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## Non-equilibrium dynamics of the open quantum O(n)-model with non-Markovian noise: exact results

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Abstract. The collective and purely relaxational dynamics of quantum many-body systems after a quench at temperature T = 0, from a disordered state to various phases is studied through the exact solution of the quantum Langevin equation of the spherical and the O(n)-model in the limit  $n \to \infty$ . The stationary state of the quantum dynamics is shown to be a non-equilibrium state. The quantum spherical and the quantum O(n)-model for  $n \to \infty$  are in the same dynamical universality class. The long-time behaviour of single-time and two-time correlation and response functions is analysed and the universal exponents which characterise quantum noise is elucidated by comparing it with an effective Markovian noise having the same scaling behaviour and with the case of non-equilibrium classical dynamics.

*Keywords*: dissipative many-body quantum dynamics, ageing, quantum Langevin equations, exactly solvable models

#### 1. Introduction

Investigating the non-equilibrium quantum dynamics of complex many-body systems is of fundamental importance for understanding the cooperative behaviour that may emerge from a large number of strongly interacting degrees of freedom. Experimentally accessible systems include cold atoms [1, 2, 3], scaled-up quantum circuits [4, 5], ultrafast pump-probe measurements in correlated materials [6, 7] and quark-gluon plasma [8]. Among the central questions are those about the nature of the stationary states after the system has been forced out equilibrium. One needs to carefully verify whether the system "thermalises" towards an equilibrium state or not, and how to describe the relaxation towards stationarity [9, 10, 11, 12]. A systematic approach to these issues is to prepare the system in some (in general non-equilibrium) initial state and to subsequently quench at least one macroscopic control parameter and to let the system relax [13, 14, 15, 16].

In particular, the possible presence of criticality in the system is likely to affect the non-equilibrium dynamics and the relaxation after a quench onto the critical point, i.e. the so-called *non-equilibrium critical dynamics*, or into the ordered phase, corresponding to *coarsening*. In these scenarios, novel qualitative features, distinct from, e.g., those of the equilibrium dynamics may be observed [12, 15, 16, 17]. Quite generically, systems quenched onto or across a critical point, will fail to thermalise and rather undergo an "ageing dynamics" which never reaches a stationary state. This dynamics is characterised by the three properties [16]: (i) slow dynamics, (ii) absence of time-translation-invariance and (iii) dynamical scaling. These features are mainly studied through the long-time behaviour of two-time correlation functions and responses [see Eqs. (22), (23), (24)]. Ageing dynamics is usually characterised by a single, emergent and time-dependent length scale L which generically grows as  $L(t) \sim t^{1/z}$  at long times t, where z is the so-called dynamical exponent. Quantities like correlators and response functions then display dynamical scaling in which the associated exponents and scaling functions are universal, i.e., largely independent of the microscopic details of the system. In contrast to equilibrium systems, dynamical scaling after a quench is observed in large portions of the parameter space [17].

Ageing effects and their reproducible and universal aspects were first studied in glassy systems [17] in contact with a thermal bath at temperature T, before it became apparent that analogous phenomena also arise in much simpler systems without disorder or frustration [18, 19, 20, 21]. The majority of systems studied in the literature have classical dynamics [16] with some notable exceptions concerning anomalous coarsening in pre-thermal phases [22, 23]. If these systems are quenched into the ordered phase with  $T < T_c$ , where  $T_c$  denotes the critical temperature of the system, the long-time behaviour is fully characterised by the gross features of the initial state, while the coupling to the external heat bath at temperature T > 0 turns out to be irrelevant. Conversely, for a critical quench onto  $T = T_c > 0$ , the leading behaviour is governed by the thermal noise and the initial state correlations are largely irrelevant as long as they are short-ranged [16, 24, 25]. One system used for the theoretical analysis of generic non-equilibrium dynamics and ageing is the spherical model [26, 27], first introduced as a simple exactly solvable model of a magnetic phase transition in d spatial dimensions with a nonmean-field critical behaviour for 2 < d < 4. Its classical, purely relaxational dynamics (model A), described by a Langevin equation with a Gaussian white noise, can be solved exactly [18, 20, 28] and does confirm the generic scaling behaviour expected as indicated above. The successful confirmation of classical, dynamical scaling makes this model a promising candidate for similar studies in the quantum realm. In particular, the noisy description of open quantum systems differs qualitatively from the classical case and one may ask whether it is possible to extend the classical characterisations towards quantum systems. We shall attempt to answer this question by analytically studying the long-time dynamics of the simplest open quantum model with non-trivial many-body interactions.

The non-equilibrium dynamics of isolated quantum systems has been analysed intensively, see, e.g., Refs. [29, 30, 31, 22, 32, 2, 33, 1, 3] and references therein. Much less is known, in general, about nonequilibrium open quantum systems [34, 35, 36, 37, 38, 39]. Partially, this might be due to the widespread expectation, summarised in Ref. [35], that "...a large class of coarsening systems (classical, quantum, pure, and disordered) should be characterised by the same scaling functions.". While there are good reasons to accept this statement in the case of finite temperatures, this is not obvious in the limit  $T \rightarrow 0$  where quantum fluctuations govern the bath structure. An important distinction is that zero-temperature quantum noise is necessarily non-Markovian [40, 41, 42, 43, 44, 45, 9, 10, 46] and the resulting memory effects might become important in the long-time quantum ageing behaviour. Comparative studies, see, e.g., Ref. [10], of the classical and quantum Brownian motion lead, respectively, to growth laws  $L_{\rm cl}(t) \sim t^{1/2}$  and  $L_{\rm qu}(t) \sim \ln t$  for the typical length scale L, with  $L_{\rm cl}(t) \gg L_{\rm qu}(t)$  at long times. In a certain sense, this suggests that quantum noise can be considered "weaker" than the classical white noise. Accordingly, one might expect that the relative importance of the initial and bath correlations could be different when comparing quantum and classical dynamics.

Exactly solvable models are useful in this context, as they permit mathematically controlled statements on a well-defined physical system, see, e.g., Ref. [47]. Here, we shall analyse the non-equilibrium quantum dynamics of two closely related models:

(a) The quantum O(n)-model in the large-n limit [48, 49] which provides the simplest approximation of nonlinear interactions on top of a free quantum field theory. (b) The quantum spherical model [50, 51, 52], which is a mathematical extension of the quantum Ising model to obtain analytical insights beyond the latter.

These models have the appealing feature that the many-body dynamics for arbitrary spatial dimension d can be reduced to the solution of a single integro-differential equation, from which all observables of physical interest can be determined. We shall describe the non-equilibrium dynamics of these models by a quantum Langevin equation, which is known to guarantee physically desirable properties for a relaxation process, including the validity of the quantum fluctuation-dissipation theorem [40, 41, 42, 43, 44, 45, 46, 53]. Since the emerging equations are linear and we focus on observables which are at most quadratic in the fluctuating fields, this scheme is self-consistent and more advanced field-theoretical treatments, that are usually needed in order to describe interacting models [9, 12], are not required. We study a quantum bath at temperature<sup>1</sup> T = 0 and shall address the following questions:

- (i) Are the leading long-time dynamics of the two models mentioned above equivalent?
- (ii) Do these systems eventually relax to an equilibrium state?
- (iii) What are the (quantum) phase transitions in these systems?
- (iv) What is the relative importance of the spatial correlations existing in the initial state and the bath noise correlators?
- (v) What are the differences between the actual quantum noise and a suitable effective Markovian noise? In which observables could such differences be seen?
- (vi) Is there a "quantum ageing" distinct from ageing in classical dynamics? For isolated systems quantum ageing after a quench has been found in the pre-thermal phase [22, 54]. Despite being distinct from classical ageing, its actual quantum character can be debated as the ageing occurs in highly excited states.

As an intermediate step, it will be useful to study a model with an effective Markovian noise, introduced artificially and tailored such that the leading scaling behaviour in the presence of the actual quantum noise is reproduced. However, the treatment of the non-Markovian noise requires the introduction of suitable mathematical tools which are discussed below, see also Appendix D, Appendix E and Appendix F. We find that the overdamped quantum Langevin dynamics at zero temperature shows several qualitative differences from classical dynamics. These concern the non-equilibrium nature of the stationary state (even for relaxations occuring in the disordered phase), the inequivalence of regimes of non-equilibrium quantum dynamics and those of the classical dynamics and the relevance of the non-Markovian quantum noise for the scaling of the single-time correlators. For clarity, we summarise these findings in Sec. 4 without focussing on technical details.

This work is organised as follows. In Sec. 2 we introduce the quantum spherical and the quantum O(n)model, in the limit  $n \to \infty$  at thermal equilibrium and we recall the main features of their quantum phase diagrams. In Sec. 3 we formulate the quantum non-equilibrium dynamics and review the scaling argument by which these models can be reduced to a single over-damped quantum Langevin equation, in which the different types of dynamics (classical, quantum, etc.) are solely distinguished by the specific expression of the noise correlation functions. In Sec. 4 we summarise our predictions for one- and two-time correlation and response functions of the fluctuating fields, obtained from the exact solution of the non-equilibrium dynamics and we discuss their physical interpretation. This is followed in Sec. 5 by the detailed solution of the spherical constraint, for the non-Markovian quantum noise. Finally, Sec. 6 discusses the derivation of the time-dependent physical observables from the formal solution of the dynamical constraints of the models while we present our conclusions in Sec. 7, notably via a detailed comparison with classical dynamics. Several appendices discuss the technical details of our analysis.

#### 2. Equilibrium behaviour of the spherical and O(n) model for $n \to \infty$

The spherical model and the O(n)-model with  $n \to \infty$ , are introduced as two exactly solvable quantum statistical systems that show non-mean-field phase transitions:

(a) The O(n)-model is described by the quantum  $\phi^4$  field theory [48, 49]

$$H_n = \frac{1}{2} \int_{\boldsymbol{x}} \left[ \pi^2 + (\nabla_{\boldsymbol{x}} \phi)^2 + r_0 \phi^2 + \frac{u}{12n} (\phi^2)^2 \right], \tag{1}$$

with the bosonic *n*-component vector field  $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)$ . The canonically conjugate momentum  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$  satisfies  $[\phi_a(\boldsymbol{x}), \pi_b(\boldsymbol{x}')] = i\hbar\delta(\boldsymbol{x} - \boldsymbol{x}')\delta_{ab}$ . The integral notation is to be understood as  $\int_{\boldsymbol{x}} = \int_{\mathbb{R}^d} d^d x$ . The parameter *u* controls the strength of the anharmonic coupling, with u = 0 corresponding to the Gaussian model, and  $r_0$  is the bare square mass of the theory. In the limit  $n \to \infty$  of the number

<sup>&</sup>lt;sup>1</sup>It is conceivable that the long-time limit and the limit  $T \to 0$  may not commute.

of components of the field, the anharmonic interaction can be decoupled and accounted for by adding fluctuations to  $r_0$ . The effective Hamiltonian then describes the scalar field theory [48]

$$H_{\infty} = \frac{1}{2} \int_{\boldsymbol{x}} \left[ \pi^2 + (\nabla_{\boldsymbol{x}} \phi)^2 + r\phi^2 \right], \quad \text{with} \quad r = r_0 + \frac{u}{6} \left\langle \phi^2 \right\rangle, \tag{2}$$

where  $\langle \cdots \rangle$  indicates the expectation value with respect to the system density matrix. In this limit, the equilibrium critical properties can be determined analytically by formally solving the external constraint on the effective parameter r.

(b) The quantum spherical model [50, 51, 52] is described by the lattice Hamiltonian

$$H_{\rm sm} = \sum_{n \in \mathscr{L}} \left[ \frac{\lambda}{2} p_n^2 + \frac{\sigma}{2} s_n^2 - J \sum_{\langle n, m \rangle} s_n s_m \right],\tag{3}$$

where the "spin" operator<sup>2</sup>  $s_n$  is located at the site  $n \in \mathscr{L}$  of the hypercubic lattice  $\mathscr{L} \subset \mathbb{Z}^d$  and  $p_n$  is its canonically conjugate momentum operator, i.e.,  $[s_n, p_m] = i\hbar\delta_{nm}$ . The exchange coupling is J > 0 and the parameter  $\lambda$  quantifies the strength of quantum fluctuations in the system with  $\lambda = 0$  corresponding to the *classical* spherical model [26, 27]. The parameter  $\sigma$  is a Lagrange multiplier imposing the *spherical constraint* 

$$\sum_{n \in \mathscr{L}} \left\langle s_n^2 \right\rangle = \mathscr{N},\tag{4}$$

where  $\mathcal{N} = |\mathcal{L}|$  is the number of sites of the lattice. This constraint distinguishes the spherical model in Eq. (3) from a set of non-interacting quantum harmonic oscillators. We rescale this "standard" formulation of the spherical model as  $s_n/\sqrt{\lambda} \to s_n$ ,  $\sqrt{\lambda}p_n \to p_n$  in such a way that the canonical commutation relation is preserved. The rescaled interaction constant reads  $2J\lambda$  and is set to 1. With the substitution  $r := \sigma\lambda - d$  we obtain

$$H_{\rm sm} = \frac{1}{2} \sum_{n \in \mathscr{L}} \left[ p_n^2 + (r+d)s_n^2 - \sum_{\langle n,m \rangle} s_n s_m \right] \quad \text{with} \qquad \sum_{n \in \mathscr{L}} \left\langle s_n^2 \right\rangle = \mathscr{N}/\lambda. \tag{5}$$

In the thermodynamic limit  $\mathscr{N} \to \infty$  the quantum spherical model and the O(n) model for  $n \to \infty$ , are characterised by a non-trivial equilibrium phase diagram [51, 55, 52, 56, 57]. In particular, for spatial dimensions d > 1, a quantum critical point  $r_0^c$  (respectively  $\lambda_c$ ) is present at T = 0, separating a ferromagnetic and a paramagnetic phase. For d > 2 such a phase transition occurs also at T > 0 along the line of critical points  $r_0^c(T)$  (respectively  $\lambda_c(T)$ ). The qualitative phase diagram is shown in Fig. 1 [57]. The critical behaviour at these equilibrium transitions is exactly solvable since the complex many-body problem is reduced to the solution of a single transcendental equation. The phase transition at  $T \neq 0$  belongs to the same universality class as the classical finite-temperature phase transition, while the phase transition occurring at T = 0 in dspatial dimensions belongs to the same universality class as the classical thermal transition in d + 1 spatial dimensions [51, 52, 58]. The close relationship between these models is apparent from the comparison of the Hamiltonians in Eqs. (2) and (5), each of which is subject to an external constraint. The universality classes of the corresponding transitions in the bulk are the same [58, 49] and the phase diagrams of the models look qualitatively similar, even though microscopic details, such as the exact critical values of the relevant parameters may vary. We shall show below that this analogy carries over to the leading relaxation behaviour out of equilibrium.

#### 3. Non-Equilibrium Dynamics

Here, we discuss the effects of a coupling to a dissipative environment and formulate the non-equilibrium quantum dynamics of the statistical systems introduced in Sec. 2. This dynamics is governed by stochastic *quantum Langevin equations* which describe the dissipative aspects of an environment via a viscous and a random force.

Since the models presented in Sec. 2 are translational invariant, it is convenient to consider the Fourier components of the time-dependent fields  $\phi(t, \mathbf{r})$ , i.e.,

$$\phi_{\boldsymbol{k}}(t) = \int_{\boldsymbol{x}} \phi(t, \boldsymbol{x}) e^{-i\boldsymbol{k}\boldsymbol{x}}.$$
(6)

<sup>&</sup>lt;sup>2</sup>The operator  $s_n$  is referred to as spin operator, motivated by the analogy to the classical spin model in terms of which the spherical model was defined. In the quantum model,  $s_n$  are position operators.



Figure 1: Schematic equilibrium phase diagram of the quantum spherical model and the O(n)-model at large n, for various spatial dimensions d, scaled such that  $\lambda_c = 1/(r_0)_c$  at zero temperature. For 1 < d < 2 the models undergo a quantum phase transition at zero temperature. For d > 2, a thermal critical line appears [51, 52, 56, 59, 57].

Hereafter, we focus specifically on the O(n)-model with  $n \to \infty$ , but the discussion can be repeated for the spherical model by simply replacing  $\phi_{\mathbf{k}} \mapsto s_{\mathbf{k}}$  and  $\pi_{\mathbf{k}} \mapsto p_{\mathbf{k}}$ . For the model Hamiltonians (2) and (5), generically indicated below by H, the dynamics of the Fourier components reads [46]

$$\partial_t \phi_{\boldsymbol{k}} = \frac{\mathrm{i}}{\hbar} \left[ H, \phi_{\boldsymbol{k}} \right] + \eta_{\boldsymbol{k}}^{(\phi)},\tag{7a}$$

$$\partial_t \pi_{\boldsymbol{k}} = \frac{\mathrm{i}}{\hbar} \left[ H, \pi_{\boldsymbol{k}} \right] - \gamma \pi_{\boldsymbol{k}} + \eta_{\boldsymbol{k}}^{(\pi)},\tag{7b}$$

with two distinct noise operators  $\eta_{k}^{(\phi)}$  and  $\eta_{k}^{(\pi)}$  whose properties will be specified further below. The damping parameter  $\gamma$  is positive and we restrict our analysis to Ohmic damping, i.e. we assume that the damping is frequency-independent. All the information on the environment and its coupling to the system is contained in the correlation functions of the noises. There are two mathematically equivalent ways to specify these correlation functions. First, following Refs. [44, 10, 41, 42, 40], one may take the environment into account explicitly and indicate its quantum Hamiltonian  $H_{env}$  and its interaction  $H_{int}$  with the system. The composite system  $H_{\text{tot}} = H + H_{\text{env}} + H_{\text{int}}$  then evolves unitarily and by admitting the environment to be a bath of thermal harmonic oscillators much larger than the system, the equation of motion for the degrees of freedom of the environment can be explicitly solved. The average over the distribution of the environment can then be carried out, provided one specifies an adequate spectral function for the bath [44]. Alternatively, one may consider the system H to be a set of harmonic oscillators. Rather than specifying the properties of the environment explicitly, one may model the dissipative aspects by an Ohmic damping and two noises (one for each conjugate bosonic variable). The correlations of these ad hoc noises cannot explicitly contain system parameters and should be determined in such that the resulting evolution of the system satisfies the following fundamental properties [46]: (i) canonical equal-time commutation relation, (ii) Kubo formula, (iii) the virial theorem and (iv) the quantum fluctuation-dissipation theorem for any T > 0. Both procedures lead exactly to the same noise specifications. For times  $t \neq t'$  the non-vanishing two-time noise correlators are given by

$$\left\langle \left\{ \eta_{\boldsymbol{k}}^{(\phi)}(t), \eta_{\boldsymbol{k}'}^{(\pi)}(t') \right\} \right\rangle = \gamma T \coth\left(\frac{\pi}{\hbar} T(t-t')\right) \delta(\boldsymbol{k}+\boldsymbol{k}'), \tag{8a}$$

$$\left\langle \left[ \eta_{\boldsymbol{k}}^{(\phi)}(t), \eta_{\boldsymbol{k}'}^{(\pi)}(t') \right] \right\rangle = \mathrm{i}\hbar\gamma\,\delta(t-t')\delta(\boldsymbol{k}+\boldsymbol{k}'). \tag{8b}$$

Here, the average is done on these noises, and all other averages of the noise (anti-)commutators vanish.<sup>3</sup> We shall be mainly interested in the long-time behaviour of the models after a quench. The analysis of the ensuing dynamics will be greatly simplified if one eliminates the equation of motion for the momentum  $\pi_k$ , which can be done by taking a formal scaling limit [46]

$$\lambda \to 0, \quad t \to \infty, \quad \text{with} \quad \tilde{t} = \lambda t = \text{cst.},$$
(9)

with the fixed and finite damping constant  $\tilde{\gamma} = \lambda \gamma$  and the rescaled temperature  $\tilde{T} = T/\lambda$ . Although the limit  $\lambda \to 0$  would correspond to a classical dynamics for t = cst. we emphasise here that this scaling limit

<sup>&</sup>lt;sup>3</sup>The construction of the noise anti-commutators is done in frequency space. Thus, Eq. (8) is correct up to a set of time differences of measure zero [46]. In particular, this implies that the equal-time anticommutators may remain finite. The noise correlators have to be understood as distributions.

does not reduce to that case since time is also scaled appropriately. We now relabel the rescaled variables by dropping all tildes from the variables  $\tilde{t}$ ,  $\tilde{\gamma}$  and  $\tilde{T}$  and focusing on the zero-temperature case T = 0. The Langevin equations (7) then reduce to a single over-damped Langevin equation with a "composite" noise  $\xi_k$ [44, 46, 10, 41, 42, 40], i.e.,

$$\gamma \partial_t \phi_{\mathbf{k}}(t) + \left( r(t) + k^2 \right) \phi_{\mathbf{k}}(t) = \xi_{\mathbf{k}}(t), \tag{10}$$

where r = r(t) is to be found self-consistently from the constraint [cf. Eqs. (5) and (2)] and  $\gamma$  is the rescaled damping parameter. Additional details on this long-time limit are provided in Appendix A. The properties of the noise  $\xi_k$  will largely determine the (quantum) character of the resulting dynamics and since we shall consider various cases, we shall specify its properties below in more detail.

