al.-2011)-with-matrices-of-system-parameters-are-given-below-

$$A = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{0_{12}} & \mathbf{0_{13}} \\ \mathbf{0_{21}} & \mathbf{A_{22}} & \mathbf{0_{23}} \\ \mathbf{0_{31}} & \mathbf{0_{32}} & \mathbf{0_{33}} \end{bmatrix}, \quad N = diag([N_1 \quad 0^- \ N_2]) + \begin{bmatrix} N_{11} & \mathbf{0}^- \\ N_{21} & N_{22} \end{bmatrix} . \quad (6.8) - \begin{bmatrix} N_{11} & \mathbf{0}^- \\ N_{21} & N_{22} \end{bmatrix} .$$

$$\begin{split} A_{11} = -\beta_1 * I_{(20\times 20)}, A_{12} &= 0_{(20\times 1)}, A_{13} = 0_{(20\times 20)}, A_{21} = 0_{(1\times 20)}, A_{22} = -1 - \beta_2, A_{23} = 0_{(1\times 20)} \text{ and } N_1 = -\alpha \times I_{(20\times 1)}, N_2 = -\alpha \times I_{(20\times 1)}, N_{11} = 0_{(1\times 40)}, N_{21} = -diag([\alpha \times I_{(1\times 19)} \quad 0^- \quad \alpha \times I_{(1\times 20)}]), N_{22} = 0_{(40\times 1)}, \text{ where } \alpha = 8.22 \times 10^{-3}, \beta_1 = 1.19 \times 10^{-3}, \beta_2 = 5. \text{ The matrices } B \text{ and } C \text{ are} \end{split}$$

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 1 \\ C_2 \end{bmatrix}, \quad (6.9)$$

 $B_{11} = \beta_1 \times T_1 \times I_{(20\times1)}, B_{12} = \gamma_1 \times I_{(20\times1)}, B_{21} = \beta_2 T_2, B_{31} = 0_{(20\times1)}, B_{32} = 0_{(20\times1)}, C_1 = 0_{(1\times19)}, C_2 = 0_{(1\times21)}, \text{ with } \gamma = 0.541, T_1 = 303.15 \text{ and } T_2 = 375.15.5$

In- (Stuetzle, Blair, Mitchell- & Beckman 2004) - this plant model was usedto-develop-a-control-algorithm-for-achieving-desired-system-response. A-linearmodel-predictive-controller-was implemented on the plant at different weatherconditions and in-(Stuetzle, Blair, Beckman & Mitchell 2004) - the gross outputof-this approach is analysed.

The order of the model being discussed is n = 41. This higher order model is obtained as a result of discretization. However, much higher order models could be obtained if the discretization points are increased for higher accuracy of state estimation. To reduce the order of the resulting model, the implementation of the standard Krylov subspace techniques discussed (Phillips 2000, Bai & Skoogh 2006, Feng & Benner 2007) would be impossible as A is a singular matrix.

6.4.2 MOR procedure

The-model-order-reduction-procedure-uses-the-Krylov-subspaces-defined-in-(6.5)-- (6.7).- The-projection-matrix-has-been-computed-using-Algorithm-5.2-whilstthe-reduced-order-models-have-been-computed-using-Algorithm-3.4.- The-resultspresented-are-for-7th, 9th and-11th order-models-using-the-simulation-inputs-

- $u_1 = 0.6$ for: $t \in [0 \cdot s : -200 \cdot s]$ (6.10)-
- $u_1 = 8.0$ for: $t \in [200 \cdot s : -1750 \cdot s]$ (6.11)

$$u_2 = 1.4$$
 for: $t \in [0.s: 1000 \cdot s]$ (6.12)

$$u_2 = 0.6$$
 for: $t \in [1000 \cdot s : \cdot 1750 \cdot s]$ (6.13)

$$u = [u_1 \quad u_2]. \tag{6.14}$$

The higher order model and the reduced order models have been simulated with a nonzero initial condition of x(0) = 465. The results are presented in the next subsection.

