Improving Algebraic Tools to Study Bifurcation Sequences of Population Models

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Abstract. Since being introduced by Collins in the 1970s Cylindrical Algebraic Decomposition has found many applications. We focus on its use to decomposition the parameter space of a parametric system of polynomial equations, and possibly some polynomial inequality constraints, with respect to the number of real solutions that the system attains. Previous studies simplify the CAD computation by first computing the discriminant variety of the system, which usually involves Gröbner Basis computation. However, on some even very small applied examples this itself can become expensive. Thus we consider development of new algorithms to reduce the complexity of this approach, by adopting numerical ideas and some recent technical developments in CAD theory. In this extended abstract we outline our recent progress as evaluated on an example from population dynamics with the Allee effect.

Keywords: cylindrical algebraic decomposition, discriminant variety, equational constraints, population dynamics, Allee effect

1 Population Models with the Allee Effect

A well-known population model is logistic growth, where due to limitation of resources the population can not exceed a certain level. At the beginning when the size of population is small, because of an abundance of resources the growth of the population is high; but as time passes and the population increases, the amount of available resources per individual decreases and the speed of growth reduces until eventually the population reaches a steady state which is called the carrying capacity of the system [14].

The Allee effect is less-known phenomenon in biology where the population is not only competing for the resources as in logistic growth models, but also has cooperative behavior increasing the chance of survival. A strong Allee effect happens when the population needs to be above a threshold, called the Allee threshold, to be safe from extinction. A simple population model with the strong Allee effect with carrying capacity 1 and Allee threshold b (where $0 < b < 1$) can be described by $dx(t)/dt = x(t)\big(1-x(t)\big)\big(x(t)-b\big)$ where $x(t)$ is the population size at time t. From here on we simply drop the emphasis on t and write x and $\dot{x}$ instead of $x(t)$ and $dx(t)/dt$.

The dynamical behavior of a single population with the strong Allee effect is clear. There are three steady states, two of which stable, extinction and the carrying capacity, and the Allee threshold which is unstable. It becomes more interesting when several populations of the same species with strong Allee effect are connected. The study of dynamical behavior of connected populations with Allee effects is an ongoing research topic [5,6,7,12,13]. Consider n populations for $n \in \mathbb{N}$ and denote the size of the i-th population with x_i. The simplest scenario is to connect all populations to each other with a complete digraph and the same dispersal rate for each path. Let a be the dispersal rate and assume that all populations have the same Allee threshold, b. The ODE system governing the dynamical behavior of this model is as follows [12, Equation (3)]:

$$\dot{x}_i = x_i(1 - x_i)(b - x_i) - (n - 1)ax_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} ax_j, \quad i = 1, \ldots, n. \tag{1}$$

To study the steady states of this model, one has to study the non-negative real solutions to the parametric polynomial system of equations obtained by setting all $\dot{x}_i$ equal to zero in (1). Here a and b are parameters that can be chosen from $\mathbb{R}_{\geq 0}$ and x_i's are variables.

2 Recent Prior work with CAD for the Discriminant Variety

In [12] this problem was addressed using Cylindrical Algebraic Decomposition (CAD) [1] with respect to the discriminant variety [9], to decompose the parameter region to disjoint connected subsets such that the number of solutions to the system is invariant over each of these subsets. We note a similar combination of methods applied in the work of [8].

This method has two steps: first computing the discriminant variety, and then computing the CAD of the parameter space with respect to the discriminant variety. The discriminant variety is usually computed using Gröbner bases techniques [1]. Unfortunately in some cases the Gröbner basis computation needed to compute the discriminant variety is not feasible on a normal computer[2]. This was the case for the simple connected population with the strong Allee effect introduced at (1) for $n \geq 3$. In [12], a combination of this algebraic method, CAD with respect to the discriminant variety, with a numerical sampling approach was developed, to build an approximation of the requested decomposition of the parameter space. We note a similar approach in the work of [4].

[1] For example the `DiscriminantVariety` command from `RootFinding[Parametric]` package in Maple, see https://www.maplesoft.com/support/help/Maple/view.aspx?path=RootFinding%2fParametric%2fDiscriminantVariety.

[2] Computations performed on Windows 10, Intel(R) Core(TM) i7-2670QM CPU @ 2.20GHz 2.20 GHz, x64-based processor, 6.00GB (RAM)

3 New approach using CAD with Equational Constraints

We now report on a purely algebraic approach to the problem, which does not use Gröbner bases. Consider (1) with $n = 3$ and denote the polynomials on the right by f_i, $i = 1, 2, 3$. Let d be the determinant of the Jacobian matrix of $f = (f_1, f_2, f_3)$ with respect to $x = (x_1, x_2, x_3)$. Then the ideal associated with the discriminant variety of this system is $\langle f_1, f_2, f_3, d\rangle \cap \mathbb{R}[a, b]$. Instead of using Gröbner bases to find a basis for this elimination ideal, we directly apply the projection step of CAD on the polynomial set $\{f_1, f_2, f_3, d\}$ with respect to the variables x. This is feasible because we may identify the f_i's as equational constraints (ECs) of the system, and use EC propagation to identify further ECs with different main variables [3,10,11]. This greatly reduces the number of polynomials identified by projection, in fact, for this example the reduction means the complexity is only single exponential in the number of variables [3].