The non-equilibrium dynamics has been reduced to an effective over-damped Langevin equation, which is formally identical to the classical Langevin equation.<sup>4</sup> Any distinction between classical and quantum dynamics will now solely emerge from the form of the noise correlation functions. In this work, we shall distinguish the following three cases:

(i) Quantum dynamics, derived from Eqs. (8), and described by [40, 41, 43, 44, 45, 10, 46]

$$\langle \{\xi_{\boldsymbol{k}}(t),\xi_{\boldsymbol{k}'}(t')\}\rangle = \frac{2\gamma\hbar}{\pi} \int_0^\infty \mathrm{d}\omega \,\omega \coth\left(\frac{\hbar\omega}{T}\right) \cos(\omega(t-t')) \,\delta(\boldsymbol{k}+\boldsymbol{k}'),\tag{11a}$$

$$\langle [\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}'}(t')] \rangle = 2i\hbar\gamma \left(\frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t')\right) \,\delta(\boldsymbol{k}+\boldsymbol{k}'),\tag{11b}$$

where T is the bath temperature and the noise correlation function is non-Markovian.

(ii) Classical dynamics, obtained from Eq. (11) in the limit  $\hbar \to 0$ , leading to

$$\langle \{\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}'}(t')\} \rangle = 4T\gamma\delta(t-t')\delta(\boldsymbol{k}+\boldsymbol{k}'), \quad \text{with} \quad \langle [\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}'}(t')] \rangle = 0.$$
(12)

This is the well-studied Markovian white noise, see, e.g., Refs. [18, 20, 28]. The central question in this work essentially concerns the consequences of the non-Markovian quantum noise in Eq. (11) in comparison with the Markovian classical white noise in Eq. (12).

 (iii) Effective dynamics [60], inspired by a simple scaling argument of the zero-temperature limit of Eqs. (11), i.e.,

$$\langle \{\xi_{\boldsymbol{k}}(t),\xi_{\boldsymbol{k}'}(t')\}\rangle = \mu |\boldsymbol{k}|^2 \delta(t-t') \delta(\boldsymbol{k}+\boldsymbol{k}'), \quad \text{with} \quad \langle [\xi_{\boldsymbol{k}}(t),\xi_{\boldsymbol{k}'}(t')]\rangle = 0,$$
(13)

with a dimensionless control parameter  $\mu$ . This is a classical noise with a momentum-dependent effective temperature  $T_{\text{eff}} = \mu |\mathbf{k}|^2/2$ . As we shall see below, the analysis of the simplified noise correlators in Eq. (13) is an efficient short-cut for studying ageing, since it readily reproduces the ageing behaviour which usually follows from a technically demanding analysis of the actual quantum noise in Eq. (11). In particular, the effective description of the noise in Eq. (13) circumvents the difficulties due to the non-locality in time of the actual correlator by introducing a more complicated spatial structure of the noise, which is however, amenable to analytical calculations. In this spirit, the factor  $|\mathbf{k}|^2$  in Eq. (13) is the result of the underlying spatio-temporal scaling of these models described by the dynamical exponent z = 2. Heuristically, the effective scaling dependence  $\sim (t - t')^{-2}$  of the r.h.s. of Eq. (11a) is replaced in Eq. (13) by  $|\mathbf{k}|^2 \times \delta(t - t')$ where each of the two factors brings in a scaling dependence  $\sim (t - t')^{-1}$ .

Non-Markovian effects are most prominent at zero temperature and we shall therefore focus our analysis on this case. The limit  $T \rightarrow 0$  in Eqs. (11) does not affect the noise commutator, while the anticommutator in Eq. (11a) is given by a singular integral. Following standard procedures [44] this singular integral is regularised by introducing an additional microscopic time-scale  $t_0 > 0$  such that the noise anticommutator reads

$$\langle \{\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}'}(t')\} \rangle = \frac{\gamma \hbar}{\pi} \int_{-\infty}^{\infty} d\omega \, |\omega| \, e^{i\omega(t-t')} e^{-t_0|\omega|} \, \delta(\boldsymbol{k} + \boldsymbol{k}').$$
(14)

While for  $t_0 = 0$ , the integral formally diverges, it is possible to interpret it as a distribution [61]. We prefer to avoid the explicit use of distributions and consider below the noise correlator Eq. (14) in its regularised form. Equation (14) then gives explicitly [44]

$$\langle \{\xi_{\mathbf{k}}(t), \xi_{\mathbf{k}'}(t')\} \rangle = 2\frac{\gamma\hbar}{\pi} \frac{\mathbf{t}_0^2 - (t - t')^2}{\left[\mathbf{t}_0^2 + (t - t')^2\right]^2} \,\delta(\mathbf{k} + \mathbf{k}').$$
(15)

In Fig. 2 this regularised form is compared with that of equal width of the corresponding regularised classical

 $^{4}$ Despite the formal similarity of the classical and the quantum equation, the latter is an operator equation. To simplify the notation we shall not emphasise this distinction.



Figure 2: Dependence of the noise correlations  $\langle \{\xi_k(t), \xi_{k'}(t')\} \rangle$  in Eq. (11a), with an exponential regularisation parameter  $t_0$ , on the dimensionless time difference  $\tau \equiv (t - t')/t_0$ , normalised by its value at t' = t. We compare the case of the quantum noise (QN) at zero temperature, corresponding to  $T \to 0$  in Eq. (11a) with the case of the Wiener process (WP) [62] with the same width, corresponding to  $\hbar \to 0$  in Eq. (11a), which is equivalent to a classical white noise.

Wiener process obtained by formally taking the limit  $\hbar \to 0$  of Eq. (11) after introducing the factor  $e^{-t_0|\omega|}$  in Eq. (11a), as done above for the case  $T \to 0$ . This process is known to describe Brownian motion in the long-time limit, as the noise correlator is  $\propto \delta(t - t')$  as  $t_0 \to 0$ . While the central peaks in Fig. 2 look quite similar, the quantum noise decays with a power-law tail for large time differences  $\tau = (t - t')/t_0$  and, furthermore, it is anti-correlated (but within a narrow region, of width  $\sim t_0$  around  $\tau = 0$ ), which is distinct from the exponential decay of the positively correlated Wiener process. The present work investigates the consequences of the differences in the noise anti-commutators illustrated in Fig. 2 on the long-time dynamics of the two models described in Sec. 2. In particular, we shall address the extent to which the non-Markovian character of the quantum noise is important and if the effective noise is successful in reproducing quantum properties.

The role of the cut-off  $t_0$ , which introduces a new time scale into the dynamics, can be better understood as follows. In Appendix B we analyse the case of a single quantum harmonic oscillator, focusing on the equal-time commutation relation and on the virial theorem that are both known to hold true for the quantum Langevin dynamics. In the limit  $T \rightarrow 0$ , the inertial term is responsible for these quantities to be well-defined by regularising the integrals over the bath degrees of freedom. We show that, for the physical quantities we focus on, the inertial term can be effectively substituted by the regularised noise correlator in the equations of motion. Although the noise structure is inherently responsible for conserving these properties, the elimination of the inertial terms substantially weakens this effect and a cut-off is needed to ensure a sensible quantum dynamics. In this way, the damping rate  $\gamma$  also sets the cut-off scale as  $t_0 \sim 1/\gamma$ .

Finally, we notice that the over-damped Langevin equation will lead to a dynamical exponent z = 2, in contrast to closed quantum systems where the unitary evolution generically leads to z = 1 [63, 64]. As stated in point (iii) above a simple dimensional analysis with z = 2 naturally leads to the effective noise correlator (13).

In summary, we have seen that the long-time behaviour of the spherical model and O(n) model for  $n \to \infty$  can always be described by the over-damped Langevin equation (10), in which the actual physical nature of the dynamics at temperature T = 0 only enters via the noise correlators. In particular: (i) for quantum dynamics these correlators are given by Eqs. (11b) and (15), (ii) for classical white noise by Eq. (12), and (iii) for the effective noise, by Eq. (13).

#### 4. Relaxation and Ageing - Analytical Predictions

In this section, we summarise and interpret our main results. The detailed analysis is presented in Secs. 5 and 6. The formal solution of Eq. (10), which provides the basis of all further analyses, is

$$\phi_{\mathbf{k}}(t) = \frac{\exp(-k^2 t/\gamma)}{\sqrt{g(t)}} \left[ \phi_{\mathbf{k}}(0) + \frac{1}{\gamma} \int_0^t \mathrm{d}s \,\sqrt{g(s)} \,\exp\left(k^2 s/\gamma\right) \,\xi_{\mathbf{k}}(s) \right],\tag{16}$$

where we introduced

$$g(t) := \exp\left(\frac{2}{\gamma} \int_0^t \mathrm{d}s \, r(s)\right),\tag{17}$$

in analogy with the treatment of the classical non-equilibrium spherical and O(n)-models for  $n \to \infty$  [20, 28, 39]. Once the function g(t) is determined from the consistency conditions in Eqs. (2) and (5), the non-equilibrium dynamics of these models is solved.

First, we focus on the upper and lower critical dimensions of these models, denoted by  $d_u$  and  $d_l$ , respectively, in comparison with the classical critical behaviour. At equilibrium, the T = 0 quantum phase transition in d spatial dimensions belongs to the same universality class as the thermal phase transition of the corresponding (d + 1)-dimensional classical system [65, 51, 52]. The upper and lower critical dimensions for the classical system are  $d_l^{(cl,eq)} = 2$  and  $d_u^{(cl,eq)} = 4$  [26] and thus their quantum analogues are  $d_l^{(qu,eq)} = 1$  and  $d_u^{(qu,eq)} = 3$ . As we shall see, the coupling to an external reservoir does affect this behaviour. However, in the diffusive scaling limit  $k^2t = \text{cst.}$  which is most natural for the over-damped Langevin equation, the lower and upper critical dimensions are shifted as  $d_l^{(qu)} = 0$  and  $d_u^{(qu)} = 2$  for quantum noise. This differs from both the classical dynamics — for which the lower and upper critical dimensions are the same as at equilibrium — and from the quantum behaviour at equilibrium. It follows that the stationary state of the dynamics driven by the quantum Langevin equation (10) with noises (15), and (11b) is neither a classical nor a quantum equilibrium state at the specific time- and length-scales dictated by the diffusive scaling limit. This statement is supported by the analyses presented below, from which it turns out that it is not possible to satisfy a fluctuation-dissipation relation, even for a quench to the disordered phase. Accordingly, any phase transitions eventually found in the dynamics should be interpreted as a sort of kinetic phase transition. In particular, for  $0 < d \leq 1$ , there is no equilibrium analogue of an ordered phase.

Our analysis of the dynamics consists in the calculation of the following quantities. First, we study the equal-time correlation function  $C_{\mathbf{k}}(t)$  defined by

$$\delta(\boldsymbol{k} + \boldsymbol{k}')C_{\boldsymbol{k}}(t) := \left\langle \left\{ \phi_{\boldsymbol{k}}(t), \phi_{\boldsymbol{k}'}(t) \right\} \right\rangle.$$
(18)

In order to obtain spatio-temporal information about the non-equilibrium state at different time and length scales we shall study  $C_{\mathbf{k}}(t)$  in the scaling limit

$$t \to \infty, \quad k \to 0, \quad \rho := \frac{k^2 t}{\gamma} = \text{cst.}$$
 (19)

Next, we investigate the two-time linear response functions

$$R_{\boldsymbol{k}}(t,s) := \frac{\delta \langle \phi_{\boldsymbol{k}}(t) \rangle}{\delta h_{\boldsymbol{k}}(s)} \Big|_{\boldsymbol{h}=0} \quad \text{and} \quad R(t,s) := \int_{\boldsymbol{k},(\Lambda)} \frac{\delta \langle \phi_{\boldsymbol{k}}(t) \rangle}{\delta h_{\boldsymbol{k}}(s)} \Big|_{\boldsymbol{h}=0}, \tag{20}$$

where, assuming spatial rotational invariance, we introduce the short-hand  $\int_{\mathbf{k},(\Lambda)} = \int_0^{\Lambda} \int_{S^d} d\mathbf{k}/(2\pi)^d$  for the momentum integration, over a hyper-sphere  $S^d$  with radius up to  $\Lambda$ .  $R_{\mathbf{k}}(t,s)$  and R(t,s), respectively, indicate the response of the order parameter  $\langle \phi_{\mathbf{k}}(t) \rangle$  at time t to a perturbation of its conjugate field  $h_{\mathbf{k}}(s)$  at time s and the (auto-) response of the order parameter at a certain point in space to an earlier perturbation applied at the same point. The auto-response function R(t,s) is particularly useful for studying ageing behaviour [13, 16, 15, 20]. Usually, one refers to s as the *waiting time* and to t as the *observation time*. Finally, we consider the two-time correlation functions

$$\delta(\mathbf{k} + \mathbf{k}')C_{\mathbf{k}}(t, s) := \langle \{\phi_{\mathbf{k}}(t), \phi_{\mathbf{k}'}(s)\} \rangle \quad \text{and} \quad C(t, s) := \int_{\mathbf{k}, (\Lambda)} C_{\mathbf{k}}(t, s),$$
(21)

which describe, respectively, how correlations propagate across the system and how the auto-correlation evolves.<sup>5</sup>

In Fig. 3, we illustrate the three different quench protocols that we consider below. First, for quenches which remain within the one-phase region, i.e., for  $r_0 > r_0^c$ , we find a rapid relaxation towards a stationary state and we analyse the influence that the quantum noise statistics has on its properties, in particular how they differ from the equilibrium case. The ageing behaviour which is studied via the scaling of  $R_k(t,s)$  and  $C_k(t,s)$ , defined in Eqs. (20) and (21) arises for quenches onto or below the critical point, i.e., for  $r_0 \leq r_0^c$ . In this case, we analyse the dynamics in the long-time scaling limit [16] in which both times t and s are simultaneously large such that

$$s \to \infty, \quad t \to \infty \quad \text{with fixed} \quad y := t/s > 1.$$
 (22)

Then the two-time auto-response and auto-correlation functions are expected to scale as

$$R(t,s) = s^{-1-a} f_R(t/s)$$
 and  $C(t,s) = s^{-b} f_C(t/s)$ . (23)

<sup>5</sup>Since the equation of motion (10) is linear and we consider quantities at most quadratic in the order parameter  $\phi_{k}$ , only the second moments of the noises are required.



Figure 3: Schematic illustration of the three distinct quench protocols considered in the present work. The gray shaded area indicates qualitatively the stationary ordered phase of the models under investigation. The system is prepared in a certain excited state with vanishing order parameter and subsequently quenched either to the ordered regime, the critical point or the disordered phase. All quenches lead to a formal zero-temperature state which, however, is not an equilibrium state.

In the limit  $y \to \infty$  one expects for the asymptotics of the scaling functions  $f_{R,C}$ 

$$f_R(y) \simeq f_{R,\infty} y^{-\lambda_R/z}$$
 and  $f_C(y) \simeq f_{C,\infty} y^{-\lambda_C/z}$ , (24)

which define the auto-response exponent  $\lambda_R$ , the auto-correlation exponent  $\lambda_C$  and the ageing exponents a and b. Their values depend on whether  $r_0 < r_0^c$  or  $r_0 = r_0^c$ . We shall also see that z = 2 throughout.

We shall study the coarsening as well as the ageing behaviour by preparing the system in an initial state with

$$\langle \phi_{\boldsymbol{k}}(0) \rangle = 0 \quad \text{and} \quad C_{\boldsymbol{k}}(0) \stackrel{\kappa \searrow 0}{=} c_{\alpha} k^{\alpha},$$
(25)

where  $C_{\mathbf{k}}(t)$  is defined in Eq. (21) such that the initial order parameter vanishes but yet we admit long-range initial correlations  $C(0, \mathbf{r}) \sim |\mathbf{r}|^{-d-\alpha}$  as  $|\mathbf{r}| \to \infty$  with  $d + \alpha > 0$  such that the initial correlations decay upon increasing the distance. While the case  $\alpha = 0$  describes short-ranged initial correlations [20], long-ranged initial correlations are obtained for  $\alpha < 0$  [28]. Because of the vanishing initial order parameter, we interpret the initial state (25) as being disordered, see Fig. 3.

With the initial conditions (25) and a generic bath correlator

$$\langle \{\xi_{\boldsymbol{k}}(t'), \xi_{\boldsymbol{k}'}(s')\} \rangle = m_{\boldsymbol{k}}(t'-s')\delta(\boldsymbol{k}+\boldsymbol{k}'), \qquad (26)$$

the two-time response and correlation functions, defined according to Eqs. (20) and (21), can be expressed as

$$R_{k}(t,s) = \frac{1}{\gamma} \sqrt{\frac{g(s)}{g(t)}} e^{-k^{2}(t-s)/\gamma} \Theta(t-s),$$
(27a)

$$C_{k}(t,s) = \gamma^{2} R_{k}(t,0) R_{k}(s,0) C_{k}(0) + \int_{0}^{t} \mathrm{d}t' \int_{0}^{s} \mathrm{d}s' R_{k}(t,t') R_{k}(s,s') m_{k}(t',s'),$$
(27b)

where the Heaviside function  $\Theta(t-s)$  imposes the causality condition t > s for the response function. By setting t = s in (27b), one obtains the equal-time (or one-time) correlator  $C_{\mathbf{k}}(t) = C_{\mathbf{k}}(t,t)$ .

#### 4.1. Effective Dynamics

Compared to solving the dynamics of the model in the presence of the actual quantum noise, it turns out that it is considerably easier to determine the dynamical behaviour driven by the Markovian effective noise (13), starting from the initial conditions (25). Since the effective noise is constructed such that its scaling dimension is the same as the one of the actual quantum noise (15), the leading scaling behaviour and exponents which characterise the emerging ageing behaviour in the presence of the effective Markovian dynamics will also hold for the actual quantum dynamics, as we shall see below. For concreteness, we use here the language of the O(n)-model in the  $n \to \infty$  limit, but all universal quantities concerning the long-time behaviour will be the same as for the spherical model. The constraint in Eq. (2) reduces to a linear integro-differential equation for the function g(t), defined in Eq. (17). Standard techniques for the solution are available [16, 20, 28], the main steps of which we recall in Appendix C. We obtain the (non-universal) critical point of the dynamics

$$r_0^c = -\frac{u}{12} \frac{\mu}{\gamma} \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^d}{d},\tag{28}$$

where  $\Omega_d$  is the *d*-dimensional solid angle. For a fixed UV cutoff  $\Lambda < \infty$ ,  $r_0^c$  is finite for all spatial dimensions d > 0, which confirms the above argument on the lower critical dimension  $d_l = 0$ . Depending on the sign of the difference  $r_0 - r_0^c$  we can distinguish the various cases schematically represented in Fig. 3, which we discuss below.

(1) For a quench remaining in the disordered phase  $r_0 > r_0^c$ , we find an exponential long-time growth

$$g(t) \sim \exp(t/\tau_{\rm r})$$
 (29)

of the function g(t), with a characteristic time scale  $\tau_{\rm r}$ . This time scale diverges as the quench parameter  $r_0$  approaches criticality from above, i.e., as  $r_0 - r_0^c \to 0^+$ . At long times after the quench, i.e., in the stationary limit, the equal-time correlation function  $C_k(\infty)$  becomes

$$C_{\boldsymbol{k}}(\infty) \simeq \frac{\mu}{\gamma^2} \frac{k^2}{1/\tau_{\rm r} + 2k^2/\gamma}.$$
(30)

This reproduces the standard classical Ornstein-Zernicke form (see, e.g., Ref. [20]), up to the momentumdependent effective temperature  $T_{\text{eff}} = T_{\text{eff}}(k) = \mu k^2/2$  which comes from the noise correlator in Eq. (14). In the same stationary limit, the two-time response and correlation functions rapidly converge to

$$R_{\boldsymbol{k}}(t,s) \simeq \frac{1}{\gamma} \exp\left(-\left(\frac{1}{2\tau_{\rm r}} + \frac{k^2}{\gamma}\right)(t-s)\right),\tag{31a}$$

$$C_{\boldsymbol{k}}(t,s) \simeq \frac{\mu k^2}{\gamma^2} \frac{1}{\frac{1}{\tau_{\rm r}} + 2\frac{k^2}{\gamma}} \exp\left(-\left(\frac{1}{2\tau_{\rm r}} + \frac{k^2}{\gamma}\right)(t-s)\right),\tag{31b}$$

which only depend on the time difference  $\tau = t - s$  (with exponentially small corrections in t, s and  $\tau$ ). Note that all these expressions do not contain any reference to the parameter  $\alpha$  which describes the initial correlations, indicating that the memory of the initial state is lost in the long-time limit. In addition, the expressions (31) satisfy an effective version of the classical fluctuation-dissipation theorem for every mode, i.e., for  $\tau > 0,^6$ 

$$\frac{\partial C_{\boldsymbol{k}}(\tau)}{\partial \tau} = -\frac{\mu}{2\gamma} k^2 R_{\boldsymbol{k}}(\tau) = -\frac{T_{\text{eff}}(k)}{\gamma} R_{\boldsymbol{k}}(\tau), \qquad (32)$$

with the same mode-dependent effective temperature as the one determined above. This classical behaviour controlled by  $T_{\text{eff}}(k)$  is a direct consequence of the form of the noise (13).