6.4.3 Results

The first-row of Figure 6.3 shows the output plots of the 7^{th} and the 9^{th} order models compared to the solar panel model output. The second row of Figure 6.3

Table-6.1:-*RT*²,-MSE,-IAE,-NIAE-and-SSE-values-for-model-order-reduction-of-solar-panel-model-using-the-Parametrised-Linear-Approximation-

1 0			11			
	RT^2	MSE-	IAE-	NIAE-	SSE-	
7^{th} order-model-	99.84-	5.0398e+05	1.7284+06-	542.1694-	1.6067e + 09-	
9^{th} order-model-	99.91-	3.0202e+05-	0.9560+06-	299.8695-	0.9628e+09-	
11^{th} order-model-	99.98-	$0.4977e + 05^{-1}$	0.3512+06-	110.1527-	0.1587e+09-	

shows-the-input-values- u_1 and u_2 as-defined-in-(6.10)-- (6.14)-over-a-time-length-of-1700-seconds.

 $\label{eq:Figure-6.4-shows-the-output-of-the-solar-panel-model-compared-to-a-reduced-order-model-of-11^{th} order.-The-second-row-of-Figure-6.4-shows-the-absolute-error-$



Figure-6.3: Input-and-output-plot-of-solar-panel-model-(SPM)-compared-to- 7^{th} and 9^{th} order-models-using-PLA-type-projection-

values-divided-by-100-of-the- 7^{th} , 9^{th} , and 11^{th} reduced-order-models. As-can-beobserved-in-this-figure, the-accuracy-of-the-reduced-order-model-increases-as-theorder-increases. This-trend-can-also-be-observed-in-Table-6.1-which-shows-thecoefficient-of-determination- (RT^2) , integral-of-absolute-error-(IAE), integral-ofabsolute-error-divided-by-number-of-samples-(NIAE), mean-square-error-(MSE), and-the-sum-of-square-of-error-(SSE)-values-for-the-reduced-order-models.

In (Couchman et al. 2011), the order of this same model has been reduced using an input constraint with balanced truncation to 3rd, 7th and 10th order. As is the case here, reducing the model to orders below 7 reduces its accuracy considerably.



Figure 6.4: Plot-of- 11^{th} order-model-compared-to-solar-panel-model-(SPM)-and-absolute-error-plots-for- 7^{th} , 9^{th} and -11^{th} order-models-

6.5 MOR with optimisation

In order to demonstrate the use of optimisation algorithms for model order reduction, consider an arbitrary biliniear system of the form presented in (Linet-al. 2007, Linet-al. 2009) where the number of inputs is 2- and the number of outputs is 3. The bilinear model structure considered here has 2-bilinear state matrices i.e. $m_1 = -2$.
$$A = \begin{bmatrix} -10^{-} & 2^{-} & 0^{-} & \cdots & 0^{-} \\ 2^{-} & -10 & 2^{-} & \ddots & \vdots^{-} \\ 0^{-} & \ddots & \ddots & \ddots & 0 \\ \vdots^{-} & \ddots & 2^{-} & -10^{-} & 2^{-} \\ 0^{-} & \cdots & 0 & 2^{-} & -10 \end{bmatrix}$$
(6.15)-

$$N_{2} = \begin{bmatrix} -10^{-} & 2^{-} & 0^{-} & \cdots & 0 \\ 2^{-} & -10^{-} & 2^{-} & \ddots & \vdots^{\frac{1}{2}} \\ 0^{-} & \cdots & 0^{-} & \ddots & 0^{-} \\ \vdots^{\frac{1}{2}} & \cdots & 2^{-} & -10^{-} & 2^{-} \\ 0^{-} & \cdots & 0 & 2^{-} & -10^{-} \end{bmatrix}, \quad N_{1} = \begin{bmatrix} 0^{-} & -1 & 0^{-} & \cdots & 0^{-} \\ 1 & 0^{-} & -1 & \ddots & \vdots^{-} \\ 0^{-} & \cdots & 0^{-} & 0^{-} \\ \vdots^{\frac{1}{2}} & \ddots & 1 & 0^{-} & -1 \\ 0^{-} & \cdots & 0 & 1 & 0^{-} \end{bmatrix}$$
(6.16)