The product of the polynomials in the last step of the projection will define a variety in the parameter space containing the discriminant variety, and hence a CAD with respect to these also gives us a decomposition of the parameter space with the properties that we desire. We applied this method successfully in Maple on the same computer that the computation of the Gröbner basis for the discriminant variety was infeasible. We could also produce an Open CAD for the polynomials defining the variety in a few minutes, and from that determine the regions in which there are consistent numbers of steady states of the system. In Figure 1 below the plot of this variety, which may be compared to that produced by the numeric method in [12, Figure 3b]. The boundary between the parameter regions with different numbers of solutions is defined by one of these graphs.

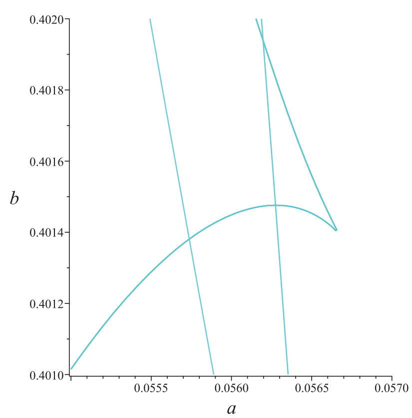

Fig. 1. Graph of the product of the polynomials from CAD projection of $\{f_1, f_2, f_3, d\}$ with respect to variables x_1, x_2 and x_3 and using ECs. Plotted in the region $[0.055, 0.057] \times [0.401, 0.402]$. To be compared with [12, Figure 3b].

We have demonstrated how CAD with ECs can push the limit on the size of models that are possible to handle using purely symbolic methods, free of numeric approximations. Furthermore, one can equip the numerical approach in [12] with this version of the algebraic method and study more complex dynamics.

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References

1. Collins, G.: Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In: Proceedings of the 2nd GI Conference on Automata Theory and Formal Languages. pp. 134–183. Springer-Verlag (reprinted in the collection [2]) (1975), https://doi.org/10.1007/3-540-07407-4_17
2. Caviness, B. F. and Johnson, J. R.: Quantifier Elimination and Cylindrical Algebraic Decomposition. Springer-Verlag Wien, (1998). https://doi.org/10.1007/978-3-7091-9459-1
3. England, M. and Bradford, R. and Davenport, J. H.: Cylindrical algebraic decomposition with equational constraints. *Journal of Symbolic Computation*, 100:38–71, 2020. https://doi.org/10.1016/j.jsc.2019.07.019
4. England, M., Errami, H., Grigoriev, D., Radulescu, O., Sturm, T., and Weber, A.: Symbolic versus numerical computation and visualization of parameter regions for multistationarity of biological networks. In: Proc. CASC '17, LNCS 10490, pp. 93–108. Springer, 2017. https://doi.org/10.1007/978-3-319-66320-3_8
5. Gyllenberg, M. and Hemminki, J. and Tammaru, T.: Allee effects can both conserve and create spatial heterogeneity in population densities. In: Theoretical population biology. 56(3): pp. 231–242, (1999), https://doi.org/10.1006/tpbi.1999.1430
6. Knipl, D. and Röst, G.: Large number of endemic equilibria for disease transmission models in patchy environment. In: Mathematical Biosciences. 258: pp. 201–222. (2014), https://doi.org/10.1016/j.mbs.2014.08.012
7. Knipl, D. and Röst, G.: Spatially heterogeneous populations with mixed negative and positive local density dependence. In: Theoretical Population Biology. 109: pp. 6–15. (2016), https://doi.org/10.1016/j.tpb.2016.01.001
8. Lichtblau, D.: Symbolic analysis of multiple steady states in a MAPK chemical reaction network. In: Journal of Symbolic Computation, 105, pp. 118–144, 2021. https://doi.org/10.1016/j.jsc.2020.06.004
9. Lazard, D. and Rouillier, F.: Solving parametric polynomial systems. In: Journal of Symbolic Computation, 42(6), pp. 636–667, 2007. https://doi.org/10.1016/j.jsc.2007.01.007
10. McCallum, S.: On Projection in CAD-Based Quantifier Elimination with Equational Constraint. In: Proc. ISSAC '99, pp. 145–149. ACM (1999), https://doi.org/10.1145/309831.309892
11. McCallum, S.: On Propagation of Equational Constraints in CAD-Based Quantifier Elimination. In: Proc. ISSAC '01, pp. 223–231.ACM (2001), https://doi.org/10.1145/384101.384132
12. Röst, G. and Sadeghimanesh, A.: Exotic bifurcations in three connected populations with Allee effect. *Preprint: bioRxiv 2021.02.03.429609*, 20 pages, (2021), https://doi.org/10.1101/2021.02.03.429609
13. Vortkamp, I., Schreiber, S. J. and Hastings, A. and Hilker, F. M.: Multiple attractors and long transients in spatially structured populations with an Allee effect. In: Bulletin of Mathematical Biology. 82(6): pp. 1522-–9602. (2020), https://doi.org/10.1007/s11538-020-00750-x
14. Logistic Population Growth. (2020, August 15). Retrieved May 16, 2021, from https://bio.libretexts.org/@go/page/14190