- (2) For a critical quench, g(t) displays a very different long-time behaviour, being algebraic rather than exponential. This is expected since the relaxation time-scale  $\tau_r$  diverges as  $r_0 \rightarrow r_0^c$  approaches criticality. Depending on the spatial dimension d, it turns out that three different cases must be distinguished, as schematically shown in Fig. 4:
  - region I: 0 < d < 2 and  $0 < d + \alpha$ . The fluctuations introduced in the dynamics by the quantum reservoir are relevant.
  - region II: 2 < d and  $0 < d + \alpha < 2$ . The quantum fluctuations due to the reservoir are irrelevant but the initial fluctuations are relevant.
  - region III: 2 < d and  $2 < d + \alpha$ . The quantum fluctuations due to the reservoir and those due to the initial state are both irrelevant and the scaling behaviour of the system is accurately described by the mean-field theory.

<sup>6</sup>One should not confuse the equal-time correlator  $C_{k}(t)$  with the time-translation-invariant two-time correlator  $C_{k}(\tau) = C_{k}(\tau + s, s)$  in those cases where it is independent of the waiting time s.

Table 1: Non-equilibrium exponents for critical and subcritical quenches. In the former case, one distinguishes three possible regions I – III (see Fig. 4) depending on the dimension d. The exponent F is defined in Eq. (33).

quench		F	$\lambda_C$	$\lambda_R$	a	b
$r_0 = r_0^c$	I. $0 < d < 2$	$-\frac{\alpha}{2}$	$d + \frac{\alpha}{2}$	$d - \frac{lpha}{2}$	$\frac{d}{2} - 1$	$\frac{d}{2}$
	II. $2 < d, d + \alpha < 2$	$1 - \frac{d+\alpha}{2}$	$1 + \frac{d+\alpha}{2}$	$\frac{d-\alpha}{2}+1$	$\frac{d}{2} - 1$	1
	III. $2 < d, d + \alpha > 2$	0	$d + \alpha$	d	$\frac{d}{2} - 1$	$\frac{d+\alpha}{2}$
$r_0 < r_0^c$		$-\frac{d+\alpha}{2}$	$\frac{d+\alpha}{2}$	$\frac{d-\alpha}{2}$	$\frac{d}{2} - 1$	0



Figure 4: Illustration of the different regions as a function of the dimension d and the exponent  $\alpha$ , corresponding to the initial correlations different asymptotic dynamics for a quench to the critical point as summarised in Table 1.

In all three critical cases we find

$$g(t) \sim t^{\mathcal{F}}.\tag{33}$$

The values of the exponent F are listed in Table 1. Region III corresponds to having F = 0, i.e., the spherical constraint does not affect the dynamics, which is therefore the same as that of a set of uncorrelated free bosonic modes.

Using the formal solution (16), we obtain the equal-time correlation function, directly in the scaling limit (19)

$$C_{\boldsymbol{k}}(t) \simeq \rho^{-F} e^{-2\rho} \left[ \mathscr{C}_{(1)} c_{\alpha} k^{\alpha+2F} + \frac{\mu}{\gamma} \int_{0}^{\rho} \mathrm{d}x \, e^{2x} x^{F} \right], \tag{34a}$$

where  $\mathscr{C}_{(1)}$  is a known constant and  $\rho$  is defined in Eq. (19). Depending on the sign of  $\alpha + 2F$ , either the initial correlations (first term) or the reservoir fluctuations (second term) dominate Eq. (34a) in the scaling limit. The exponents in table 1 show that in region I, both terms contribute while in regions II and III, the initial correlations are dominant. For the two-time auto-response and autocorrelation functions we find, in the scaling regime (22), with y = t/s > 1,

$$R(ys,s) = R_{(0)}s^{-d/2}y^{-F/2}(y-1)^{-d/2},$$
(34b)

$$C(ys,s) \simeq s^{-d/2} \left[ \mathscr{C}_{(2),1} c_{\alpha} s^{-F-\alpha/2} \frac{y^{-F/2}}{(1+y)^{(d+\alpha)/2}} + \mathscr{C}_{(2),2} \frac{\mu}{\gamma} \int_{-1}^{1} \mathrm{d}x \, \frac{(1-x)^{F}}{(y+x)^{d/2+1}} \right],\tag{34c}$$

where  $R_{(0)}$ ,  $\mathscr{C}_{(2),1}$  and  $\mathscr{C}_{(2),2}$  are known constants whose values will not be needed here. Up to an overall normalisation, the auto-response function is universal and is independent of both the initial and the noise correlations. For the autocorrelator, instead, we find that both sources of fluctuations contribute to the scaling function in region I, although the leading asymptotic behaviour of  $f_C(y)$  (see Eq. (23)) for  $y \to \infty$  is dominated by the initial noise correlations. On the other hand, in regions II and III, the initial correlations dominate for  $\alpha < 0$ . Based on their definitions in Eqs. (23) and (24), the auto-response, autocorrelation and ageing exponents can now be easily determined from the asymptotic behaviours of Eq. (34), resulting in the values listed in Table 1.

(3) For a quench across the critical point and into the ordered phase, i.e., with  $r_0 < r_0^c$ , the long-time behaviour is again algebraic as in Eq. (33) with  $\mathcal{F} = -(d + \alpha)/2$ . In terms of this  $\mathcal{F}$ , the relevant correlation and response functions are still given by Eq. (34), provided that  $d+\alpha < 2$ . In this case, Eqs. (34) hold exactly as the asymptotic form of g(t) is integrable in the origin. For  $d+\alpha > 2$  the noise contributions in Eqs. (34) are formally not defined, which is merely an artifact of substituting the asymptotic limit too early. However, this does not actually matter as the k-exponent in the initial term remains negative and thus the initial contributions keep on dominating (as opposed to the critical case I). Critical and subcritical quenches are distinct in one important aspect. Namely, for subcritical quenches, the initial correlations always dominate the long-time asymptotic behaviour, which is not always the case for critical quenches. Comparing the auto-response and auto-correlation exponents  $\lambda_R$  and  $\lambda_C$ , defined in Eq. (24) respectively, for quenches with  $r_0 \leq r_0^c$ , we see that one always has

$$\lambda_C = \lambda_R + \alpha. \tag{35}$$

This relationship is known to hold also for classical dynamics, and it has been derived both from the analysis of explicit models and from general arguments, although within a different range of dimensions [13, 28, 66, 16].

Summarising, is there any evidence for a clear quantum effect on the ageing behaviour? The answer is certainly affirmative because of the dimensional shift  $d \mapsto d-2$  when going from classical dynamics at T > 0 to quantum dynamics at T = 0. But is there any additional contribution coming from the noise correlations of the quantum bath? Our results for the effective quantum noise suggest that this can happen only in case I of the critical quench, i.e., for dimensions d < 2. In addition, in the presence of long-range initial correlations with  $\alpha < 0$ , the leading asymptotics of the scaling function  $f_C(y)$  for  $y \to \infty$  depends only on the initial noise correlator, see Eq. (34). However, for fully disordered initial conditions with  $\alpha = 0$ , both quantum and initial noise correlators do contribute to the scaling function  $f_C(y)$  at a generic value of y. Accordingly, also the asymptotic amplitude  $f_{C,\infty}$  contains a non-vanishing contribution from the bath noise correlations and therefore a signature of the original quantum nature of the system.

#### 4.2. Quantum Noise

The quantum noise in Eqs. (11a) and (15) is not Markovian. This implies that the techniques used above and in Appendix C for solving the spherical constraint in Eq. (2) in the case of the effective dynamics are no longer applicable. In fact, the non-Markovianity leads to a *non*-linear integro-differential equation for g(t), rather than a linear one, as we shall see in Eqs. (50a) and (50b).

The approach we used to solve it is explained in Secs. 5 and 6. Here we provide an overview of the conclusions of this analysis. Various aspects of the results we present below for the emerging scaling behaviour of the dynamics can be expressed in terms of the asymptotics of the auxiliary function (see Eqs. (5.2.13), (5.2.16), and (5.2.35) in Ref. [67]) with x > 0,

$$g_{\rm AS}(x) := \int_0^\infty dt \, \frac{\cos t}{t+x} \simeq \begin{cases} -\left(C_E + \ln x\right) & \text{for } x \ll 1, \\ 1/x^2 & \text{for } x \gg 1, \end{cases}$$
(36)

where  $C_E = 0.5772...$  is Euler's constant. Using the formulation in terms of the O(n)-model with  $n \to \infty$ , the critical point at T = 0 and in the limits  $\gamma \to \infty$ ,  $\Lambda \to \infty$ ,  $t_0 \to 0$  with  $\Lambda^2 t_0 / \gamma = \text{cst.}$  can be evaluated. For any d > 0 we find

$$r_0^c = \frac{u}{12} \frac{4\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^d}{d} \left[ \ln\left(\Lambda^2 \frac{\mathbf{t}_0}{\gamma}\right) + \mathcal{C}_E - \frac{2}{d} \right].$$
(37)

Note that  $r_0^c$  is finite for any d and therefore we conclude that  $d_l = 0$ , while the actual value of  $r_0^c$  now depends on both cut-off parameters  $\Lambda$  and  $t_0$ .

As in Sec. 4.1, we discuss separately the different possible quenches.

(1) For a quench to the disordered phase  $r_0 < r_0^c$ , we still find the exponential long-time behaviour of g(t) as in Eq. (29). Hence, we can identify a finite time-scale  $\tau_r$  which is distinct from that of the effective quantum noise but still diverges as  $r_0 \to r_0^{c,+}$ . The equal-time correlation function then reads, in the stationary limit,

$$C_{\boldsymbol{k}}(\infty) \simeq \frac{\hbar}{\pi\gamma} g_{\rm AS} \left( \mathsf{t}_0 \left( k^2 / \gamma + (2\tau_{\rm r})^{-1} \right) \right), \tag{38}$$

instead of Eq. (30). In Fig. 5 (left panel), this stationary correlator is shown as a function of the momentum k. For small values of k, its shape is almost identical to the one of the classical Ornstein-Zernicke form. However, at larger momenta a crossover towards a different behaviour occurs. The expression for the stationary correlations in (30), corresponding to the effective dynamics is, however, completely different, as shown in the figure. If the Ornstein-Zernicke form were exact, the stationary spatial correlator  $C(\infty, R) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}}C_{\mathbf{k}}(\infty)$  in d = 1 would become exponential, i.e.,  $C(\infty, R) = e^{-(\gamma/\tau_r)^{1/2}|R|}$  as a function of the distance R. The right panel in Fig. 5 shows the stationary correlator  $C(\infty, R)$ , obtained from Eq. (38), as a function of the distance R, for several values of the relaxation time  $\tau_r$ . Indeed, although there is an exponential decay for sufficiently large values of |R|, the correlator  $C(\infty, R)$  is rounded, compared to the exponential, for  $|R| \to 0$  which comes about since the spherical spins are softer than, e.g., those of the d = 1 Glauber-Ising model [16]. Since the Langevin equation (10) is linear, the two-time response function is given by the same expression as in Eq. (31), found for the effective noise. However, the two-time correlation function shows a different behaviour, which depends on the separation  $\tau = t - s$  of the involved times t and s, with t > s. We focus here on the case  $t \gg s \gg 1$  with  $\tau$  being kept fixed and large. Then

$$C_{\mathbf{k}}(s+\tau,s) \stackrel{s\gg\tau}{\simeq} -\frac{2\hbar}{\pi\gamma} \frac{1}{\left[(2\tau_{\rm r})^{-1} + k^2/\gamma\right]^2} \frac{1}{\tau^2},$$
 (39a)

$$R_{\mathbf{k}}(\tau) \simeq \frac{1}{\gamma} \exp\left(-\left(\frac{1}{2\tau_{\rm r}} + \frac{k^2}{\gamma}\right)\tau\right). \tag{39b}$$



Figure 5: Equal-time correlation function  $C_k(\infty)$  for a quench to the disordered phase in d = 1 spatial dimension and with  $\gamma = 1$ . Left panel: stationary correlation function in momentum space for the effective dynamics (eff) and the quantum noise (QN), compared to the Ornstein-Zernicke form (OZ). For illustration purposes we introduced a normalisation parameter A and choose  $A = C_{\infty}(\infty)$  for the effective noise and  $A = C_0(\infty)$  for the OZ form and the quantum noise. Right panel: stationary real-space correlator  $C(\infty, R)$ , derived from Eq. (38), as a function of the distance R and normalised such that  $C(\infty, 0) = 1$ .

Although these expressions are stationary, being dependent only on  $\tau$ , the correlator features an algebraic rather than an exponential decay for  $\tau \gg 1$ . Thus, no obvious fluctuation-dissipation ratio emerges. This shows that the steady state of the zero-temperature quantum dynamics after a quench in the one-phase region cannot be an equilibrium state.

(2) For a critical quench, we need to distinguish the different cases I-III listed in Table 1. Since the relaxational time scale  $\tau_{\rm r}$  which characterises the sub-critical quench diverges when criticality is approached, we now have an algebraic long-time behaviour as in Eq. (33), with a multiplicative prefactor which we denote by  $g_c$  and an exponent F which is the same as for the effective dynamics. In order to calculate the remaining exponents  $\lambda_{C,R}$ , a, and b, from Eqs. (24) and (23) we discuss below the correlation and response functions. The equal-time correlation function  $C_k(t)$  can be decomposed into two contributions,

$$C_{k}(t) = C_{k}^{(ic)}(t) + C_{k}^{(n)}(t),$$
(40a)

the first being determined by the correlations in the initial state (ic) and the second by the quantum noise (n). In turn, as we shall show in Sec. 6, these quantities admit the following scaling behaviours in the scaling limit (19):

$$C_{\boldsymbol{k}}^{(\mathrm{ic})}(t) \simeq \frac{c_{\alpha}}{g_c} (k^2 t)^F e^{-2k^2 t/\gamma} k^{F+\alpha/2}, \tag{40b}$$

$$C_{\boldsymbol{k}}^{(\mathrm{n})}(t) \simeq -\frac{4\hbar}{\pi\gamma} \left[ C_E + \ln\left(\frac{\mathsf{t}_0 k^2}{\gamma}\right) \right] - \frac{2\hbar}{\pi\gamma} \left(\frac{k^2 t}{\gamma}\right)^{-F} \Psi\left(\frac{k^2 t}{\gamma}, \frac{k^2 t}{\gamma}\right), \tag{40c}$$

with a universal scaling function  $\Psi(\rho, \rho)$  whose form – reported in Eq. (100) – depends only on F and  $c_{\alpha}$  and is given by the initial conditions in Eq. (25). From the integral representation (100) derived in Sec. 5 below, it follows that  $\Psi(\rho, \rho) \stackrel{\rho \gg 1}{\sim} \rho^{F-1}$  for large arguments  $\rho$ . By itself, this scaling form fixes the dynamical exponent z = 2, because the relevant variable turns out to be  $k^2 t$  and therefore  $t \sim k^{-2}$ . In case I, both terms in Eq. (40a) equally contribute to the final expression and their relative importance is fixed by the non-universal amplitude  $c_{\alpha}/g_c$ . In cases II and III, instead, the initial term in Eq. (40b) dominates over the quantum noise contribution in the scaling limit. The two-time auto-response and autocorrelation functions are obtained from a careful asymptotic analysis of the double integral involving the quantum noise memory kernel. In the scaling limit, see Eq. (22), we find

$$R(t,s) = R_{(0)}s^{-d/2} \left(\frac{t}{s}\right)^{-F/2} \left(\frac{t}{s}-1\right)^{-d/2}, \quad C(t,s) \sim s^{-d/2} \frac{(t/s)^{-F/2}}{(1+t/s)^{(d+\alpha)/2}}.$$
(41)

Using Eq. (24), we read off the autoresponse and autocorrelation exponents

$$\lambda_R = d + F$$
, and  $\lambda_C = d + \alpha + F$ , (42)



Figure 6: Equal-time correlation function  $C_k(t) - C_k(\infty)$  for  $\gamma = 1$  as a function of the dimensionless parameter  $\rho$  (see Eq. (19) for the case I of a quench to the critical point, in the presence of either the quantum noise in Eq. (40) (left panel) or the effective dynamics in Eq. (34a) (right panel). The decay of the initial correlations in (25) has been chosen with with  $\alpha = -1$ . The contributions  $C_k^{(ic)}(t)$  coming from the initial noise correlators and  $C_k^{(n)}(t)$  coming from the bath noise correlators are indicated separately.

respectively. These results are identical to those of effective dynamics reported in Table 1.

In Fig. 6 we plot the equal-time correlators for the critical quench in case I, for both the quantum noise (left panel) and the effective dynamics (right panel) with the time-independent term  $C_{\mathbf{k}}(\infty)$  in Eqs. (34a) and (40c) being subtracted. The qualitative behaviour is very similar. In both cases, the behaviour of the correlations for small values of the scaling variable  $\rho$  (see Eq. (19)) is dominated by the contribution of the initial noise  $C_{\mathbf{k}}^{(ic)}(t)$  which depends on a non-universal amplitude  $c_{\alpha}/g_c$ , whereas for larger values of  $\rho$ , the corresponding universal bath noise term  $C_{\mathbf{k}}^{(n)}(t)$  dominates. Qualitatively, these bath noise terms are distinct, see the red lines in Fig. 6. While for  $\rho \gg 1$ , they are both anticorrelated and decay to zero as  $\sim \rho^{-1}$  (note that anticorrelations are absent for the classical dynamics), their difference emerges for finite values of  $\rho$ . For the effective noise,  $C_{\mathbf{k}}^{(n)}(t)$  grows monotonically as a function of  $\rho$  while for the quantum noise the shape of the scaling function is not monotonic and for  $\rho \simeq 1$  the contribution is positively correlated and has a maximum. Although the effective noise (right panel in Fig. 6) does faithfully reproduce the qualitative scaling behaviour of the two-time observables and provides exactly the same exponents as the actual quantum noise (left panel), it is not adequate for a precise quantitative description of single-time observables.

(3) For a quench across the critical point, that is for  $r < r_0^c$ , the self-consistent function g(t) still shows an algebraic behaviour that coincides with the one of effective dynamics discussed above and even with what happens in the presence of classical white noise [28]. Equation (33) holds with a multiplicative prefactor  $g_d$  and the exponent  $F = -(d+\alpha)/2$ . This is actually not surprising since the quantum noise is sub-dominant compared to the classical white noise and the white noise itself is, in turn, sub-dominant compared to the propagation of the initial correlations for a deep quench. Using again Eq. (40), the equal-time correlation function in the scaling limit  $(k \to 0, t \to \infty \text{ and } \rho = \text{cst.})$  becomes

$$C_{\boldsymbol{k}}(t) \simeq \frac{c_{\alpha}}{g_d} k^{-d} \rho^{(d+\alpha)/2} e^{-2\rho}, \qquad (43)$$

in terms of the scaling variable  $\rho$  in Eq. (19), which also implies z = 2. For any correlated initial state with  $\alpha < 0$ , the propagation of the initial correlation dominates over the one of the contribution due to the quantum noise. Only for a fully uncorrelated initial state, corresponding to  $\alpha = 0$ , both initial and quantum noise correlations would contribute similarly to the scaling function.

Another commonly used prescription for quantum dynamics is a master equation of the Lindblad form. It is instructive to compare Eq. (43) with the corresponding prediction when the quantum Langevin dynamics considered here is replaced by a Lindblad master equation (Li). The Lindblad dynamics of the quantum spherical model has been analysed in Ref. [39]: the lower critical dimension turns out to be  $d_l^{(\text{Li})} = 1$ , the upper critical dimension  $d_u^{(\text{Li})} = 3$ , and the critical point of the dynamics is the same as in equilibrium. In addition, at the leading non-trivial order of the semi-classical expansion, the fluctuation-dissipation theorem applies in the single-phase region and the long-time behaviour of the single-time correlator becomes independent of the dissipative dynamical coupling  $\gamma$ . In d = 2 dimensions, one finds a scaling form  $C_{\mathbf{k}}(t) = k^{-2}\Psi(|\mathbf{k}|t)$ , hence the dynamical exponent is z = 1, as expected for a closed quantum system with unitary dynamics [63, 64]. For d > 2 dimensions, instead, logarithmic corrections to scaling are found.<sup>7</sup> This explicit example shows that quantum Langevin dynamics and Lindblad dynamics lead to completely different results.

Returning to the quantum Langevin equation, for the sub-critical quench we are considering here, the two-time autoresponse and autocorrelation function read

$$R(t,s) = R_{(0)}s^{-d/2} \left(\frac{t}{s}\right)^{(d+\alpha)/4} \left(\frac{t}{s} - 1\right)^{-d/2}, \quad C(t,s) \sim \frac{(t/s)^{(d+\alpha)/4}}{[1 + (t/s)]^{(d+\alpha)/2}},\tag{44}$$

with the same  $R_{(0)}$  as in Eq. (41). The corresponding exponents can be determined by comparing these expressions with Eqs. (23) and (24) as

$$\lambda_R = \frac{d - \alpha}{2}, \qquad \lambda_C = \frac{d + \alpha}{2}. \tag{45}$$

All these predictions are identical to those of the case of the effective dynamics, reported in Tab. 1.