Also, $N_1, N_2 \in \mathbb{R}^{n \times n}$. B is an $n \times m$ matrix while C-is a $p \times n$ matrix and are in the form given below

$$B = \begin{bmatrix} 1 & 1^{-} \\ 0 & 1^{-} \\ \vdots & \vdots^{-} \\ 0 & 1^{-} \\ 0 & 1^{-} \\ 0 & 1^{-} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1^{-} & \cdots & 1 & 1^{-} \\ 0.8 & 0.8^{-} & \cdots & 0.8 & 0.8^{-} \\ 0.5 & 0.5^{-} & \cdots & 0.5 & 0.5^{-} \end{bmatrix}. \quad (6.17)^{-}$$

Using Algorithm 5.1, an optimisation scheme called the Nelder-Mead simplexmethod which is inbuilt in MATLAB has been used to estimate the parameters η_1 and η_2 . As an initial guess, the parameter values $\eta_1 = 0.79$ and $\eta_2 = 0.79$ has been used. The reduced order model to be optimised is of dimension 25 where $q_1 = -5$, $q_2 = -5$ and $p_2 = -4$.

At-each-iteration-step,-the-scheme-returns-a-cost-for-the-estimated-reducedorder-model.- In-this-case,-the-squared-error-(SE)-has-been-used-as-the-costfunction:-

$$SE = \sum_{i=1}^{n_s} (\hat{y}_i - y_i)^2 \tag{6.18}$$

where y_i is the output of the higher order bilinear model at time i, \hat{y}_i is the output of the reduced order bilinear model at time i, and n_s is the number of samples collected for each output. The input which has been used for simulation in the optimisation scheme is given. This is called the estimation input.

u_1	18.3745-	5.9107-	51.7177-	51.7490-	70.2652-
Time-range-(s):-	$t \in [0-:-0.9]^{-1}$	$t \in [1-1.9]^{-1}$	$t \in [2 - : -2.9]$ -	$t \in [3 - : -3.9]$ -	$t \in [4 - : -4.9]$ -
u_1	30.3039-	40.1577-	62.7450-	5.0065-	6.3760-
Time-range-(s):-	$t \in [5 : -5.9]^{-1}$	$t \in [6 - : -6.9]$ -	$t \in [7 - :-7.9]$ -	$t \in [8 - : -8.9]$ -	$t \in [9 - : -9.9]$ -
u_1	40.9033-				
Time-range-(s):-	$t \in [10]^{-1}$				

Table-6.2: Input-values- u_1 for-parameter-estimation.-

$$u_2 = (\sin(2\pi \times t) + 90)/10 + 0.1 - (6.19) - (6.$$

$$u = [u_1 \quad u_2]^T \tag{6.20}$$

The results to be shown are for simulations using the estimation data set and then the validation data set i.e. input and output values. The validation input is given as follows.

$$u_2 = (\sin(2\pi \times t) + 90)/10 + 0.2$$
 (6.21)

$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T \tag{6.22}$$

6.5.1 Results

Figure 6.5-displays the outputs y_1 of the reduced order models derived from using the PLA and the optimised PLA and the higher order bilinear model (HOBM). The second row of this figure shows the absolute error values of the reduced order models. In these plots, the optimised PLA shows a better fit for the higher order bilinear model.

Figure 6.6 shows the input u_1 as defined in Table 6.2 in the first row. The second row of Figure 6.6 shows the second input u_2 as defined in (6.20)

u_1	14.8532-	7.1170-	35.5489-	47.0612-	47.0612-
Time-range-(s):-	$t \in [0-:-0.9]^{-1}$	$t \in [1-1.9]^{-1}$	$t \in [2 - : -2.9]$ -	$t \in [3 - : -3.9]$ -	$t \in [4 - : -4.9]$ -
u_1	22.2576-	28.3737-	42.3934-	6.5557-	7.4058-
Time-range-(s):-	$t \in [5 : -5.9]$ -	$t \in [6 - : -6.9]$ -	$t \in [7-:-7.9]^-$	$t \in [8 - : -8.9]$ -	$t \in [9 - : -9.9]$ -
u_1	28.8365-				
Time-range-(s):-	$t \in [10]^{-1}$				

Table-6.3: Input-values u_1 for model-validation.