In summary, our explicit calculations have shown that:

- (i) As far as the occurrence of a phase transition is concerned, both the effective dynamics and the quantum dynamics lead to a dimensional shift  $d \mapsto d-2$  with respect to the classical dynamics.
- (ii) The stationary state at temperature T = 0 reached after a quench cannot be an equilibrium state. This is due to the fact that the overdamped limit (see Eq. (9)) prevents the emergence of any kind of equipartition.
- (iii) The effective dynamics, characterised by a rescaled Markovian noise in Eq. (13) instead of the actual quantum noise in Eq. (11), turns out to be sufficient in order to reproduce correctly all the universal exponents which describe the ageing after a quantum quench. The additional long-time memory effects in the quantum noise are not strong enough to yield further modifications. The entries of Table 1 are therefore the same for both the effective and the quantum dynamics.
- (iv) The scaling of the equal-time correlator, which describes the coarsening occurring after a quantum quench, is sensibly determined by the long-time memory effects of the quantum noise. In particular, this influence is manifest above the critical point (see, e.g., Fig. 5) and in region I (see Fig. 4) at criticality. In the critical regions II, III (see Fig. 4) and for sub-critical quenches, the quantum noise merely leads to corrections to the leading scaling behaviour.

Finally, let us reconsider the role of the cutoff scale  $t_0$ , introduced into the quantum noise correlator in Eq. (14) at temperature T = 0. In those cases and observables for which the quantum noise turned out to be irrelevant, the actual value of  $t_0$  has no influence on the long-time behaviour of the dynamics. The cases in which the quantum noise is important concern the equal-time correlator  $C_k(t)$ , either for region I of a critical quench or for a quench to the disordered phase. For case I of the critical quench, according to the classification of Tab. 1, Eq. (40c) shows that  $t_0$  merely enters a simple additive and time-independent term. This term describes the logarithmic singularity which arises if one attempts to take the limit  $t_0 \rightarrow 0$ . However, the other aspects of the dynamics do not depend explicitly on  $t_0$ . For a quench to the disordered phase, Eq. (38) shows that  $t_0$  is merely setting a scale for the stationary correlator. Here it is adequate to recall the heuristic argument from Appendix B that this scale is set at  $t_0 \sim \gamma^{-1}$ . In any case, the value of  $t_0$  does not influence the long-time quantum dynamics of the quantum spherical model.

A detailed discussion and comparison with the results of classical dynamics is provided in Sec. 7.

#### 5. Non-Markovian equation: details of the solution

We analyse here the effects of the quantum spherical constraint on the dynamics of the model. We shall first show that the function g(t) defined in Eq. (17) obeys a non-linear integro-differential equation. Focusing on the non-Markovian quantum noise correlations (15), we then show how this equation can be embedded into a larger set of linear Volterra integro-differential equations for a function G(t,s) of two variables, such that g(t) = G(t,t). The asymptotic solution of this linear integro-differential equation renders the sought long-time behaviour in Eqs. (81), (85), and (87) of the function g(t). This piece of information is necessary for the calculation of physical observables, which we discuss in Sec. 6.

<sup>7</sup>Indeed, if z = 2 a dimensionless scaling variable can only be of the form  $k^2 t/\gamma$ , assuming the microscopic velocity v is set to v = 1. For z = 1, a scaling variable kt is dimensionless and no further time scale  $1/\gamma$  is needed.

#### 5.1. Formulation of the quantum spherical constraint

The solution of the non-equilibrium dynamics of either the spherical model or the O(n)-model at large n relies on the fact that the field  $\phi_{\mathbf{k}}(t)$  basically evolves according to a (time-dependent) linear equation, which is essential for finding the general formal solution in Eqs. (16) and (27). All aspects of the many-body interaction are self-consistently introduced through the spherical constraints of these models, represented by g(t), which, in turn, is determined by the equal-time correlator C(t, t). In order to determine the evolution of the constraint, we begin by rewriting Eq. (27b) in the following compact form

$$C(t,t) = \frac{1}{g(t)} \left[ A(t) + (g_2 * F)(t,t) \right],$$
(46)

where we introduced the two-variable function

$$g_2(t,s) := \sqrt{g(t)g(s)} , \qquad (47)$$

two auxiliary functions A(t) and F(t,s) and the double convolution denoted by \*\*, specified further below in Eq. (49). The functions A(t) and F(t,s) describe the propagation of initial and quantum noise correlation and are respectively given by

$$A(t) := \int_{\boldsymbol{k}} \exp\left(-2\frac{k^2}{\gamma}t\right) c_{\alpha}k^{\alpha}, \quad F(t,s) := \gamma^{-2} \int_{\boldsymbol{k}} \exp\left(-\frac{k^2}{\gamma}(t+s)\right) m_{\boldsymbol{k}}(t-s), \tag{48}$$

with the memory-kernel  $m_k$  of the quantum noise in Eq. (26). Herein, we do not yet specify whether the momentum integrals  $\int_{\mathbf{k}}$  require a cutoff at some large momentum  $|\mathbf{k}| = \Lambda$ , or not, see below. The double convolution \*\* in Eq. (46) is defined for two functions  $h_1, h_2 : \mathbb{R}^2_+ \to \mathbb{C}$ , as (see also Appendix D)

$$(h_1 * * h_2)(t, s) = \int_0^t \mathrm{d}x \int_0^s \mathrm{d}y \, h_1(x, y) h_2(t - x, s - y). \tag{49}$$

The spherical constraints Eqs. (2) and (5) applied to Eq. (46) now produce integro-differential equations for g(t). For the spherical model, the constraint  $C(t,t) = 1/\lambda$  leads to

$$\frac{1}{\lambda}g(t) = A(t) + (g_2 * F)(t, t),$$
(50a)

while for the O(n)-model at large n, the constraint  $r(t) = r_0 + uC(t,t)/12$  becomes

$$\frac{6\gamma}{u}\frac{\mathrm{d}g(t)}{\mathrm{d}t} - \frac{12r_0}{u}g(t) = A(t) + (g_2 * *F)(t,t).$$
(50b)

For the effective or classical dynamics, where the noise  $m_{\mathbf{k}}(\tau)$  is  $\delta$ -correlated in the time  $\tau$ , Eqs. (50) become (generalised) linear Volterra integral (or integro-differential) equations for g(t). For completeness and comparison, this case is discussed and solved in Appendix C.

#### 5.2. Equivalence of spherical and O(n)-models, at large n

As anticipated above, determining the dynamics of the various models in the presence of the different type of noises discussed in the previous sections is essentially reduced to solving one of Eqs. (50) for the constraint, depending on which one of the specific models introduced in Sec. 2 one is focusing on. Eqs. (50a) and (50b) do not coincide, as they refer to two *a priori* different models. However, the formal limit  $\gamma \to 0$  together with the identification  $1/\lambda = -12r_0/u$  transforms Eq. (50b) into Eq. (50a). Accordingly, the spherical model dynamics can always be obtained from the solution of Eq. (50b) via this limit. We shall see later in Sec. 5.3 that the extra terms contained in Eq. (50b) do not modify the universal features of the long-time behaviour. In addition, asymptotic solutions of the form  $g(t) \sim e^{t/\tau_r}$  or  $g(t) \sim$  cst. occur for certain parameter ranges in both equations.

Because of the equivalence of the long-time behaviour of the two models, we shall focus henceforth on the O(n)-model in the large-n limit and therefore concentrate on solving Eq. (50b).

#### 5.3. Formal solution of the spherical constraint

For the quantum noise, such as that in Eq. (15), which is not  $\delta$ -correlated in time, the constraint in Eq. (50b) is a *non-linear* integro-differential equation for the function g(t), without obvious explicit solutions. The non-linearity can be formally avoided via the introduction of the function  $g_2(t,s) = \sqrt{g(t)g(s)}$ , according to Eq. (47), depending on two variables t and s, combined with a double convolution. In principle, the knowledge of either

function g or  $g_2$  determines the other one via Eq. (47). Accordingly, it might appear promising to try to determine the symmetric function  $g_2(t,s) = g_2(s,t)$  directly.

However, instead of solving directly the non-linear equation (50b), we rather consider a *different* integrodifferential equation for a symmetric function G(t,s) = G(s,t). The function g = g(t), whose long-time behaviour is sought, is recovered in the equal-time limit g(t) = G(t,t). The equation reads

$$\frac{3\gamma}{u} \left[ \frac{\partial G(t,s)}{\partial t} + \frac{\partial G(t,s)}{\partial s} \right] - \frac{12r_0}{u} G(t,s) = A\left(\frac{t+s}{2}\right) + (G^{**}F)(t,s)$$
(51)

and has two desirable properties: (a) it is *linear* in terms of the function G(t,s) and (b) for  $s \to t$  it reduces to Eq. (50b) as  $\lim_{s\to t} G(t,s) = G(t,t) = g(t)$ , although, generically,  $G(t,s) \neq g_2(t,s)$ . For solving Eq. (51), we impose the initial condition G(0,0) = g(0) = 1 implied by Eq. (17). In addition, a boundary condition for G(t,0) = G(0,t) will be required.<sup>8</sup>

For given initial and boundary conditions, Eq. (51) has a unique solution G(t,s) = G(s,t) and furnishes g(t) = G(t,t). In addition  $g_2(t,s) = g_2(s,t)$  is another solution of either Eq. (51) or (50b), but a priori with different boundary conditions. It will turn out that one finds in the long-time limit (i) either a leading exponential behaviour  $g(t) \sim e^{t/\tau}$  or (ii) a leading algebraic behaviour  $g(t) \sim t^F$ . As we shall show below, the boundary conditions do not modify the value of F. In addition, the boundary conditions do not affect which of the above two possibility occurs. Analogously, setting  $\gamma = 0$  in Eq. (51) and substituting  $\lambda = -u/12r_0$ , the solution of Eq. (51) will produce the leading long-time exponent F of Eq. (50a). Accordingly, we now show that:

The leading long-time behaviour of g(t) = G(t,t) derived from Eq. (51) is independent of the boundary conditions on G(t,0) = G(0,t). It therefore agrees with the leading long-time behaviour of  $g_2(t,t)$  found from Eq. (50b) for  $\gamma \neq 0$  or Eq. (50a) in the  $\gamma \rightarrow 0$  limit.

In order to prove these statements, we begin with the formal solution of Eq. (51). This is readily obtained, via a two-dimensional Laplace transform [68, 69, 70] according to the definition

$$\overline{\overline{f}}(p,q) := \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s \ f(t,s) e^{-pt-qs}.$$
(52)

Using the properties listed in Appendix D, we straightforwardly obtain the Laplace transform  $\overline{G}(p,q)$  of the solution G(t,s) of Eq. (51), as

$$\overline{\overline{G}}(p,q) = \frac{\overline{\overline{A}}(p,q) + (3\gamma/u) \left[\overline{G}_0(q) + \overline{G}_0(p)\right]}{(3\gamma/u) \left(p+q\right) - (12r_0/u) - \overline{\overline{F}}(p,q)},\tag{53}$$

with the following boundary terms (recall that G(t, 0) = G(0, t))

$$\overline{G}_0(p) := \int_0^\infty dt \, e^{-pt} G(t,0) = \int_0^\infty ds \, e^{-ps} G(0,s).$$
(54)

These boundary terms require a specific analysis, using the symmetry G(t,s) = G(s,t) and the requirement that G(t,s) is partially differentiable with respect to t and s. It will turn out to be convenient to set

$$G(t,s) =: H\left(\frac{t+s}{2}, \frac{t-s}{2}\right) = H(\alpha, \beta) \quad \text{with} \quad \alpha := \frac{t+s}{2} \quad \text{and} \quad \beta := \frac{t-s}{2}.$$
(55)

The symmetry condition under the exchange of t and s implies

$$H(\alpha,\beta) = H\left(\frac{t+s}{2},\frac{t-s}{2}\right) = H\left(\frac{t+s}{2},\frac{s-t}{2}\right) = H(\alpha,-\beta).$$
(56)

Since G(t, s) is partially differentiable, apply the symmetry relation (56) to find

$$\frac{1}{2}(\partial_{\alpha} + \partial_{\beta})H(\alpha, \beta) = \partial_{t}G(t, s) = \partial_{t}G(s, t) = \frac{1}{2}(\partial_{\alpha} - \partial_{\beta})H(\alpha, -\beta),$$
(57)

or equivalently

$$\partial_{\beta} \left[ H(\alpha, \beta) + H(\alpha, -\beta) \right] = \partial_{\alpha} \left[ H(\alpha, -\beta) - H(\alpha, \beta) \right] = 0,$$
(58)

 $^8 {\rm This}$  requirement is actually unnecessary in the  $\gamma \to 0$  limit.

where we used the symmetry (56). This implies  $\partial_{\beta} H(\alpha, \beta) = 0$ . Accordingly,

$$H = H(\alpha) = H\left(\frac{t+s}{2}\right),\tag{59}$$

i.e., H actually depends only on a single variable. After these preliminaries, we consider the function  $G_0(t) := G(t,0) = H(t/2)$ . Returning to Eq. (51), the double convolution term will vanish when the limit  $s \to 0$  is considered and  $(\partial_t + \partial_s) = \partial_\alpha$ . This leads to

$$\lim_{s \to 0} \frac{3\gamma}{u} \partial_{\alpha} H\left(\frac{t+s}{2}\right) = \frac{3\gamma}{u} \partial_{\alpha} H(\alpha) = \frac{12r_0}{u} H(\alpha) + A(\alpha) \text{ where } \alpha = \frac{t}{2}.$$
(60)

Laplace-transforming this equation immediately produces  $-\frac{3\gamma}{u}H(0) + p\overline{H}(p) = \frac{12r_0}{u}\overline{H}(p) + \overline{A}(p)$ , hence with H(0) = G(0,0) = 1, one finds

$$\overline{H}(p) = \frac{(3\gamma/u) + \overline{A}(p)}{p - 12r_0/u}.$$
(61)

Since we had  $G_0(t) = G(t,0) = H(t/2)$ , it follows that  $\overline{G}_0(p) = 2\overline{H}(2p)$ . At long last, the formal solution (53) of Eq. (51) explicitly becomes in double Laplace space

$$\overline{\overline{G}}(p,q) = \frac{\overline{A}(p,q) + (6\gamma/u) \left[\overline{H}(2q) + \overline{H}(2p)\right]}{(3\gamma/u) \left(p+q\right) - (12r_0/u) - \overline{\overline{F}}(p,q)},\tag{62}$$

In Eqs. (62)  $\overline{A}(p,q)$  is the double Laplace transform of the function A((t+s)/2), which can be derived from that of A(t), see Eq. (64) below.

Well-known Tauberian theorems for the Laplace transform of a single variable [71] can be generalised to the present case of a double Laplace transforms. For homogeneous functions, this generalisation is formulated and proven in Appendix F and, accordingly, the long-time behaviour of G(t,s) for large t and s we are interested in is related to the behaviour of  $\overline{\overline{G}}(p,q)$  for small p and q. In order to determine it, the asymptotic expansions of the functions  $\overline{\overline{A}}(p,q)$ ,  $\overline{\overline{F}}(p,q)$  and  $\overline{H}(p)$  for small p and q are needed.

First, the long-time asymptotics of the function  $A(t) = c_{\alpha}A_{\alpha}(t)$ , and hence also  $\overline{A}(p)$  for  $p \ll 1$ , is derived in Appendix C, for the effective dynamics. There, it is shown that

$$\overline{A}_{\alpha}(p) \simeq a_{\alpha} p^{\frac{d+\alpha}{2}-1} + \sum_{n=0}^{\lfloor \frac{d+\alpha}{2}-1 \rfloor} (-1)^n A_n^{(\alpha)} p^n,$$
(63)

with explicit expressions for the constants  $a_{\alpha}$  and  $A_n^{(\alpha)}$ . Since we have chosen in Eq. (51) the two-time form as A(t,s) = A((t+s)/2), we can use an identity proven in Appendix D, which states that

$$\overline{\overline{A}}(p,q) = \frac{\overline{A}(p/2) - \overline{A}(q/2)}{(p-q)/2},\tag{64}$$

in order to determine the small-p and q behaviour of  $\overline{\overline{A}}(p,q)$  in terms of that of  $\overline{A}$  given in Eq. (63). This allows us to conclusively examine the relevance of the terms  $\sim \overline{H}(p)$  in Eq. (62) coming from the boundary conditions. The general results from appendix Appendix F on inverting double Laplace transformation state that the leading long-time behaviour can be read from the small-p scaling of the generic form  $p^a f(\frac{q}{p})$ . The leading small-p contributions of the numerator in Eq. (62) are proportional to either  $\overline{\overline{A}}(p,q) \sim p^{\frac{d+\alpha}{2}-2}$ , using (63,64), or to  $\overline{H}(2p) + \overline{H}(2q) \sim p^{\frac{d+\alpha}{2}-1}$ , using (61,63), times scaling functions in q/p. Clearly, the contribution of the boundary term merely generates a correction to the leading scaling behaviour. Thus, in the subsequent analysis the boundary terms need not be included.

For the function  $\overline{F}(p,q)$ , the non-Markovian character of the noise makes the analysis of the function considerably more difficult than that of  $\overline{\overline{A}}$ . We outline here only the main points of this analysis and refer to Appendix E for further details and explicit calculations. The double Laplace transform of F(t,s) is explicitly obtained as

$$\overline{\overline{F}}(p,q) = \frac{2\hbar}{\pi\gamma} \int_{\mathbf{k},(\Lambda)} \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s \ e^{-pt-qs} e^{-\frac{k^2}{\gamma}(t+s)} \frac{\mathsf{t}_0^2 - (t-s)^2}{\left[\mathsf{t}_0^2 + (t-s)^2\right]^2},\tag{65}$$

where we now also indicate the need of a cutoff at large momentum  $|\mathbf{k}| = \Lambda$ . Using the symmetry of the integrand in t and s for p = q = 0, the integration domain in (t, s) can be reduced to a wedge domain with

s < t. Then, we can introduce diagonal coordinates (x, y) = (t - s, t + s) such that the double Laplace transform reduces to a single integral of the form

$$\mathcal{J} = \int_0^\infty \mathrm{d}x \, f(x) \frac{\mathsf{t}_0^2 - x^2}{\left(\mathsf{t}_0^2 + x^2\right)^2} \tag{66}$$

with an explicitly known function f(x). In Appendix E we present a general approach for evaluating such an integral, based on interpreting the quantum noise correlator as a generalised function acting on the test function f(x). For a generic function f, we shall see in Appendix E that it is useful to introduce the formal series<sup>9</sup> in x

$$f(x) = \sum_{j}' a_{j} x^{\alpha_{j}} + \sum_{n=0}^{\infty} b_{2n+1} x^{2n+1} , \qquad (67)$$

where the first sum contains all even powers and all non-analytic contributions in x and the second sum contains all odd powers in x. We further show that the integral in Eq. (66) may be evaluated as

$$\mathcal{J} = -\frac{\pi}{2} \sum_{j}' \frac{\mathsf{t}_{0}^{\alpha_{j}-1} a_{j} \alpha_{j}}{\cos\left(\pi \alpha_{j}/2\right)} + \sum_{n=0}^{\infty} \left(-\mathsf{t}_{0}^{2}\right)^{n} \left[ (1 + (2n+1)\ln\mathsf{t}_{0})b_{2n+1} - (2n+1)B_{n} \right]$$
(68)

with the constants

$$B_n = \lim_{z \to -(2n+1)} \frac{\mathrm{d}}{\mathrm{d}z} \left[ (z+2n+1) \int_0^\infty \mathrm{d}y \, y^{z-1} f(y) \right]. \tag{69}$$

For the particular case of an exponential function  $f(x) = e^{-\nu x}$ , we show in Appendix E how to evaluate explicitly the series in Eq. (68) in terms of the auxiliary function  $g_{AS}$  defined in Eq. (36). Exploiting the known asymptotics of  $g_{AS}$  (see Eq. (36)) one can determine the leading terms for  $p, q \to 0$  of the double Laplace transform as

$$\overline{\overline{F}}(p,q) \simeq -\frac{2\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} \int_0^\Lambda \mathrm{d}k \; k^{d-1} \left[ \mathscr{F}(p,q) + \mathscr{F}(q,p) \right],\tag{70a}$$

where

$$\mathscr{F}(p,q) := \frac{\left(k^2/\gamma + p\right) \left[C_E + \ln\left(\mathsf{t}_0\left(k^2/\gamma + p\right)\right)\right]}{k^2/\gamma + (p+q)/2}.$$
(70b)

As further shown in Appendix E, the integral in Eq. (70a) decomposes into a non-universal, regular (i.e. analytic) part and a universal, irregular (i.e., non-analytic) part

$$\overline{\overline{F}}(p,q) = \overline{\overline{F}}_{reg}(p,q) + \overline{\overline{F}}_{irr}(p,q) .$$
(71)

These can be evaluated explicitly, to their respective lowest order in  $t_0$ 

$$\overline{\overline{F}}_{\rm reg}(p,q) \simeq -\frac{4\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} \left\{ \frac{\Lambda^d}{d} \left[ \ln\left(\Lambda^2 \frac{\mathbf{t}_0}{\gamma}\right) + \mathcal{C}_E - \frac{2}{d} \right] + \frac{1}{2} \frac{\gamma \Lambda^{d-2}}{d-2} (p+q) + \mathcal{O}(\mathbf{t}_0) \right\},\tag{72a}$$

$$\overline{\overline{F}}_{\rm irr}(p,q) \simeq -\frac{4\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} (\gamma p)^{\frac{d}{2}} \mathbb{F}(q/p) + \mathcal{O}(p,q),$$
(72b)

with the scaling function

$$\mathbb{F}(z) = \int_0^\infty \mathrm{d}x \, x^{d-1} \frac{(x^2+z)\ln\left(1+z/x^2\right) + (x^2+1)\ln\left(1+1/x^2\right)}{(x^2+z) + (x^2+1)}.$$
(72c)

Equations (71) and (72), provide the asymptotic expansion of  $\overline{\overline{F}}(p,q)$  we are interested in. In particular, by using Eq. (53), it turns out that the critical point of the O(n)-model is  $r_0^c = -(u/12)\overline{\overline{F}}(0,0)$ , the value of which follows from Eq. (72a), resulting in Eq. (37) of Sec. 4.