Figure 6.5:- Plot-of-outputs- y_1 and absolute-error-for-PLA, optimised-PLA-and-high-order-bilinear-model-using-estimation-input-

In-Table-6.4, the performance criteria values of the coefficient of determination (RT^2) , the mean square error (MSE), the integral of absolute error (IAE), the integral of absolute error divided by number of samples (NIAE) and the sum of square of error (SSE) are displayed.

Table-6.4: Table-of- RT^2 , MSE, IAE, NIAE- and SSE- values- for- model- orderreduction-of-solar-panel-model-using-parametrised-linear-approximation-

	RT^2	MSE-	IAE-	NIAE-	SSE-
PLA-	99.86-	0.2603-	139.6369-	0.4036-	8.9180-
PLA-optimised-	99.99-	0.0258-	40.9249-	0.1183-	90.0735-



Figure 6.6: Plot-of-inputs- u_1 and u_2 used-for-parameter-optimisation-of-the-PLA-parameters-and-the-simulation-output-given-in-Figure 6.5-as-defined-in-Table 6.2-and-(6.19).

The results presented in Figures 6.5 and Table 6.4 are for the input used for parameter estimation. These validation results using a new set of inputs are shown in Figures 6.7.

6. Applications of PLA and IP Projection

Table 6.5: Table of RT^2 , MSE, IAE, NIAE and SSE values for model order reduction of solar panel-model using Parametrised Linear Approximation

	RT^2	MSE-	IAE-	NIAE-	SSE-
PLA-	99.99-	0.0165-	43.1672-	0.1680-	4.2396-
PLA-optimised-	99.98-	0.0545-	24.4731-	0.0952-	14.0130-



Figure 6.7:- Plot-of-outputs- y_1 and absolute-error-for-PLA, optimised-PLA-and-higher-order-bilinear-model-using-validation-input-



Figure 6.8:- Plot of inputs u_1 and u_2 used for validating the optimised PLA parameters.

Figure 6.7-displays-the-output-plots-of-the-reduced-order-models-comparedwith-that-of-the-higher-order-bilinear-model-(HOBM).-Figure 6.8-shows-the-inputplots- u_1 (top)-and- u_2 (bottom).- These-inputs-are-as-defined-in-Table 6.3-and-(6.21)-respectively.-

In-Table 6.5, the performance criteria values of the coefficient of determination (RT^2) , the mean square error (MSE), the integral of absolute error (IAE), the integral of absolute error divided by number of samples (NIAE) and the sum of square of error (SSE) of the validation output values are shown.

6.6 Conclusion

In this chapter, two-unique applications of the parametrised-linear approximation-(PLA)- approach- to- applying- Krylov- subspaces- for- model- order- reduction- are-discussed- and-demonstrated. The-first-is- the-use-of-PLA- for- the- reduction- of-pseudo-singular-systems. These-are-systems/models-which-have-singular-system-matrices. This-means-that-the-Krylov-subspace-model-order-reduction-methods-discussed- in- Chapters- 3,- 4- and- 5- i.e- the- Phillips- type- (Phillips- 2000),- Feng-and-Benner-type-(Feng-&-Benner-2007),-Bai-type-(Bai-&-Skoogh-2006)-and-the-Improved-Phillips-type-projection-methods-will-not-be-applicable-for-model-order-reduction.- However-using-the-PLA-approach,-a-pseudo-singular-model-which-has-been-used-in-(Couchman-et-al.-2011)-has-been-reduced.- In-order-to-reduce-this-system-using-PLA,-the-Krylov-subspaces- $K_{q1}(\mathbb{N},\mathbb{M})$ -and- $K_{q2}(\mathbb{N},\mathbb{M})$ -were-defined-using-the-parameterised-linear-approximation-which-is-invertible. The-case-study-on-Solar-Panel-Model-demonstrated-the-advantage-of-the-use-of-PLA-approach.-Not-only-is-it-useful-for-reducing-models-with-noninvertible-matrices,-it-can-also-achieve-this-at-accuracy-of- RT^2 values-of-99.98.-

In the second application and example, a hybrid model order reduction approach has been proposed. This approach suggests that the PLA parameters η_1, \ldots, η_m can be optimised by using an optimisation scheme. Using the example provided in Chapter 5, the optimised reduced order model was compared to the case where the PLA parameters are equal and the optimised reduced order model shows a better input-output preservation for the two sets of test input, i.e. the input-used for optimisation and the input-used for validation.

Chapter 7

Conclusion and Further Work

This-chapter-will-summarise-the-findings-and-contributions-of-this-thesis. Alsoit-will-discuss-future-research-ideas-which-are-the-result-of-the-findings-therein. In-the-next-subsection, a-brief-description-of-the-research-objectives-is-revisited. This-is-followed-by-the-research-conclusion, contributions-and-further-work.

7.1 Conclusion

This-thesis-presents-novel-approaches-in-the-reduction-of-bilinear-models.-Singleinput-single-output-(SISO),-multi-input-multi-output-(MIMO)-and-their-pseudosingular-model-cases-are-considered.-

In- Chapters- 2- and- 3,- the- basic- concepts- of- Krylov- subspace- model- orderreduction-approaches-are-discussed.- Chapter-3-discusses-the-state-of-the-art-ofone-sided-Krylov-subspace-projection-for-bilinear-models-(Phillips-2000,-Feng-&-Benner-2007)-and-their-application-to-MIMO-bilinear-models-(Lin-et-al.-2007,-Linet- al.- 2009,- Lin- et- al.- 2007,- Lin- et- al.- 2009).- These- Krylov- projection- typesdiscussed- multimoments- of- bilinear- models/systems- and- in- order- to- do- this,matrix-inversion-is-necessary.- This-limits-their-application-only-to-models-whichhave-invertible-matrices.- Also,- the-error-of-the-derived-models-tend-to-becomelarger-as-the-input-increases-and-are-not-flexible-in-their-implementation.-

In-Chapter-4,-a-new-set-of-Krylov-subspaces-for-matching-multimoments-hasbeen-proposed. The new approach is called the Improved Phillips (IP) type projection-and-takes-advantage-of-the-fact-that-model-order-reduction-of-bilinearmodels-are-dominated-by-the-linear-approximation-of-the-bilinear-model-as-hasbeen-discussed-in- (Baur-et-al.-2014). This-new-method-therefore-achieves-alinear-approximation-by-matching-the-multimoments- $m(l_1, l_2), l_1 = 1, \ldots, q_1, l_1 = 1, \ldots, q_2 - 1$. Going-further, to continue exploiting the linear approximation of the bilinear-model, another new approach is proposed called the Parametrised Linear-Approximation- (PLA).- This- Parametrised- Linear- Approximation- is- shown- topreserve-the-input-output-relationship-of-a-bilinear-system-using-three-exampleswhich show the advantages of using the Improved Phillip type projection, the Parametrised-Linear-Approximation-and-the-combination-of-both.- The-inputoutput-preservation-of-the-reviewed-methods-in-Chapter-3- and-the-proposedmethods-in-Chapter-4-have-been-analysed-using-input-output-plots,-coefficientof determination (RT^2) , mean square error (MSE), integral of absolute error (IAE), sum of square error (SSE), integral of absolute error divided by numberof-samples-(NIAE)-and-plots-of-absolute-error-against-ascending-input-values.-

Subsequently, the Feng and Benner type projection (Feng & Benner 2007), Improved Phillips type and the Parametrised Linear Approximation type projection have been extended to MIMO structures in Chapter 5. The multimomentmatching for Feng and Benner type (Feng & Benner 2007) has been analysed for MIMO models and the analysis can be extended to other types of Krylov subspace projections in literature i.e projection for MIMO bilinear models as proposed in (Lin et al. 2007, Lin et al. 2009), where the Phillip type projection has been extended to MIMO cases in (Lin et al. 2007) and the Bai type projection in (Lin et al. 2009). Using the same criteria that has been used in Chapter 4, the Feng and Benner type projection for MIMO bilinear models, Improved Phillips-type-projection-for-MIMO-bilinear-models-and-the-parametrised-linearapproximation-for-MIMO-bilinear-models-have-been-compared-to-the-work-donein- (Lin- et- al. 2007, Lin- et- al. 2009). In- the-numerical-simulation-results, ithas-been-found-that-the-parametrised-linear-approximation-(PLA)-shows-goodinput-output-preservation-when-compared-to-the-other-types. This-is-expectedbecause-the-PLA-uses-a-so-called-better-linear-approximation-for-the-SISO-and-MIMO-bilinear-model-reduction.