We can now analyse the relative importance of the various contributions to the exact formal solution in Eq. (62). Considering, first, the denominator in Eq. (62), we see from the leading terms of the expansions in Eqs. (71) and (72) that there are two distinct non-constant terms. There is a term  $\sim p^{d/2}$  which comes exclusively from the quantum noise and there is a term  $\sim p$  which has contributions from several different sources. Hence, for d < 2 the leading behaviour will be determined by the quantum noise, but for d > 2,

<sup>&</sup>lt;sup>9</sup>These constants  $a_j$  must not be confused with the constants  $a_{\alpha}$  in Eq. (63).

the leading exponent is independent of it. However, the associated amplitude may receive a non-vanishing contribution from the quantum noise.

Finally, we briefly return to the relationship between the spherical model and the O(n)-model for  $n \to \infty$ . The difference between them comes from the terms in Eq. (62), which contain a factor  $\gamma$ .<sup>10</sup> Clearly, these terms can at most renormalise certain amplitudes. Accordingly, one can draw the conclusion that the leading universal properties of the spherical and of the O(n)-model for  $n \to \infty$  are identical also in these non-equilibrium conditions.

#### 5.4. Asymptotic behaviour of the self-consistent solution for d < 2

For d < 2, given that  $\alpha \leq 0$ , one also has  $d + \alpha < 2$ . Using Eqs. (63) and (64), the function  $\overline{A}(p,q)$  may then be written in terms of a scaling function  $\mathbb{A}(z)$  according to

$$\overline{\overline{A}}(p,q) \simeq c_{\alpha} a_{\alpha} 2^{2-\frac{d+\alpha}{2}} p^{\frac{d+\alpha}{2}-2} \mathbb{A}(q/p), \quad \text{with} \quad \mathbb{A}(z) = \frac{1-z^{(d+\alpha)/2-1}}{1-z}.$$
(73)

Equation (62) can then be rewritten asymptotically as follows, using Eqs. (71) and (72),

$$\overline{\overline{G}}(p,q) \simeq \frac{c_A p^{\frac{a+\alpha}{2}-2} \mathbb{A}(q/p)}{M^2 + c_F p^{d/2} \mathbb{F}(q/p)},\tag{74}$$

where  $M^2 = r_0^c - r_0$  quantifies the distance from the critical point  $r_0^c = -(u/12)\overline{F}(0,0)$ , already quoted above in Eq. (37). In order to simplify the notation, in the equation above we introduced the constants  $c_A$  and  $c_F$  which account for all numerical factors in the corresponding functions, following from Eq. (62). We now consider iteratively the different quench protocols schematically illustrated in Fig. 3.

(A) For a quench above criticality, i.e., with  $r_0 > r_0^c$ , the constant  $M^2$  introduced in Eq. (74) is formally negative, i.e.,  $M^2 < 0$ . Accordingly, the denominator in Eq. (74) has a line of poles in the (p,q) plane. Each of these poles corresponds to a mode of relaxation. To be explicit, consider a quench close to the critical point: the denominator vanishes along a curve  $\Gamma$  of points  $(p_0, q_0)$  which yields a curve of poles for the solution  $\overline{\overline{G}}(p,q)$ . Sufficiently close to criticality,  $\Gamma$  is defined by the asymptotic equation for  $p, q \to 0$ , i.e.,

$$M^{2} + c_{F} \left[ p_{0}^{d/2} + q_{0}^{d/2} \right] = 0.$$
(75)

The function g(t) is obtained by formally inverting the double Laplace transform, i.e.,

$$g(t) = G(t,t) = \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{\mathrm{d}p}{2\pi \mathrm{i}} \int_{c_2 - i\infty}^{c_2 + i\infty} \frac{\mathrm{d}q}{2\pi \mathrm{i}} \overline{\overline{G}}(p,q) e^{(p+q)t},\tag{76}$$

and these two contour integrals in the complex plane can be evaluated with the poles on the curve  $\Gamma$ . Thus it is apparent that many relaxation times

$$1/\tau_{\rm r} \simeq p_0 + q_0 \tag{77}$$

with  $(p_0, q_0) \in \Gamma$  contribute to the integral. Generally the system will choose the slowest relaxational time scale as each of them contributes exponentially. Anyhow, all time scales satisfy

$$1/\tau_{\rm r} \simeq \left(-M^2/c_F\right)^{2/d} \tag{78}$$

and consequently,  $g(t) \sim e^{t/\tau_{\rm r}}$  is of exponential form.<sup>11</sup>

(B) For quenches onto or across criticality, i.e., with  $r_0 \leq r_0^c$  and therefore either  $M^2 = 0$  or  $M^2 > 0$ , the long-time behaviour of g(t) becomes algebraic. This can be immediately seen from Eq. (78), since the relaxational time-scale diverges as criticality is approached, i.e.,

$$\tau_{\rm r} \to \infty \quad \text{for} \quad r_0 \to r_0^c.$$
 (79)

Since d < 2, the leading singularity of  $\overline{\overline{G}}$  in Eq. (62) is no longer a pole but we have the small-p behaviour

$$\overline{\overline{G}}(p,q) \simeq \begin{cases} \frac{c_A}{c_F} p^{\frac{\alpha}{2}-2} \frac{\mathbb{A}(q/p)}{\mathbb{F}(q/p)} & \text{for } M^2 = 0, \\ \frac{c_A}{M^2} p^{\frac{d+\alpha}{2}-2} \mathbb{A}(q/p) & \text{for } M^2 > 0. \end{cases}$$
(80)

<sup>10</sup>In the limit  $\gamma \to 0$ , Eq. (51) reduces to a linear Volterra integral equation in two variables.

<sup>11</sup>A comment on stationary critical exponents: since  $\tau_{\rm r} \sim \left| r_0 - r_0^c \right|^{-\nu z}$ , Eq. (78) implies that  $\nu z = 2/d$ .

Thus, the function  $\overline{\overline{G}}(p,q) = p^{\mathfrak{g}-2}\mathbb{G}(q/p)$  assumes a scaling form with an algebraic pre-factor and a certain exponent  $\mathfrak{g}$ . In Appendix F we show that this is equivalent to a scaling form  $G(t,s) = s^{-\mathfrak{g}}\mathscr{G}(t/s)$ . In particular it follows that  $g(t) = G(t,t) = t^{-\mathfrak{g}}\mathscr{G}(1)$ . The actual value of the exponent  $\mathfrak{g}$  can be directly identified from Eq. (80). Hence, we conclude that

$$g(t) \simeq \begin{cases} g_c t^{-\alpha/2} & \text{for } M^2 = 0, \\ g_d t^{-(d+\alpha)/2} & \text{for } M^2 > 0, \end{cases}$$
(81)

where  $g_c$  and  $g_d$  denote, respectively, the non-universal pre-factors for a critical and a deep quench which can be obtained by evaluating the Laplace-inverted scaling function  $\mathscr{G}(1)$ . However, their explicit values will never be required in our analysis.

We finish with a comment on the form of the effective parameter r(t), see Eq. (2), which is implied by these results. In general, the asymptotic long-time behaviour  $g(t) = g_0 t^F + g_1 t^{-\kappa}$  is expected, where we included the leading correction  $g_1 t^{-\kappa}$  with the yet unknown exponent  $\kappa$  and we admit  $F \ge 0$  and  $\kappa > 0$ . Using Eq. (17), we find for large times the "effective mass"

$$r(t) = \frac{\gamma}{2} \frac{g'(t)}{g(t)} \simeq \frac{\gamma}{2} \left[ \frac{F}{t} - \frac{g_1}{g_0} \frac{F + \kappa}{t^{1+F+\kappa}} + \dots \right].$$
(82)

As long as  $F \neq 0$ , this produces the asymptotic scaling  $r(t) \simeq \frac{\gamma}{2} \frac{F}{t} + \dots$ . This estimate suggests the very natural *light-cone ansatz*  $r(t) \sim t^{-1}$  which is usually introduced on the basis of dimensional analysis and which simplifies calculations considerably, also for the field-theoretical Keldysh formalism [36, 37], see, e.g., Eq. (70) in Ref. [37]. Now, consider a quench onto the critical point, which for d < 2 corresponds to regime I and also focus on the limit of short-ranged spatial initial correlations (studied throughout in Ref. [36, 37]) which corresponds to  $\alpha = 0$  [28]. Equation (81) then implies  $F = -\frac{\alpha}{2} \to 0$ . Thus, Eq. (82) rather yields asymptotically  $r(t) \simeq -\frac{\gamma}{2} \frac{g_1}{g_0} \kappa t^{-1-\kappa}$  and the ansatz mentioned above for r(t) no longer applies.

#### 5.5. Asymptotic behaviour of the self-consistent solution for d > 2

In more than two spatial dimensions, the leading quantum noise contribution is regular and the self-consistency function  $\overline{\overline{G}}(p,q)$  can be written as

$$\overline{\overline{G}}(p,q) \simeq \frac{\overline{A}(q,p) + 12\gamma/u}{M^2 + \tilde{c}_F p \left(1 + q/p\right)}.$$
(83)

As seen in Appendix C for the effective noise, the result of the subsequent analysis depends on the initial condition through the value of  $d + \alpha$ . For a quench towards the disordered phase, the generic behaviour derived for d < 2 carries over to d > 2 and only the value of the relaxational time-scale  $\tau_{\rm r}$  changes. Accordingly, all qualitative behaviours emerging for d < 2 extend to d > 2.

We now look again at the **quenches onto and across the critical point** but we need to distinguish regions II and III in Fig. 4.

(B1) We start by focusing on the case  $d + \alpha < 2$ , which corresponds to spatially long-ranged correlated initial conditions. The scaling forms for the function  $\overline{\overline{G}}(p,q)$  are then written as

$$\overline{\overline{G}}(p,q) \simeq \begin{cases} \frac{c_A}{\tilde{c}_F} p^{\frac{d+\alpha}{2}-3} \frac{\mathbb{A}(q/p)}{1+q/p} & \text{for } M^2 = 0, \\ \frac{c_A}{M^2} p^{\frac{d+\alpha}{2}-2} \mathbb{A}(q/p) & \text{for } M^2 > 0. \end{cases}$$

$$\tag{84}$$

As before, the techniques of Appendix F allow the formal inversion of these scaling forms and yield the following leading long-time behaviour

$$g(t) \simeq \begin{cases} g_c t^{1-\frac{d+\alpha}{2}} & \text{for } M^2 = 0, \\ g_d t^{-\frac{d+\alpha}{2}} & \text{for } M^2 > 0. \end{cases}$$
(85)

(B2) In the case  $2 < d + \alpha$ , we must reconsider the scaling form of  $\overline{\overline{A}}(p,q)$ . Now, A(t,s) is integrable and the value of  $\overline{\overline{A}}(0,0)$  is finite. On the other hand, since the boundary conditions are given by  $\overline{G}_0(p) = \overline{G}(p,0)$ ,

see Eq. (54), and these are only defined for  $p \ge 0$ , we can analytically continue  $G_0 = G_0(t) = G_0(-t)$  to an even function. Going back to Eq. (53) and expanding, we find

$$\overline{\overline{G}}(p,q) \simeq \begin{cases} \overline{\overline{A}}(0,0) + (6\gamma/u) \left[\overline{G}_0(p) + \overline{G}_0(q)\right] \frac{1}{p+q} & \text{for } M^2 = 0, \\ \\ \overline{\overline{A}}(0,0) + (6\gamma/u) \left[\overline{G}_0(p) + \overline{G}_0(q)\right] \\ \frac{\overline{\overline{A}}(0,0) + (6\gamma/u) \left[\overline{G}_0(p) + \overline{G}_0(q)\right]}{M^2} + \frac{c_A}{M^2} p^{\frac{d+\alpha}{2}-2} \mathbb{A}(q/p) & \text{for } M^2 > 0. \end{cases}$$

$$\tag{86}$$

We first invert this in the critical case  $M^2 = 0$ . We define a new even function  $\mathscr{R}(t)$  such that  $\overline{\mathscr{R}}(p) + \overline{\mathscr{R}}(q) = \overline{\overline{A}}(0,0) + (6\gamma/u) \left[\overline{G}_0(p) + \overline{G}_0(q)\right]$ . Using the identity (D.3), we obtain  $G(t,s) = \tilde{c}_F^{-1}\mathscr{R}(t-s)$ . Because of the boundary condition  $G(0,0) = \tilde{c}_F^{-1}\mathscr{R}(0) = 1$ , we finally have  $g(t) = G(t,t) = \tilde{c}_F^{-1}\mathscr{R}(0) = 1$ , i.e., g does not depend on time.

Next, for the case  $M^2 > 0$ , the first term in Eq. (86) will merely lead to terms concentrated around t = 0 or s = 0, which we neglect in our study of the long-time behaviour. The inversion of the last term in Eq. (86) proceeds as before, so that we have the leading long-time behaviour

$$g(t) \simeq \begin{cases} g_c t^0 & \text{for} \quad M^2 = 0, \\ g_d t^{-(d+\alpha)/2} & \text{for} \quad M^2 > 0. \end{cases}$$
(87)

Writing the long-time behaviour as in Eq. (33), we see that the results in Eqs. (81), (85), and (87) for the exponent F are identical to those found in Sec. 3 for the effective Markovian dynamics either in the critical regions I, II, III or for  $r_0 < r_0^c$ . Hence the results listed in Tab. 1 hold true for both the actual quantum dynamics and the effective dynamics.

#### 6. Physical observables - the quantum noise case

Having solved the spherical constraint in the previous section, we can now determine the long-time behaviour of the physical correlation and response functions defined in Sec. 4. We shall begin with the one-time correlation function before we move on to the two-time quantities.

#### 6.1. One-time correlation function

The equal-time correlation function is found from the formal solution Eqs. (16) and (27b) as

$$C_{\boldsymbol{k}}(t) = \frac{e^{-2k^2t/\gamma}}{g(t)} \left[ c_{\alpha}k^{\alpha} + \frac{2\hbar}{\pi\gamma} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \, g_2(t_1, t_2) \, e^{k^2(t_1 + t_2)/\gamma} \frac{\mathbf{t}_0^2 - (t_1 - t_2)^2}{\left(\mathbf{t}_0^2 + (t_1 - t_2)^2\right)^2} \right] =: C_{\boldsymbol{k}}^{(\mathrm{ic})}(t) + C_{\boldsymbol{k}}^{(\mathrm{n})}(t). \tag{88}$$

Here, we decompose the total correlator into a contribution  $C_{\mathbf{k}}^{(\mathrm{ic})}(t)$  due to the initial correlator and a contribution  $C_{\mathbf{k}}^{(\mathrm{n})}(t)$  of the quantum noise. In what follows, we shall analyse the relative importance of these terms. We emphasise that it is not the function G(t,s), but rather the function  $g_2(t,s) = \sqrt{g(t)g(s)}$ , defined in Eq. (47), which arises in the quantum noise integral contribution  $C_{\mathbf{k}}^{(\mathrm{n})}(t)$ .

We proceed by considering the quenches above, onto and below the critical point  $r_0^c$ .

(A) For a **quench above the critical point**, in the previous section we had derived the following exponential long-time behaviour

$$g_2(t,s) \sim \exp\left(\frac{t+s}{2\tau_{\rm r}}\right).$$
 (89)

First, it follows that the initial correlations  $C_{\mathbf{k}}^{(\mathrm{ic})}(t)$  are exponentially suppressed at large times and it suffices to treat the quantum noise correlations  $C_{\mathbf{k}}^{(\mathrm{in})}(t)$ . Second, the main contribution to that integral in Eq. (88) comes from the upper integration bound and we may replace  $g_2(t,s)$  by its leading asymptotic form. Third, since  $g_2$  depends only on the sum t + s, we can reduce the integration domain in Eq. (88) to the triangle  $0 \leq t_2 \leq t_1 \leq t$  and then change coordinates to diagonal coordinates (see also Appendix E). This yields

$$C_{\boldsymbol{k}}^{(n)}(t) = e^{-(1/\tau_{\rm r} + 2k^2/\gamma)t} \frac{2\hbar}{\pi\gamma} \int_0^t \mathrm{d}u \int_u^{2t-u} \mathrm{d}v \ e^{\left[(2\tau_{\rm r})^{-1} + k^2/\gamma\right]v} \frac{\mathsf{t}_0^2 - u^2}{\left(\mathsf{t}_0^2 + u^2\right)^2}.$$
(90)

Calculating the v-integral we find that the quantum noise acts as a distribution on two exponential test functions

$$C_{\boldsymbol{k}}^{(n)}(t) = \frac{2\hbar}{\pi\gamma} \int_0^t \mathrm{d}u \; \frac{e^{-\left[(2\tau_{\mathrm{r}})^{-1} + k^2/\gamma\right]u} - e^{-\left[(2\tau_{\mathrm{r}})^{-1} + k^2/\gamma\right](2t-u)}}{(2\tau_{\mathrm{r}})^{-1} + k^2/\gamma} \frac{\mathsf{t}_0^2 - u^2}{\left(\mathsf{t}_0^2 + u^2\right)^2}.\tag{91}$$

The first term can be evaluated, in the limit  $t \to \infty$ , as discussed in Appendix E, in particular Eq. (E.20), while the second term is exponentially suppressed. We eventually arrive at the stationary correlator already quoted in Eq. (38)

$$C_{\boldsymbol{k}}(\infty) = \frac{2\hbar}{\pi\gamma} g_{\rm AS}(\mathsf{t}_0((2\tau_{\rm r})^{-1} + k^2/\gamma)), \qquad (92)$$

with the function  $g_{\rm AS}$  defined in Eq. (36).

(B) For a critical quench, we shall study the scaling limit defined in Eq. (19) of the equal-time correlator. We derived in Sec. 5 the leading long-time behaviour  $g(t) \simeq g_c t^F$  with  $F \ge 0$ , see also Tab. 1. The contribution from the initial correlations shows a scaling behaviour

$$C_{k}^{(\rm ic)}(t) \simeq \frac{c_{\alpha}}{g_{c}} (k^{2}t)^{-F} e^{-2k^{2}t/\gamma} k^{F+\alpha/2}.$$
(93)

The noise contribution reads

$$C_{\boldsymbol{k}}^{(n)}(t) \simeq \frac{2\hbar}{\pi\gamma} \frac{1}{g(t)} \int_{[0,t]^2} dt_1 dt_2 \ e^{-k^2(2t-t_1-t_2)/\gamma} g_2(t_1,t_2) \frac{\mathsf{t}_0^2 - (t_1-t_2)^2}{(\mathsf{t}_0^2 + (t_1-t_2)^2)^2} = \frac{2\hbar}{\pi\gamma} \frac{(g_2 * *h)(t,t)}{g(t)}, \tag{94}$$

and is expressed via a double convolution, with the auxiliary function

$$h(t_1, t_2) = e^{-\frac{k^2}{\gamma}(t_1 + t_2)} \frac{\mathbf{t}_0^2 - (t_1 - t_2)^2}{\left(\mathbf{t}_0^2 + (t_1 - t_2)^2\right)^2}.$$
(95)

In order to analyse this double convolution, we study first the two-time double convolution with two distinct arguments  $(g_2 * *h)(t, s)$  and then set them equal at the end. We emphasise that this procedure does *not* correspond to studying a two-time correlation function. The double Laplace transform of h is evaluated using Appendix E, see Eq. (E.20). For the sought long-time scaling limit, one should fix  $\bar{p} = p\gamma/k^2$  and  $\bar{q} = q\gamma/k^2$  and then expand for  $p, q, k^2$  small. We find

$$\overline{\overline{h}}(p,q) = \frac{\frac{k^2}{\gamma} + p}{\frac{k^2}{\gamma} + \frac{p+q}{2}} g_{AS} \left( \mathsf{t}_0 \left( \frac{k^2}{\gamma} + p \right) \right) + \frac{\frac{k^2}{\gamma} + q}{\frac{k^2}{\gamma} + \frac{p+q}{2}} g_{AS} \left( \mathsf{t}_0 \left( \frac{k^2}{\gamma} + q \right) \right) \\ \simeq -2 \left[ C_E + \ln \left( \frac{\mathsf{t}_0 k^2}{\gamma} \right) \right] - \frac{(1+\bar{p}) \ln (1+\bar{p}) + (1+\bar{q}) \ln (1+\bar{q})}{1 + (\bar{p} + \bar{q})/2}.$$
(96)

Then, by using the form of  $\overline{\overline{h}}$  in Eq. (96) and the identities from Appendix D, we find

$$\frac{(g_{2}**h)(t,t)}{g(t)} = \frac{1}{g(t)} \mathcal{L}_{2}^{-1} \left(\overline{g_{2}}(p,q)\overline{h}(p,q)\right)(t,t) 
= -2 \left[C_{E} + \ln\left(\frac{t_{0}k^{2}}{\gamma}\right)\right] - \frac{1}{g(t)} \mathcal{L}_{2}^{-1} \left(\overline{g_{2}}(p,q)\frac{(1+\bar{p})\ln(1+\bar{p}) + (1+\bar{q})\ln(1+\bar{q})}{1+\frac{1}{2}(\bar{p}+\bar{q})}\right)(t,t) 
= -2 \left[C_{E} + \ln\left(\frac{t_{0}k^{2}}{\gamma}\right)\right] - \left(\frac{k^{2}}{\gamma}\right)^{-2} \Gamma^{2} \left(1+\frac{F}{2}\right) \left(\frac{k^{2}t}{\gamma}\right)^{-F} \times 
\times \mathcal{L}_{2}^{-1} \left((\bar{p}\bar{q})^{-1-F/2} \frac{(1+\bar{p})\ln(1+\bar{p}) + (1+\bar{q})\ln(1+\bar{q})}{1+\frac{1}{2}(\bar{p}+\bar{q})}\right)(t,t).$$
(97)

In Appendix F, we show that the scaling function  $\Phi(\bar{p}, \bar{q}) = \Phi(p\gamma/k^2, q\gamma/k^2)$  is the double Laplace transform of the scaling function  $(k^2/\gamma)^2 \phi(k^2 t/\gamma, k^2 s/\gamma)$ , provided  $\Phi(u, v) = \overline{\phi}(u, v)$ . Applying this to the last line of Eq. (97), we find the scaling form of the noise contribution

$$C_{\boldsymbol{k}}^{(\mathrm{n})}(t) \simeq -\frac{4\hbar}{\pi\gamma} \left[ C_E + \ln\left(\frac{\mathsf{t}_0 k^2}{\gamma}\right) \right] - \frac{2\hbar}{\pi\gamma} (k^2 t/\gamma)^{-F} \Psi(k^2 t/\gamma, k^2 t/\gamma), \tag{98}$$

with a universal scaling function  $\Psi(t,t')$  which can be obtained by inverting

$$\overline{\overline{\Psi}}(p,q) = \Gamma^2 \left( 1 + \frac{F}{2} \right) (pq)^{-1 - F/2} \frac{(1+p)\ln(1+p) + (1+q)\ln(1+q)}{1 + (p+q)/2}.$$
(99)

This inversion can be carried out by using Eq. (44) at p. 186 of Ref [68] or Eq. (D.4), followed by Eq. (2.5.2.1) of Ref. [72]. This leads to the integral representation

$$\Psi(x,x) = 4 \int_0^x d\xi \ e^{-2(x-\xi)} \xi^{F+1} \left\{ \frac{{}_2F_2\left(1,1;2,2+\frac{F}{2};-\xi\right)}{1+\frac{F}{2}} + \frac{1}{\xi} \left[ \psi\left(1+\frac{F}{2}\right) - \ln\xi \right] + \frac{1}{\xi} {}_2F_2\left(1,1;2,1+\frac{F}{2};-\xi\right) + \frac{F}{2\xi^2} \left[ \psi\left(\frac{F}{2}\right) - \ln\xi \right] \right\}$$
(100a)

$$\sim \begin{cases} x^F \ln x & \text{for } x \ll 1, \\ x^{F-1} & \text{for } x \gg 1, \end{cases}$$
(100b)

where  $\psi(x)$  is the digamma function [67]. The asymptotics for small arguments is obtained by expanding the integrand above. The asymptotics of  $\Psi$  for large scaling arguments is obtained, instead, from a Laplace approximation [73], or by using the results presented in Appendix F.

Finally, we can compare the contributions due to initial correlations to those due to noise correlations. We make use of the values of F listed in table 1 to do this. In region I (see Fig. 4), both terms equally contribute and we need to consider the full equal-time correlator (88). In regions II and III instead, the initial term dominates for small momenta, because of the prefactors  $k^{1-d/2}$  or  $k^{\alpha/2}$ , respectively.

(C) For a quench across criticality the noise contribution is still given by Eq. (88) upon the replacement  $\alpha \mapsto d + \alpha$ . The initial correlations are nevertheless more relevant in this case since their scaling form reads

$$C_{\mathbf{k}}^{(\rm ic)}(t) = \frac{c_{\alpha}}{g_d} k^{-d} (k^2 t)^{(d+\alpha)/2} e^{-2k^2 t/\gamma}.$$
(101)

Because of the factor  $k^{-d}$  in this expression, the initial correlations dominate in the scaling limit.

#### 6.2. Two-time response and correlation functions

We consider first the two-time response function. Since the underlying equations of motion are linear, it is clear from Eq. (20) that the response function remains unaffected by the noise structure, be it classical, effective or quantum. It follows that in all these cases

$$R_{k}(t,s) = \gamma^{-1} \sqrt{\frac{g(s)}{g(t)}} e^{-k^{2}(t-s)/\gamma} \Theta(t-s).$$
(102)

Using g(t) as derived in Sec. 5.4 yields the results discussed in Sec. 4.2.

In contrast, the two-time correlation function

$$C_{\boldsymbol{k}}(t,s) = \frac{e^{-\frac{k^2}{\gamma}(t+s)}}{g_2(t,s)} \left[ c_{\alpha}k^{\alpha} + \frac{2\hbar}{\pi\gamma} \int_0^t \mathrm{d}t' \int_0^s \mathrm{d}s' \, e^{\frac{k^2}{\gamma}(t'+s')} g_2(t',s') \frac{\mathsf{t}_0^2 - (t'-s')^2}{\left[\mathsf{t}_0^2 + (t'-s')^2\right]^2} \right],\tag{103}$$

does depend on the noise correlator. We need to study carefully this non-Markovian integral in order to analyse its relevance with respect to the contribution of the correlations in the initial state, given by the first term in the brackets. We shall use the decomposition  $C_{\mathbf{k}}(t,s) = C_{\mathbf{k}}^{(\mathrm{ic})}(t,s) + C_{\mathbf{k}}^{(\mathrm{n})}(t,s)$  to refer to the first contribution from the initial correlations (ic) and the second contribution coming from the noise correlations (n), corresponding to the first and second term in brackets respectively.

(A) For a quench to the disordered region the self-consistent function shows an exponential behaviour. We shall evaluate the two-time correlator in the asymptotic limit with  $s \to \infty$  but fixed  $\tau = t - s$ . Later on, we shall also consider the case in which  $\tau$  becomes large. Since we are generally interested in t > s, we may separate the integration in Eq. (103) into two terms  $\int_0^t dt' \int_0^s ds' = \int_0^s dt' \int_0^s ds' + \int_s^t dt' \int_0^s ds'$ . The first term will contribute to the equal-time correlation function  $C_k^{(n)}(s)$ , see Eq. (88). The second integral retains only the dependence on  $\tau$  in the asymptotic limit. In fact, this can be seen as follows. First, we write

$$C_{\boldsymbol{k}}^{(n)}(s+\tau,s) \simeq e^{-\left[k^{2}/\gamma+(2\tau_{r})^{-1}\right]\tau} C_{\boldsymbol{k}}^{(n)}(s) + \frac{2\hbar}{\pi\gamma} \int_{s}^{s+\tau} dt' \int_{0}^{s} ds' \, e^{(k^{2}/\gamma+(2\tau_{r})^{-1})(t'+s'-2s-\tau)} \frac{\mathsf{t}_{0}^{2}-(t'-s')^{2}}{\left[\mathsf{t}_{0}^{2}+(t'-s')^{2}\right]^{2}}.$$
(104)

We identify in the first term the part  $C_{\mathbf{k}}^{(n)}(s)$  of the equal-time correlator, using Eq. (88). The second term is dominated by the contributions near to the upper limits of integration, so that we immediately

substituted the exponential form (89) for  $g_2$ . Now, we can express the first term as a response function by using Eq. (102) and again Eq. (89), in the asymptotic limit. In the the second term, the new variables  $u = t' - s - \tau$  and v = s - s' are introduced, yielding

$$C_{\boldsymbol{k}}^{(n)}(s+\tau,s) \simeq \gamma R_{\boldsymbol{k}}(\tau) C_{\boldsymbol{k}}^{(n)}(s) + \frac{2\hbar}{\pi\gamma} \int_{-\tau}^{0} \mathrm{d}u \int_{0}^{s} \mathrm{d}v \, e^{(k^{2}/\gamma + (2\tau_{\mathrm{r}})^{-1})(u-v)} \frac{\mathsf{t}_{0}^{2} - (\tau+u+v)^{2}}{\left[\mathsf{t}_{0}^{2} + (\tau+u+v)^{2}\right]^{2}} \tag{105}$$

Next, we rescale u and v such that the limit  $t_0 \to 0$  can be taken (we also let  $s \to \infty$ ). This yields

$$C_{\boldsymbol{k}}^{(n)}(s+\tau,s) \simeq \gamma R_{\boldsymbol{k}}(\tau) C_{\boldsymbol{k}}^{(n)}(\infty) - \frac{2\hbar}{\pi\gamma} \int_0^\infty \mathrm{d}v \int_{-1}^0 \mathrm{d}u \; \frac{e^{\tau (k^2/\gamma + (2\tau_r)^{-1})(u-v)}}{(1+u+v)^2}.$$
 (106)

Next, the *u*-integration above is calculated, for  $\tau$  large, by using the Laplace approximation [73] and finally, the remaining *v*-integral is estimated, again for  $\tau$  large. We eventually find

$$C_{\mathbf{k}}^{(n)}(s+\tau,s) \simeq \gamma R_{\mathbf{k}}(\tau) C_{\mathbf{k}}^{(n)}(\infty) - \frac{2\hbar}{\pi\gamma} \frac{1}{\left[(2\tau_{\rm r})^{-1} + k^2/\gamma\right]^2} \frac{1}{\tau^2}.$$
 (107)

The first term in  $C_{\mathbf{k}}^{(n)}(s+\tau,s)$  decays exponentially upon increasing  $\tau$ , see Eq. (102), while the second term depends algebraically on  $\tau$ . For large  $\tau$ , we conclude that

$$C_{k}(s+\tau,s) \simeq -\frac{2\hbar}{\pi\gamma} \frac{1}{\left[(2\tau_{\rm r})^{-1} + k^2/\gamma\right]^2} \frac{1}{\tau^2},$$
(108)

since the initial contribution  $C_{k}^{(ic)}(t,s)$  is exponentially small as  $\tau$  increases. This proves Eq. (30) in Sec. 4.

(B) For a quench to the critical point we need to reconsider the noise contribution. Since we are mainly interested in the universal exponents defined in Eq. (24) we shall immediately work with the autocorrelation function  $C(t,s) = \int_{k} C_{k}(t,s)$  from which they can be readily obtained. The noise contribution then reads

$$C^{(n)}(t,s) = \frac{2\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} (ts)^{-F/2} \int_0^t dt' \int_0^s ds' \frac{(t's')^{F/2}}{(t+s-t'-s')^{d/2}} \frac{\mathsf{t}_0^2 - (t'-s')^2}{(\mathsf{t}_0^2 + (t'-s')^2)^2}.$$
 (109)

This can be written via a weighted convolution, defined in Appendix D, as

$$C^{(n)}(t,s) = \frac{2\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} (ts)^{-F/2} (h_1 * *_w h_2) (t,s)$$
(110)

with the identifications

$$h_1(t,s) = (ts)^{F/2}, \quad h_2(t,s) = (t+s)^{-d/2} \text{ and } w(t-s) = \frac{\mathbf{t}_0^2 - (t-s)^2}{\left[\mathbf{t}_0^2 + (t-s)^2\right]^2}.$$
 (111)

Using Eq. (D.8), we need to evaluate the double Laplace transforms of  $h_2(t, s)$  and of  $h_1(t, s) \cdot w(t - s)$ , in order to factorise the weighted convolution (110) and study its asymptotics, followed by the application of a Tauberian theorem. While the former is straightforwardly evaluated as

$$\overline{\overline{h_2}}(p,q) = \Gamma\left(1 - \frac{d}{2}\right) \frac{p^{\frac{d}{2} - 1} - q^{d/2 - 1}}{q - p},$$
(112)

finding the latter is a non-trivial task. Starting from the formal definition of the double Laplace transform, using the techniques from Appendix E in order to go over to diagonal coordinates yields

$$\overline{\overline{(h_1w)}}(p,q) = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s \ (ts)^{F/2} \frac{\mathbf{t}_0^2 - (t-s)^2}{\left[\mathbf{t}_0^2 + (t-s)^2\right]^2} e^{-pt-qs}$$
$$= \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}s \ (ts)^{F/2} \frac{\mathbf{t}_0^2 - (t-s)^2}{\left[\mathbf{t}_0^2 + (t-s)^2\right]^2} \left(e^{-pt-qs} + e^{-ps-qt}\right) = \int_0^\infty \mathrm{d}u \frac{\mathbf{t}_0^2 - u^2}{\left(\mathbf{t}_0^2 + u^2\right)^2} H(u), \quad (113)$$

with the auxiliary function H(u). This is evaluated by using Eq. (2.3.6.10) of Ref. [74] and where  $K_{\nu}$  is a modified Bessel function [67]. Calculating the integral and then expanding in u, we find

$$H(u) = 2^{-F} \cosh\left(\frac{u}{2}(p-q)\right) e^{-\frac{u}{2}(p+q)} \int_{0}^{\infty} \left(v^{2} + 2uv\right)^{F/2} e^{-\frac{v}{2}(p+q)} dv$$
$$= \frac{F \Gamma\left(\frac{F}{2}\right)}{\sqrt{\pi}} \left(\frac{u}{p+q}\right)^{\frac{1}{2} + \frac{F}{2}} \cosh\left(\frac{u}{2}(p-q)\right) K_{\frac{1}{2} + \frac{F}{2}} \left(\frac{p+q}{2}u\right)$$
$$\overset{u \ll 1}{\simeq} \frac{F \Gamma\left(\frac{F}{2}\right)}{\sqrt{\pi}} \left[2^{F} \Gamma\left(\frac{1+F}{2}\right)(p+q)^{-F-1} + 2^{-F-2} \Gamma\left(\frac{-F-1}{2}\right)u^{1+F}\right].$$
(114)

We are now able to evaluate the double Laplace transform using the small-u expansion of H(u) in Eq. (114), according to the analysis presented in Appendix E. First, the u-independent constant term does not contribute to the integral for  $\overline{\overline{h_1w}}$ , see Eq. (E.7) with s = 0. Next, the lowest-order correction in u is independent of p and q. Thus, using Eqs. (112) and (D.2), we conclude that to leading order the weighted convolution behaves as

$$(h_1 * *_w h_2)(t,s) \sim (t+s)^{-d/2},$$
(115)

which implies that  $C^{(n)}(t,s)$  is less relevant than the initial correlations  $C^{(ic)}(t,s)$ , if  $\alpha < 0$ . We should remark, however, that this argument does not hold if F = 0, e.g., in region I with  $\alpha = 0$  (see Fig. 4). Then the term we calculated above is constant as well and the next-to-leading contribution in u must be worked out by expanding H(u) in Eq. (114) to the next order in u and using again Appendix E. Then the scaling of the weighted convolution behaves as

$$(h_1 * *_w h_2)(t,s) \sim (t+s)^{-d/2-1} \tag{116}$$

for t and s large. Accordingly, the initial correlations always dominate the asymptotic behaviour of the two-time correlation function. The two-time autocorrelation function thus reads

$$C(t,s) \simeq C^{(\rm ic)}(t,s) \simeq s^{-d/2} f_C(t/s) \quad \text{with} \quad f_C(x) = \frac{c_\alpha}{g_c} \frac{\Omega_d}{(2\pi)^d} \frac{x^{-F/2}}{(1+x)^{(d+\alpha)/2}}.$$
(117)

(C) For a quench across criticality all the steps presented above for the analysis of the noise contribution of the critical auto-correlator apply upon replacing  $\alpha \mapsto \alpha + d$ . In particular, the initial correlations remain the dominating contribution such that finally

$$C(t,s) \simeq s^0 f_C(t/s)$$
 with  $f_C(x) = \frac{c_\alpha}{g_d} \frac{\Omega_d}{(2\pi)^d} \frac{x^{(d+\alpha)/4}}{(1+x)^{(d+\alpha)/2}}.$  (118)

Summarising, for quantum quenches onto or across the critical point, the ageing scaling behaviour of the two-time auto-responses and auto-correlators is the same as the one derived for the effective dynamics in Sec. 4.

#### 7. Conclusions

We have presented a detailed study of the relaxational dynamics and of the ageing phenomena in a many-body quantum system in contact with an external bath at temperature T = 0. In comparison with classical dynamics, we have considered two distinct types of noise correlators, characterised by the corresponding correlators:

- (a) Non-Markovian, quantum noise the correlators of which are given in Eqs. (11a) and (11b) and are derived from the system-interaction-bath method which is known to reproduce all physically desirable properties of the system, including the quantum fluctuation-dissipation theorem for temperatures T > 0 [40, 41, 42, 46]. Setting T = 0, a regularisation, such as that in Eq. (14), is necessary [44].
- (b) A Markovian *effective noise*, with correlators (13) which resembles a classical white noise, but with a momentum-dependent effective temperature  $T_{\text{eff}} = \frac{\mu}{2} |\mathbf{k}|^2$ . This was chosen such as to reproduce the scaling dimensions of the actual quantum noise.

Comparing these two noises allows us to study the non-Markovian memory effects, which are present in the quantum noise correlators, but not in the Markovian effective noise. In addition, the scaling of these two noises is different from the one of the classical white noise. We chose to investigate two paradigmatic and exactly solvable models of statistical mechanics, namely the quantum spherical model and the quantum O(n)-model, with  $n \to \infty$ . For simplicity, only "ferromagnetic" nearest-neighbour interactions were considered, in d spatial dimensions, but generalisations should not be very difficult. Accounting for the effects of a "paramagnetic" initial state with spatially long-ranged correlations as in Eq. (25) [28] turned out to be an important for understanding the ensuing dynamics.

In analogy with what was already known at T > 0 from the quantum equilibrium state and the classical dynamics of these two models, the long-time relaxational behaviour, at T = 0, of the quantum spherical and the O(n)-model for  $n \to \infty$  belongs to the same universality class, for both cases of noise considered. These predictions were not obtained for the full quantum Langevin equation, but rather for its over-damped limit (10) which has been derived in a particular scaling limit [46]. That these models are solvable is due to the exact reduction of the many-body dynamics to a single integro-differential equation for a function g(t). Since that equation is strongly non-linear, it is solved by embedding it into a system of linear equations for a function G(t,s) of two variables, for which G(t,t) = g(t) holds. Then the long-time behaviour of g(t) has been derived via Mellin transform methods and Tauberian theorems, discussed in detail in the appendices. From this, the long-time behaviour of the correlation and response functions of the order parameter can be obtained.

region		F	$\lambda_C$	$\lambda_R$	a	b
$I_c$	$2 < d < 4,  0 < d + \alpha < 2$	$-1 - \frac{\alpha}{2}$	$d + \frac{\alpha}{2} - 1$	$d - \frac{\alpha}{2} - 1$	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$
$II_c$	$4 < d,  0 < d+\alpha < 2$	$1 - \frac{d+\alpha}{2}$	$1 + \frac{d+\alpha}{2}$	$\frac{d-\alpha}{2}+1$	$\frac{d}{2} - 1$	1
$III_c$	$2 < d < 4,  d+\alpha > 2$	$\frac{d}{2} - 2$	$\frac{3}{2}d - 2$	$\frac{3}{2}d - 2$	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$
$IV_c$	$4 < d, d+\alpha > 2, \alpha > -2$	0	d	d	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$
$V_c$	$4 < d,  d+\alpha > 2,  \alpha < -2$	0	$d + \alpha$	d	$\frac{d}{2} - 1$	$\frac{d+\alpha}{2}$
$T < T_c$	2 < d	$-\frac{d+\alpha}{2}$	$\frac{d+\alpha}{2}$	$\frac{d-\alpha}{2}$	$\frac{d}{2} - 1$	0

Table 2: Non-equilibrium exponents for classical dissipative dynamics, taken from Ref. [28]. For a critical quench to  $T = T_c$ , one should distinguish a number of cases, denoted by  $I_c - V_c$ . For a sub-critical quench to  $T < T_c$  such a distinction is not necessary.

The meaning of our results becomes clearer from a comparison with those of classical dynamics, which are summarised in Table 2.

The main results of our analysis can be stated as follows.