Whilst-the-Feng-and-Benner-type-projection-(Feng-&-Benner-2007)-matchesmore-multimoments-when-compared-to-the-Phillips-type-projection-(Phillips-2000,-Lin-et-al.-2007)-and-the-Improved-Phillips-type-for-both-SISO-and-MIMOmodels,-it-is-not-always-the-case-that-the-reduced-order-models-produced-givea-better-approximation.- This-is-because-during-the-Krylov-subspace-reductionprocess,-the-multiplication-of-matrices-which-are-nonsingular-produce-loss-of-information- and-therefore-the-Feng- and-Benner-approach-(Feng-&-Benner-2007)cannot-guarantee-a-better-reduced-order-model-and-in-some-cases-it-will-be-impossible-to-compute-the-projection-matrix-if-there-is-a-total-loss-of-rank-in-theresulting-matrices.- It-has-been-said-that-the-Improved-Phillips-type-projectioncombines-the-advantages-of-the-Phillips-type-projection-(Phillips-2000)-and-the-Feng-and-Benner-type-projection-(Feng-&-Benner-2007)-by-matching-more-moments-of-the-linear-approximation-of-the-bilinear-model-and-avoiding-the-loss-ofinformation-when-computing-the-projection-matrix.-

The application of the model order reduction techniques developed so far were applied to a pseudo-singular system in Chapter 6. Therein, a pseudo-singular system has been defined as a system with system matrices that cannot be inverted. The application of the Improved Phillips type projection and the PLA approach shows the viability in the use of the proposed Krylov subspaces for reducing systems of this kind. As opposed to the other systems where the alternate linear approximation is applied to only match the linear moments, in this

case-the-linear-approximation-has-shown-to-be-useful-for-computing-subsequent-Krylov-subspaces-for-the-bilinear-approximation.- This-means-that-the-singularsystem-matrix-can-be-replaced-by-a-non-singular-alternate/approximate-whichexhibits-the-same-characteristics.- The-reduced-order-models-derived-show-goodaccuracy.- A-second-example-which-has-been-considered-in-Chapter-6-exploitsthe-hybrid-combination-of-optimisation-techniques-and-the-PLA-approach.- Thishas-been-shown-to-improve-the-reduced-order-model-when-compared-to-the-PLAwithout-an-optimisation-scheme.-

In-summary,-the-contributions-of-the-research-reported-in-this-thesis-are-as-follows:-

- 1.- The-matching-of-a-higher-number-of-multimoments-whilst-avoiding-themultiplication-of-nonsingular-matrices.- This-has-been-called-the-Improved-Phillip-type-projection-(Chapter-3).-
- 2.- The proposal of a reduced order modelling approach using Kylov subspacesby applying a so-called better-linear approximation. This approach is called the Parametrised Linear Approximation (PLA).
- 3.- The analysis of multimoment matching for the Feng and Benner type projection (Feng & Benner 2007), Phillip type projection (Phillips 2000) and the Improved Phillip type projection.
- 4.- The extension of the Improved Phillip type projection, Parametrised Linear-Approximation-projection and the Feng and Benner type projection (Feng-& Benner 2007) to MIMO cases.
- 5.- The analysis of multimoment matching for MIMO bilinear model reduction using Krylov subspaces.
- 6.- The use of PLA for the reduced order modelling of pseudo-singular bilinear systems to enable the reduction of systems with nonsingular system-

matrices.-

7.- The use of an optimisation scheme for finding parameters which form an alternate linear approximation of a bilinear system/model and the use of these parameters for model order reduction.