- (1) The stationary state for a quench in the one-phase region at temperature T = 0 is not an equilibrium state. This holds for both the quantum and the effective dynamics. This conclusion is based on the following evidence:
  - (a) For the existence of a critical point, between classical and quantum dynamics there is a dimensional shift  $d + 2 \mapsto d$ . We find a dynamical quantum phase transition, with a finite critical coupling  $r_0^c$ , for any dimension d > 0. However, an *equilibrium* quantum phase transition only exists for d > 1.
  - (b) For quenches into the single-phase region, both responses and correlators rapidly become stationary and do not depend on the initial conditions. The two-time response  $R_{\mathbf{k}}(s+\tau,s)$  is equal to the classical one and decays exponentially upon increasing  $\tau$ . For the effective dynamics, this is also the case for the two-time correlator  $C_{\mathbf{k}}(s+\tau,s)$  and the *classical* fluctuation-dissipation theorem with the effective temperature  $T_{\text{eff}}(\mathbf{k}) \neq 0$  is satisfied.

For the quantum noise, instead, the two-time correlator  $C_k(s+\tau, s)$  decays algebraically upon increasing  $\tau$ . This mismatch of functional forms makes it impossible to satisfy the quantum fluctuation-dissipation theorem.

- (2) "Quantum ageing" may be characterised via the scaling behaviour in Eq. (23) of the two-time response and correlation functions for quantum quenches onto or across the quantum phase transition. The universal exponents which describe ageing turn out to be the same for the effective and the quantum dynamics and they are listed in Table 1. Accordingly, quantum memory effects do not appear to be relevant for the ageing dynamics.
- (3) The scaling properties observed during "quantum ageing" are subtly different from those of classical dynamics, as it can be inferred from comparing the relevant characteristic exponents reported in Tables 1 and 2.
  - (a) For a critical quench, Table 1 distinguishes the cases I III (see Fig. 4) of "quantum ageing", and Table 2 those  $I_c$ - $V_c$  of classical ageing. The classical cases  $III_c$  and  $IV_c$ , which correspond to the paradigmatic instance of fully uncorrelated initial conditions [18, 20], do not have an analogue in "quantum ageing".
  - (b) We observe the correspondences  $I \leftrightarrow I_c$ ,  $II \leftrightarrow II_c$  and  $III \leftrightarrow V_c$ . The exponents  $\lambda_{C,R}$  in the cases  $I_c$  and I are related by the dimensional shift  $d-1 \mapsto d$ . This relationship however does not extend to the exponents a and b.

In the remaining two correspondences, the respective exponents are all identical. The difference between classical and the quantum ageing which we observe here, merely comes from the dimensional shift  $d + 2 \mapsto d$ .

- (4) For critical quenches, both the effective and the quantum noises appear to be weaker than classical white noise. This comes about since those classical cases  $III_c$  and  $IV_c$ , where the bath noise dominates the contributions of the initial correlations, do not have a quantum analogue. Only in the region I (and analogously,  $I_c$  for classical noise) are their corresponding contributions of the same order. Accordingly, critical quantum systems appear to be more sensitive than classical ones to spatially long-ranged initial correlations.
- (5) For a critical quantum quench, spatially long-ranged initial correlations with  $\alpha < 0$  are necessary for a long-time scaling behaviour distinct from that predicted by mean-field theory. This is not the case for classical dynamics.

- (6) Quantum memory effects are apparent for the equal-time structure factor.
  - (a) For a critical quench in region I, Fig. 6 shows that although the time-dependence of  $C_{\mathbf{k}}(t)$  for the quantum noise is qualitatively very similar to the one of the effective dynamics, there are also quantitative differences, notably for smaller values of the scaling variable  $\rho = k^2 t / \gamma$ .
  - (b) The stationary structure factor  $C_{\mathbf{k}}(\infty)$  resulting from the quantum noise is well approximated by an Ornstein-Zernicke form in the single-phase region, see Fig. 5. The different form for the effective dynamics comes from the momentum-dependence of  $T_{\text{eff}}(\mathbf{k})$ .
- (7) In the single-phase region, for large waiting times s, the two-time correlator  $C_{\mathbf{k}}(s + \tau, s)$ , becomes independent of s and decays upon increasing  $\tau$  algebraically for quantum noise and exponentially for the effective dynamics. This evidence for dynamical scaling above the quantum critical point is surprising. Its origin may require further investigations in the future.
- (8) The dynamical exponent z = 2 of the dissipative dynamics of open quantum systems is distinct from the value z = 1 of the unitary dynamics of closed quantum systems [63, 64] (or of Markovian approximations of quantum dynamics via Lindblad equations [39]). If one considers a dynamics where  $T \to 0$  such that the stationary state is an equilibrium state, dissipative quantum dynamics still leads to z = 2 but different values of the exponents of ageing are obtained [36, 37].
- (9) For quantum quenches across criticality, the long-time dynamics is dominated by the initial correlations and is identical to classical dynamics, see Tables 1 and 2 This is somewhat expected, since the quantum noise can be considered to be weaker than the classical white noise.

It would be interesting to see which of the above conclusions are valid also for different non-equilibrium universality classes.

In summary, it appears that the main differences between "quantum" and classical ageing come from the different scaling of the noises. Our exact results for the quantum spherical model at T = 0 suggest that it should be often sufficient to replace the scaling properties of the quantum noise correlators (11) – with its temporal non-locality – by a well-chosen effective Markovian noise (13), with a spatial non-locality. Both of them are distinct from the classical white noise (12). Non-Markovian long-term memory effects generically appear as relatively minor quantitative details, notably for the structure factor, and hardly ever lead to qualitative changes of the long-time behaviour. It would be interesting to investigate if similar phenomenological prescriptions could be formulated beyond the model-specific context of the present work.

The coarsening dynamics and defect formation after quenches across critical points can also be described by the *Kibble-Zurek mechanism* [75, 76, 77, 78]. Because of the divergence of length- and time-scales at criticality (known as the critical slowing-down), even for "slow" quenches across criticality the system's adiabatic dynamics can no longer equilibrate. Then the dynamics is analogous to the one of a rapid quench such that our predictions might be seen as a reliable benchmark for Kibble-Zurek studies of transitions at zero temperature, such as the one recently carried out for the classical spherical model in Ref. [79].

A different question for future work concerns the possible consequences of different long-time memory effects in stochastic complex systems, for which some of the new mathematical tools developed here might become useful.

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#### Appendix A. Overdamping as long-time scaling limit

We illustrate the main steps connected to the scaling limit indicated in Eq. (9) that yields the overdamped Langevin equation (10), following the steps outlined in Ref. [46]. For clarity, we focus on a single degree of freedom which suffices for the analysis of the spherical model and the O(n) model for  $n \to \infty$  as the equations of motion decouple in Fourier space. The corresponding quantum Langevin equations read

$$\partial_t \phi(t) = \lambda \pi(t) + \eta^{(\phi)}(t), \tag{A.1a}$$

$$\partial_t \pi(t) = -\frac{1}{\lambda} r(t)\phi(t) - \gamma \pi(t) + \eta^{(\pi)}(t), \qquad (A.1b)$$

which yield a single second-order quantum Langevin equation,

$$\partial_t^2 \phi(t) = -r(t)\phi(t) - \gamma \lambda \pi(t) + \lambda \eta^{(\pi)}(t) + \partial_t \eta^{(\phi)}(t).$$
(A.2)

We apply the following scaling transformation to Eq. (A.2)

$$\tilde{t} = \lambda t, \quad \eta^{(\phi)}(t) = \lambda^0 \tilde{\eta}^{(\phi)}(\tilde{t}), \quad \eta^{(\pi)}(t) = \lambda^0 \tilde{\eta}^{(\pi)}(\tilde{t}), \quad \phi(t) = \lambda^1 \tilde{\phi}(\tilde{t}), \tag{A.3}$$

such that the quantum Langevin equation takes the form

$$\lambda^2 \partial_{\tilde{t}}^2 \tilde{\phi}(\tilde{t}) = -r(\tilde{t}) \tilde{\phi}(\tilde{t}) - \tilde{\gamma} \partial_{\tilde{t}} \tilde{\phi}(\tilde{t}) + \tilde{\xi}(\tilde{t})$$
(A.4)

with  $\tilde{\gamma} = \gamma \lambda$  and the composite noise  $\tilde{\xi}(\tilde{t}) = \tilde{\eta}^{(\pi)}(\tilde{t}) + \tilde{\gamma}\lambda^{-2}\tilde{\eta}^{(\phi)}(\tilde{t}) + \partial_{\tilde{t}}\tilde{\eta}^{(\phi)}(\tilde{t})$ . In the scaling limit (9) one lets  $\lambda \to 0$  such that the second-order time derivative on the left-hand-side of Eq. (A.4) is suppressed and we obtain the overdamped quantum Langevin equation

$$\tilde{\gamma}\partial_{\tilde{t}}\tilde{\phi}(\tilde{t}) = -r(\tilde{t})\tilde{\phi}(\tilde{t}) + \tilde{\xi}(\tilde{t}).$$
(A.5)

It remains to analyse the effects of the overdamped limit on the noise correlations. To this end we report the complete noise (anti-) commutators [46]

$$\left\langle \left\{ \eta^{(\phi)}(t), \eta^{(\pi)}(t') \right\} \right\rangle = \frac{\hbar\gamma}{2\pi} J\left(\frac{\hbar}{2T}, t - t'\right), \quad \left\langle \left[ \eta^{(\phi)}(t), \eta^{(\pi)}(t') \right] \right\rangle = \mathrm{i}\hbar\gamma\delta(t - t'), \tag{A.6}$$

with the function  $J(a, \tau) = -i \int_{\mathbb{R}} d\nu \coth(a\nu) e^{i\nu\tau}$ . For the noise correlators to be well-defined, we observe that the temperature T needs to be rescaled in the overdamped as  $\tilde{T} = T/\lambda$ . A careful analysis reveals that the (anti-) commutation relation of the composite noise in the overdamped Langevin equation is given by

$$\left\langle \left\{ \tilde{\xi}(\tilde{t}), \tilde{\xi}(\tilde{t}') \right\} \right\rangle = \frac{\hbar \tilde{\gamma}}{\pi} I\left(\frac{\hbar}{2\tilde{T}}, \tilde{t} - \tilde{t}'\right), \quad \left\langle \left[ \tilde{\xi}(\tilde{t}), \tilde{\xi}(\tilde{t}') \right] \right\rangle = 2i\hbar \tilde{\gamma} \delta'(\tilde{t} - \tilde{t}'), \tag{A.7}$$

with  $I(a,\tau) = \partial_{\tau} J(a,\tau)$ . Two limiting cases are of special interest here, namely

- a)  $\tilde{T} \to \infty$ , which reproduces the classical white noise.
- b)  $\tilde{T} \to 0$ , in which the zero-temperature noise correlation function reads

$$\left\langle \left\{ \tilde{\xi}(\tilde{t}), \tilde{\xi}(\tilde{t}') \right\} \right\rangle = \frac{\tilde{\gamma}\hbar}{\pi} \int_{-\infty}^{\infty} |\omega| e^{i\omega(\tilde{t}-\tilde{t}')} d\omega.$$
(A.8)

In order to simplify the notation, in the main text we drop all tildes from the variables  $\tilde{t}$ ,  $\tilde{\gamma}$  and  $\tilde{T}$ . By considering now the case with many degrees of freedom discussed in the main text, one can reproduce the argument above for the Fourier modes of the field  $\phi_{\mathbf{k}}(t)$ . We find the quantum Langevin equation (10)

$$\gamma \partial_t \phi_{\mathbf{k}}(t) + \left( r(t) + k^2 \right) \phi_{\mathbf{k}}(t) = \xi_{\mathbf{k}}(t), \tag{A.9}$$

with the quantum noise correlation function in Eq. (11a)

$$\langle \{\xi_{\boldsymbol{k}}(t),\xi_{\boldsymbol{k}'}(t')\}\rangle = \frac{2\gamma\hbar}{\pi} \int_0^\infty d\omega \,\omega \coth\left(\frac{\hbar\omega}{T}\right) \cos(\omega(t-t')) \,\delta(\boldsymbol{k}+\boldsymbol{k}'). \tag{A.10}$$

#### Appendix B. Regularised quantum noises in the over-damped limit

We focus on the following quantum Langevin equation for the harmonic oscillator with position operator x and friction coefficient  $\gamma$ , i.e.,

$$\epsilon \ddot{x} + \gamma \dot{x} + \Omega^2 x = \xi, \tag{B.1}$$

where the "mass"  $\epsilon$  allows us to keep track of the impact of the inertial term  $\epsilon \ddot{x}$ , while  $\Omega$  quantifies the strength of the harmonic potential. We now proceed to study the equal-time commutation relation of x and the canonically conjugate variable  $p = \epsilon \dot{x}$ . This is useful as we shall see that the inertia term acts as a regulator to guarantee that the canonical commutation relation is satisfied.

Assuming that the initial conditions of the dynamics are in the very remote past, and that they relax in time due to dissipation [44], the solution of the homogeneous equation vanishes and one is left only with the contribution generated by the noise. This can be readily determined by using a Fourier transform in time according to  $\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dt \, e^{-i\omega t} x(t)$  [44], which yields

$$\tilde{x}(\omega) = \frac{\tilde{\xi}(\omega)}{\Omega^2 - \epsilon \omega^2 + i\gamma\omega} \quad \text{and} \quad \tilde{p}(\omega) = \frac{i\omega\epsilon\,\tilde{\xi}(\omega)}{\Omega^2 - \epsilon\omega^2 + i\gamma\omega},$$
(B.2)

where  $\tilde{p}$  is defined in analogy with  $\tilde{x}$ . Using the quantum noise correlator in Eq. (11b), it is readily checked that the equal-time commutator does not depend on  $\epsilon$ , i.e.,

$$\langle [x(t), p(t)] \rangle = \int_{\mathbb{R}^2} \frac{\mathrm{d}\omega \mathrm{d}\omega'}{2\pi} \left\langle [\tilde{x}(\omega), \tilde{p}(\omega')] \right\rangle e^{\mathrm{i}(\omega + \omega')t} = \frac{\mathrm{i}\hbar\gamma}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}\omega \frac{\omega^2}{\Omega^2 - \epsilon\omega^2 + \mathrm{i}\gamma\omega} \frac{\epsilon}{\Omega^2 - \epsilon\omega^2 - \mathrm{i}\gamma\omega} = \mathrm{i}\hbar. \quad (B.3)$$

We thus see this choice of the noise correlator guarantees that the canonical commutation relation is satisfied at all times. Note that this noise correlation diverges at short times and this is where the inertia term is relevant to ensure the convergence of the integral for large values of  $|\omega|$ , i.e., at short times. Heuristically, however, the late-time dynamics of the system we are interested in is expected to be effectively dominated by the dissipative terms and should be rather insensitive to what happens at short times, especially as far as the emergence of collective behaviours is concerned. This suggests that the limit  $\epsilon \to 0$  could be taken from the outset in Eq. (B.3) if one introduces a suitable regularising function  $\Re(\omega)$  acting on the integrand evaluated for  $\epsilon = 0$ , i.e.,

$$\langle [x(t), p(t)] \rangle = i\hbar \frac{\gamma}{\pi} \int_{\mathbb{R}} d\omega \, \frac{\omega^2 \mathscr{R}(\omega)}{\Omega^4 + (\gamma \omega)^2}.$$
 (B.4)

The choice  $\mathscr{R}(\omega) = e^{-t_0|\omega|}$  (with  $t_0$  thought to be a small quantity) gives

$$\langle [x(t), p(t)] \rangle = i\hbar \frac{\gamma}{\pi} \int_{-\infty}^{\infty} d\omega \, \frac{\omega^2 e^{-t_0|\omega|}}{\Omega^4 + (\gamma\omega)^2} \simeq i\hbar \frac{2}{\pi t_0 \gamma} + O(t_0), \tag{B.5}$$

which suggests the natural choice  $t_0 = 2/(\gamma \pi) \sim \gamma^{-1}$  for the scale of the cut-off in order to preserve the canonical commutation relations. Although we tested this practical prescription on the equal-time commutator, it works also for various other quantities. For example, all the other relevant two-point functions yield the same conclusion concerning the cutoff as can be easily verified by similar calculations.

#### Appendix C. Effective dynamics: details of the analysis

We outline here the calculations for the effective dynamics, with the Markovian noise correlator in Eq. (13). Using the formal solution (27b), the equal-time autocorrelator  $C(t,t) = \int_{\mathbf{k},(\Lambda)} C_{\mathbf{k}}(t,t)$  is given by

$$C(t,t) = \frac{1}{g(t)} \int_{\boldsymbol{k},(\Lambda)} e^{-2k^2 t/\gamma} C_{\boldsymbol{k}}(0) + \frac{1}{g(t)} \frac{\mu}{\gamma^2} \int_0^t \mathrm{d}s \, g(s) \int_{\boldsymbol{k},(\Lambda)} k^2 e^{-2k^2 (t-s)/\gamma}.$$
 (C.1)

With the initial condition (25), the definition  $A_{\alpha}(t) := \int_{\mathbf{k},(\Lambda)} k^{\alpha} e^{-2k^2 t/\gamma}$  and the convolution from Appendix D, this gives

$$C(t,t) = \frac{1}{g(t)} \left[ c_{\alpha} A_{\alpha}(t) + \frac{\mu}{\gamma^2} (g * A_2)(t) \right].$$
 (C.2)

We rewrite the spherical constraint as a linear integral or integro-differential equation for the function g(t), as follows for the two models considered here.

For the spherical model, the spherical constraint in Eq. (5) reads  $C(t,t) = 1/\lambda$ . This directly produces, along with the formal exact solution in Laplace space,

$$\frac{1}{\lambda}g(t) = c_{\alpha}A_{\alpha}(t) + \frac{\mu}{\gamma^2}(g * A_2)(t) \quad \Rightarrow \quad \overline{g}(p) = \frac{c_{\alpha}A_{\alpha}(p)}{1/\lambda - (\mu/\gamma^2)\overline{A_2}(p)}.$$
 (C.3a)

For the O(n)-model with  $n \to \infty$ , the spherical constraint (2) is  $r(t) = r_0 + \frac{u}{12}C(t,t)$ . From the definition (17), one has  $\frac{g'(t)}{g(t)} = \frac{2}{\gamma}r(t)$ . This gives, again together with the formal solution and g(0) = 1

$$\frac{6\gamma}{u}g'(t) - \frac{12r_0}{u}g(t) = c_{\alpha}A_{\alpha}(t) + \frac{\mu}{\gamma^2}(g*A_2)(t) \quad \Rightarrow \quad \overline{g}(p) = \frac{c_{\alpha}\overline{A_{\alpha}}(p) + 6\gamma/u}{6\gamma p/u - 12r_0/u - (\mu/\gamma^2)\overline{A_2}(p)}.$$
 (C.3b)

In both cases, the late-time behaviour of g(t) is related, via Tauberian theorems [71], to the small-p behaviour of  $\overline{g}(p)$ . In turn, in order to determine this, we need to know the small-p expansion of  $A_{\alpha}(p)$ . Since the computation of this expansion is standard, see, e.g., Refs. [20, 28], we simply cite the results. The final expansion contains at least one  $\Lambda$ -independent term which is in general not an entire function of p and, in addition, a sum of terms with integer powers of p taking the form

$$\overline{A_{\alpha}}(p) \simeq a_{\alpha} p^{(d+\alpha)/2-1} + \sum_{n=0}^{\lfloor \frac{d+\alpha}{2}-1 \rfloor} (-1)^n A_n^{(\alpha)} p^n,$$
(C.4)

where  $\lfloor x \rfloor$  is the largest integer  $\langle x \rangle$  and the sum above is understood to be zero if its upper limit is negative. In addition, if  $d + \alpha = 2m \in \mathbb{N}$  is a positive even integer, extra logarithmic factors arise which we neglect here. The constants in Eq. (C.4) read explicitly

$$a_{\alpha} = \frac{\pi}{2} \frac{\Omega_d}{(2\pi)^d} \frac{(\gamma/2)^{(d+\alpha)/2}}{\sin\left(\frac{\pi}{2}(d+\alpha)\right)} \quad , \qquad A_n^{(\alpha)} = \frac{\Omega_d}{(2\pi)^d} \left(\frac{\gamma}{2}\right)^{n+1} \int_0^{\Lambda} \mathrm{d}k \; k^{d+\alpha-3-2n}, \tag{C.5}$$

where  $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$  is the surface of the unit hypersphere  $S^d$  in d dimensions. Clearly, the first term in Eq. (C.4) is universal, while the other terms, if they occur, depend explicitly on the momentum cutoff  $\Lambda$  and cannot be universal.