This-thesis-also-served-as-a-resource-for-understanding-and-implementationof-reduced-order-modelling-using-Krylov-subspaces.

7.2 Further work

The research done in this thesis brings about various new scopes of research which are quite interesting.

- 1.- Exploration of other nonlinear approximations: The focus of this thesis has been on bilinear and nonlinear models of a certain type, i.e. those nonlinear models which can be bilinearised. The scope can also be extended to quadratic approximations (Chen 1999) and quadratic bilinear control systems (Benner & Breiten 2012b). Quadratic approximations have been reported to be less effective when compared to bilinear approximations. The application of an alternate linear approximation for computing the projection matrices will improve the input-output preservation of quadratic and quadratic bilinear approaches to MOR.
- 2.- Extension-to-other-bilinear-systems-with-singularity-or-ill-conditioned-matrices:- This-work-makes-it-possible-to-apply-bilinear-model-order-reduction-using-Krylov-subspaces-to-systems-which-would-have-been-otherwiseimpossible-to-reduce.- A-further-application-of-the-techniques-to-othersystems-which-have-singular-and/or-ill-conditioned-matrices-either-due-totheir-derivation-from-Carleman-bilinearisation-or-through-the-discretisation-process-is-an-interesting-prospect.- More-practical-examples-of-systems-

which-result-in-singularity-or-ill-conditioned-matrices-will-further-highlightthe-effectiveness-of-the-PLA-approach.-

- 3.- Further- exploration- of- optimization- techniques:- optimisation- of- modelorder-reduction-techniques- are- also- an- interesting- area- to- look- at.- Inthis-thesis,-Krylov-subspaces-were-combined-with-parameter-estimation.-Here, the only algorithm looked at is the use of the Nelder-Mead optimisation-algorithm.- However, it-has-been-suggested-in-(Abdullah, Deris, Anwar-&-Arjunan-2013)-that-other-algorithms-such-as-the-Firefly-Algorithm-(FA)-(Yang-2009),-Particle-Swarm-Optimization-(PSO)-(Kennedy-2011, Campbell 2009) and the Hybrid Firefly Evolutionary Optimization-(Abdullah- et- al.- 2013), - significantly- outperform- the- MATLAB $^{\textcircled{R}}$ implementation-of-the-Nelder-Mead-algorithms. Candidate-algorithms-to-beinvestigated-should-include-the-aforementioned-nature-inspired-optimisation-algorithms-as-well-as-efficient-gradient-based-constrained-optimisationalgorithms-such-as-the-interior-point-or-sequential-quadratic-programming-(SQP)- (Nocedal- &- Wright- 2006)- solvers- implemented- in- the- MATLABfunction-fmincon. Exploring-other-optimisation-schemes- and- combination-of-other-parameter-estimation-techniques-with-Krylov-subspace-MORshould-improve-the-accuracy-of-the-reduced-order-models.- Such-improvement-would-enable-faster-as-well-as-more-accurate-online-simulations.- Thisshould-result-in-improved-performance-of-model-based-control-schemesexploiting-such-reduced-order-models.-
- 4.- Further-hybrid-approaches:- This-involves-the-hybrid-implementation-of-Improved-Phillip-and-PLA-with-other-classical-model-estimation-algorithmssuch-as-balanced-truncation-and- H_2 model-order-reduction.- In-the-future,the-combination-of-PLA-with-balanced-truncation- and- H_2 model-orderreduction- are-likely- to- make- the- use- of- these- techniques- more- practicaland-less- time-consuming.- For- example,- as- has- been- discussed- in- litera-

ture, Gramian-based-approaches-can-achieve-higher-accuracy-and-Krylovsubspace-techniques-can-be-used-for-models-with-much-higher-dimensions.

5.- Two-sided-projection-approaches:- the-natural-progression-after-considering-one-sided-approaches-is-to-consider-two-sided-projection.- This-can-beachieved-by-using-a-different-set-of-Krylov-subspaces-for-defining-the-leftprojection-matrix.- This-will-increase-the-number-of-multimoments-matchedby-the-IP-and-PLA-approaches.-

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