Given the expansion (C.4), we can now compare the leading behaviour of  $\overline{g}(p)$  for the two solutions of Eq. (C.3). First, for 0 < d < 2, the leading non-constant term in both denominators comes from  $\overline{A_2}(p) \sim p^{d/2}$ , such that the term  $(6\gamma/u)p$  present in Eq. (C.3b) merely provides a correction to scaling. Since  $\alpha \leq 0$ , it follows that  $d + \alpha < 2$ , thus the leading term in  $\overline{A_\alpha}(p) \sim p^{(d+\alpha)/2-1}$  in both numerators will dominate over an eventual constant present in the numerator of Eq. (C.3b). Accordingly, the leading long-time behaviour of both the spherical and the O(n)-model is the same. Second, let 2 < d. Then one has schematically the leading structure  $\overline{A_2}(p) \sim p^0 + p^1 + p^{d/2}$ , where we omitted to indicate the various constants. The extra terms in the denominator of Eq. (C.3b) can be absorbed into these, up to re-defining certain non-universal constants. For the numerators, if  $d + \alpha < 2$ , then the leading terms comes from  $\overline{A_\alpha}(p) \sim p^{(d+\alpha)/2-1}$  and the constant term present in the numerator of Eq. (C.3b) merely creates a finite-time correction. If, on the other hand,  $d + \alpha > 2$ , then one has the structure  $\overline{A_\alpha}(p) \sim p^0 + p^{(d+\alpha)/2-1}$  and the extra constant term in the numerator of Eq. (C.3b) can be absorbed constant. Again, we conclude that the leading long-time behaviour of the spherical and O(n) models is the same, for all d > 0 and all initial conditions. Although the leading exponents are the same, the corresponding amplitudes can be different, especially for d > 2 and/or  $d + \alpha > 2$ .

We now determine the critical point from the formal solutions Eq. (C.3). If the denominator vanishes for some  $p_c > 0$ , then the function  $\overline{g}(p)$  has a simple pole at  $p = p_c$  and it follows that asymptotically  $g(t) \sim \exp(t/\tau_r)$  which defines the relaxation time scale  $\tau_r$ . On the other hand, if  $p_c \to 0$ , then the behaviour of  $\overline{g}(p)$  will change to  $g(t) \sim t^F$  becoming algebraic. The condition  $p_c = 0$  fixes the critical point. Expanding for  $p \to 0$  and keeping the bath control parameter  $\mu$  fixed, gives  $1/\lambda_c = (\mu/\gamma^2)\overline{A_2}(0)$  for the spherical model and  $(12/u)r_0^c = -(\mu/\gamma^2)\overline{A_2}(0)$  for the O(n)-model. Specifically, the critical values of the control parameters are given by:

$$\frac{1}{\lambda_c} = \frac{\mu}{\gamma^2} A_0^{(2)} \qquad = \frac{\mu}{\gamma} \frac{\Omega_d}{(2\pi)^d} \int_0^{\Lambda} \mathrm{d}k \, k^{d-1} \qquad \qquad \text{for the spherical model} \tag{C.6a}$$

$$r_0^c = -\frac{u}{12}\frac{\mu}{\gamma^2}A_0^{(2)} = -\frac{u}{12}\frac{\mu}{\gamma}\frac{\Omega_d}{(2\pi)^d}\int_0^{\Lambda} \mathrm{d}k\,k^{d-1} \qquad \text{for the } O(n)\text{-model.}$$
(C.6b)

The rest of the analysis required for determining the leading relaxation time  $\tau_r$  as well as the exponents of the leading algebraic behaviours follows closely the approach used for classical dynamics [20, 28, 16] and produces the results quoted in the main text of Sec. 4.1.

#### Appendix D. Properties of double Laplace transforms

We summarise here some useful properties of double convolutions, related to the double Laplace transform. First, we recall the definition of the simple Laplace transform of a function  $h : \mathbb{R}_+ \to \mathbb{C}$  of a single variable, namely  $\overline{h}(p) = \mathcal{L}(h)(p) := \int_0^\infty dt \ e^{-pt}h(t)$ . The convolution of two functions  $h_1$ ,  $h_2$  of a single variable is defined as  $(h_1*h_2)(t) := \int_0^t dt' h_1(t')h_2(t-t')$ . An important property is the factorisation identity  $\overline{(h_1 * h_2)}(p) = \overline{h_1}(p)\overline{h_2}(p)$ , see, e.g., Refs. [67, 80, 81].

The double Laplace transform of a function  $h: \mathbb{R}^2_+ \to \mathbb{C}$  of two variables is defined as [68]

$$\overline{\overline{h}}(p,q) = \mathcal{L}_2(h)(p,q) := \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s \, e^{-pt-qs} \, h(t,s).$$
(D.1)

We refer to the literature [68, 69] for detailed discussions of the conditions under which these Laplace transforms exist and we rather concentrate here on formal identities for explicit calculations. First, if the function h depends only on the sum of its two arguments, namely h(t, s) = k(t + s), the double Laplace transform  $\overline{\overline{h}}$  is related to the simple Laplace transform  $\overline{\overline{k}}$  of k via [68, 69, 70]

$$\overline{\overline{h}}(p,q) = \frac{\overline{k}(p) - \overline{k}(q)}{p - q}.$$
(D.2)

On the other hand, if h(t, s) = k(|t - s|), one has [69]

$$\overline{\overline{h}}(p,q) = \frac{\overline{k}(p) + \overline{k}(q)}{p+q}$$
(D.3)

This latter identity also holds if h(t,s) = k(t-s), provided  $k(\tau) = k(-\tau)$  is even [70]. Second, if h(t,s) = h(s,t) is symmetric, it follows that  $\overline{\overline{h}}(p,q) = \overline{\overline{h}}(q,p)$ , i.e.,  $\overline{\overline{h}}$  is also symmetric. Equations (D.2) and (D.3) provide some examples. Third, if  $h(t,s) = h_1(t)h_2(s)$  then  $\overline{\overline{h}}(p,q) = \overline{h}_1(p)\overline{h}_2(q)$ . Fourth, we note (see, e.g., Eq. (44) at p. 186 of Ref. [68])

$$\mathcal{L}_2^{-1}\left(\frac{\overline{\overline{h}}(p,q)}{c+p+q}\right)(t,t) = \int_0^t \mathrm{d}t' \ e^{-ct'} \ h(t-t',t-t'). \tag{D.4}$$

The double convolution of two functions  $h_{1,2}$  of two variables is defined as

$$(h_1 * *h_2)(t,s) := \int_0^t \mathrm{d}t' \int_0^s \mathrm{d}s' \, h_1(t',s') h_2(t-t',s-s'). \tag{D.5}$$

The factorisation identity for the simple convolution via Laplace transform [67, 81] naturally carries over to the double Laplace transform [68, 69]

$$\overline{\overline{(h_1 \ast \ast h_2)}}(p,q) = \mathcal{L}_2(h_1 \ast \ast h_2)(p,q) = \overline{\overline{h}}_1(p,q) \overline{\overline{h}}_2(p,q).$$
(D.6)

This property allows us to solve linear Volterra integral equations in two variables, as shown in the main text. We introduce a weighted convolution, defined as

$$(h_1 * *_w h_2)(t,s) := \int_0^t \mathrm{d}t' \int_0^s \mathrm{d}s' \, h_1(t',s') h_2(t-t',s-s') w(t'-s'), \tag{D.7}$$

with the weight function w = w(t). Its double Laplace transformation factorises as

$$\mathcal{L}_2(h_1 \ast \ast_w h_2)(p,q) = \overline{\overline{h}}_2(p,q) \,\mathcal{L}_2(h_1(t,s)w(t-s))(p,q) = \overline{\overline{h}}_2(p,q) \,\overline{(h_1w)}(p,q) \tag{D.8}$$

The proof of Eq. (D.6) is given in Refs. [68, 69] and merely uses Fubini's theorem. The proof of the new identity in Eq. (D.8) is similar.

#### Appendix E. Asymptotics of the quantum noise integrals

Consider the double Laplace transform of the quantum noise correlation function, c.f. Eqs. (48) and (15),

$$\overline{\overline{F}}(p,q) = \frac{2\hbar}{\pi\gamma} \int_0^\infty dt \int_0^\infty dt' \int_{k,(\Lambda)} e^{-\frac{k^2}{\gamma}(t+t')} \frac{\mathbf{t}_0^2 - (t-t')^2}{\left[\mathbf{t}_0^2 + (t-t')^2\right]^2} e^{-pt-qt'}.$$
(E.1)



Figure E1: Illustration of the change of variables employed in Eq. (E.2) in order to to isolate the action of the quantum noise function.

Since the original function F(t,t') = F(t',t) is symmetric, this also holds for the double Laplace transform,  $\overline{\overline{F}}(p,q) = \overline{\overline{F}}(q,p)$ , see Appendix D.

We now reduce Eq. (E.1) to a form for which the asymptotic behaviour, especially for  $t_0 \rightarrow 0$ , can be easily determined. We decompose the square integration domain into two triangles, as in Fig. E1. The integration over the upper triangle indicated by the white domain in Fig. E1 can be reduced to an integration over the lower triangle, denoted by the shaded domain in Fig. E1, by Fubini's theorem, and we also use the symmetry of F(t, t'). This leads to

$$\overline{\overline{F}}(p,q) = \int_0^\infty dt \int_0^t dt' F(t,t') e^{-pt-qt'} + \int_0^\infty dt \int_t^\infty dt' F(t,t') e^{-pt-qt'} 
= \int_0^\infty dt \int_0^t dt' F(t,t') e^{-pt-qt'} + \int_0^\infty dt' \int_0^{t'} dt F(t,t') e^{-pt-qt'} 
= \int_0^\infty dt \int_0^t dt' F(t,t') e^{-pt-qt'} + \int_0^\infty dt \int_0^t dt' F(t',t) e^{-pt'-qt} 
= \int_0^\infty dt \int_0^t dt' F(t,t') \left( e^{-pt-qt'} + e^{-pt'-qt} \right).$$
(E.2)

Next, we change the integration variables according to x = t + t', v = t - t', such that the shaded domain of integration in figure E1 is rewritten as  $\int_0^\infty dt \int_0^t dt' = \frac{1}{2} \int_0^\infty dv \int_v^\infty dx$ . Because of the identity

$$e^{-pt-qt'} + e^{-pt'-qt} = 2 e^{-x(p+q)/2} \cosh\left(\frac{v}{2}(q-p)\right)$$
 (E.3)

the above change of variables casts the double integral (E.2) into a form where the quantum noise correlation acts as a distribution on a test function f(v), namely

$$\overline{\overline{F}}(p,q) = \frac{2\hbar}{\pi\gamma} \int_0^\infty \mathrm{d}v \, f(v) \frac{\mathsf{t}_0^2 - v^2}{\left(\mathsf{t}_0^2 + v^2\right)^2} \tag{E.4}$$

where the variables p and q are implicit in the test function f. The integrals (E.1) and (E.2) lead to the following integral representation of this test function

$$f(v) = \int_{v}^{\infty} \mathrm{d}x \int_{\boldsymbol{k},(\Lambda)} e^{-\frac{k^2}{\gamma}x} e^{-x\frac{p+q}{2}} \cosh\left(v\frac{q-p}{2}\right),\tag{E.5}$$

which is clearly invariant upon exchanging p and q. This appendix analyses general integrals of the form (E.4) in the limit  $t_0 \rightarrow 0$ . Note that setting  $t_0 = 0$  from the outset would in general lead to a divergent integral.

#### Appendix E.1. The quantum noise memory kernel as a generalised function

In classical dynamics, one may write the noise correlation as a generalized function by modeling a Markovian noise through a delta function, see Eq. (12). We are interested in interpreting the quantum noise correlation in a similar way. Consider the integral

$$\int_0^\infty \mathrm{d}x \, f(x) \frac{\mathsf{t}_0^2 - x^2}{(\mathsf{t}_0^2 + x^2)^2}.$$
 (E.6)



Figure E2: Integration contours for carrying out the inverse Mellin transform: moving the integration contour C to the left yields contributions from the poles that are inside the black contour.

For certain choices of f(x), this kind of integral can be calculated from the residue theorem. For example, with -1 < s < 1

$$\int_0^\infty \mathrm{d}x \, x^{-s} \frac{\mathsf{t}_0^2 - x^2}{\left(\mathsf{t}_0^2 + x^2\right)^2} = \frac{\pi}{2} \frac{s}{\cos(\pi s/2)} \mathsf{t}_0^{-1-s}.$$
(E.7)

In order to study systematically the dependence of the integral (E.6) on the cut-off parameter  $t_0$  we use the *Mellin transform* which is defined as [82]

$$\check{f}(s) = \mathscr{M}(f)(s) := \int_0^\infty \mathrm{d}x \, x^{s-1} f(x) \quad , \qquad f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \, x^{-s} \check{f}(s) \tag{E.8}$$

where the real constant c is chosen freely in the fundamental strip of the respective transform, as illustrated in Fig. E2 by the right integration path. This *fundamental strip* is defined through the convergence of the integral and thus is set by the asymptotic behaviour of f(x). For example, if  $f(x) \stackrel{x\to 0}{\sim} x^{-\alpha}$  and  $f(x) \stackrel{x\to\infty}{\sim} x^{-\beta}$ with  $\alpha < \beta$ , then the fundamental strip is a superset of the strip  $\{s = \sigma + i\tau \in \mathbb{C} \mid \tau \in \mathbb{R} \text{ and } \alpha < \sigma < \beta\}$ . On the fundamental strip,  $\check{f}(s)$  exists and is holomorphic [82]. A monomial  $f(x) = x^{\alpha}$  does not admit a Mellin transform.

For the moment, we do not specify c in Eq. (E.8) but we shall come back to this point, once we have correctly identified the necessary assumptions on the function f(x). With Eq. (E.7), and in the fundamental strip -1 < c < 1, the integral (E.6) is rewritten as

$$\int_0^\infty \mathrm{d}x \, f(x) \frac{\mathsf{t}_0^2 - x^2}{\left(\mathsf{t}_0^2 + x^2\right)^2} = \frac{\pi}{2} \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \, \breve{f}(s) \frac{s \, \mathsf{t}_0^{-1-s}}{\cos\left(\frac{\pi}{2}s\right)}.$$
(E.9)

The remaining analysis depends on the function f(s). From its definition (E.8) it is clear that the convergence of the integral for  $x \to \infty$  as well as for  $x \to 0$  has to be guaranteed. We assume here that the function f(x) does not cause any problem at infinity for some subset of  $c \in (-1, 1)$ . Furthermore we assume that f(x)has some formal series expansion (which does not necessarily represent an analytic function, but we assume  $0 < \alpha_0 < \alpha_1 < \ldots$ )

$$f(x) = \sum_{j} a_j x^{\alpha_j}, \quad \text{for } x \to 0.$$
(E.10)

According to the Direct Mapping Theorem [82], the exponents  $\alpha_j$  in the expansion (E.10) correspond to poles  $s_0$  of the Mellin transform, while the coefficients  $a_j$  are the residues

$$\operatorname{Res}_{s=-\alpha_j}\left(\breve{f}(s)\right) = a_j. \tag{E.11}$$

Under these conditions, the fundamental strip of the Mellin transform is the segment  $(\max\{-\alpha_0, -1\}, 1)$ . Assume that  $|\check{f}(\sigma + i\tau)| < \mathfrak{f}_0(\sigma)e^{\mathfrak{f}_1|\tau|}$  for  $\tau \to \pm \infty$  such that  $\mathfrak{f}_1 < \frac{\pi}{2}$  and  $\mathfrak{f}_0(\sigma)$  remains bounded for all  $\sigma \in [-\infty, 0]$ . Then, we can shift the integration contour in Eq. (E.9) to  $c \to -\infty$  and write the integral as a sum over all residues to the left of the initial contour (figure E2 shows the initial contour and the intermediate stage where one has already shifted  $c \mapsto c - 4$ )

$$\int_{0}^{\infty} \mathrm{d}x f(x) \frac{\mathsf{t}_{0}^{2} - x^{2}}{\left(\mathsf{t}_{0}^{2} + x^{2}\right)^{2}} = \frac{\pi}{2} \sum_{s_{0} \leq -\min(\alpha_{0}, 1)} \operatorname{Res}_{s=s_{0}} \left(\breve{f}(s) \frac{s \mathsf{t}_{0}^{-1-s}}{\cos\left(\frac{\pi}{2}s\right)}\right).$$
(E.12)

Besides the simple poles of the Mellin transform  $\check{f}(s)$ , the cosine function also generates simple poles, located at  $s_0 = -(2n+1)$  with  $n \in \mathbb{N}_0$ . Indeed, those poles  $s_0$  of the Mellin transform which do *not* occur at a negative

odd integer correspond to simple poles in Eq. (E.12). Those poles  $s_0$  which occur at a negative odd integer correspond, instead, to double poles in Eq. (E.12). Accordingly, we decompose the formal expansion (E.10) of f(x) according to

$$f(x) = \sum_{j}' a_{j} x^{\alpha_{j}} + \sum_{n=0}^{\infty} b_{2n+1} x^{2n+1},$$
(E.13)

where the exponents  $\alpha_j$  are ordered according to  $-1 < \alpha_0 < \alpha_1 < \ldots$  and the  $\alpha_j$  cannot be odd positive integers. Then the first formal series in Eq. (E.13) contains all even powers and non-analytic terms in x that generate first-order poles, while the second formal series in Eq. (E.13) contains all odd powers in x that generate second-order poles. The integral can then be written as

$$\int_{0}^{\infty} \mathrm{d}x \ f(x) \frac{\mathsf{t}_{0}^{2} - x^{2}}{(\mathsf{t}_{0}^{2} + x^{2})^{2}} = -\frac{\pi}{2} \sum_{j}' a_{j} \frac{\mathsf{t}_{0}^{-1 + \alpha_{j}} \alpha_{j}}{\cos\left(\frac{\pi}{2}\alpha_{j}\right)} + \frac{\pi}{2} \sum_{n=0}^{\infty} \operatorname{Res}_{s=-(2n+1)} \left( \breve{f}(s) \frac{s}{\cos\left(\frac{\pi}{2}s\right)} \mathsf{t}_{0}^{-1-s} \right).$$
(E.14)

It remains to determine the residues at the second-order poles. These can be found in general as follows, see, e.g., Ref. [16]. Consider two functions h(z) and g(z) such that, around  $z \approx z_0$ ,

$$h(z) = \frac{1}{z - z_0} \left[ P(z_0) + P'(z_0)(z - z_0) + \ldots \right],$$
(E.15a)

$$g(z) = (z - z_0) \left[ Q(z_0) + Q'(z_0)(z - z_0) + \ldots \right],$$
 (E.15b)

with entire functions P, Q and  $Q(z_0) \neq 0$ . The residue of the quotient function at  $z = z_0$  is thus

$$\operatorname{Res}_{z=z_0}\left[\frac{h(z)}{g(z)}\right] = \frac{P'(z_0)}{Q(z_0)} - \frac{P(z_0)Q'(z_0)}{Q(z_0)^2}.$$
(E.16)

In the case we are interested in, a second-order pole arises, from the second line in Eq. (E.14), if and only if  $\check{f}(s)$  has a first-order pole at  $s_0 := -(2n+1)$ . Thus  $(s+2n+1)\check{f}(s)$  is well-defined and analytic in  $s = s_0$ . For  $s \approx s_0$ , we then have

$$\frac{\check{f}(s)st_0^{-1-s}}{\cos\left(\frac{\pi}{2}s\right)} = \frac{1}{s-s_0} \frac{(s-s_0)\check{f}(s)st_0^{-1-s}}{\cos\left(\frac{\pi}{2}s\right)} \\
\simeq \frac{2}{\pi} \frac{(-1)^n t_0^{-1-s_0}}{(s-s_0)^2} \left( \left[ b_{2n+1} + B_n(s-s_0) \right] \left[ 1 - \ln\left(t_0\right)(s-s_0) \right] \left[ s_0 + (s-s_0) \right] \right) \\
\simeq \frac{2}{\pi} \frac{(-1)^n t_0^{-1-s_0}}{(s-s_0)^2} \left[ b_{2n+1}s_0 + \left( b_{2n+1} + B_ns_0 - b_{2n+1}s_0 \ln t_0 \right)(s-s_0) + O((s-s_0)^2) \right], \quad (E.17)$$

with the constants

$$b_{2n+1} = \lim_{s \to s_0} \left[ (s - s_0) \breve{f}(s) \right], \qquad B_n = \lim_{s \to s_0} \frac{\mathrm{d}}{\mathrm{d}s} \left[ (s - s_0) \breve{f}(s) \right].$$
(E.18)

The residue is read off from the pre-factor of the linear term in  $s - s_0$ , inside the brackets.

Collecting all results, the integral in Eq. (E.14), already using the formal expansion in Eq. (E.13), can now be evaluated and gives

$$\int_{0}^{\infty} \mathrm{d}x \, f(x) \frac{\mathsf{t}_{0}^{2} - x^{2}}{\left(\mathsf{t}_{0}^{2} + x^{2}\right)^{2}} = -\frac{\pi}{2\mathsf{t}_{0}} \sum_{j}' \frac{\mathsf{t}_{0}^{\alpha_{j}} a_{j} \alpha_{j}}{\cos\left(\frac{\pi}{2}\alpha_{j}\right)} + \sum_{n=0}^{\infty} \left(-1\right)^{n} \mathsf{t}_{0}^{2n} \left[ \left(1 + (2n+1)\ln\mathsf{t}_{0}\right) b_{2n+1} - (2n+1)B_{n} \right]$$
(E.19)

This equation is the central result of our approach, as it allows one to understand the behaviour as  $t_0 \rightarrow 0^+$ .

We now investigate some specific examples, which we also checked numerically. First, we study the exponential function  $f(x) = \exp(-\nu x)$ , with  $\nu > 0$ . Its Mellin transform is  $\check{f}(s) = \mathscr{M}(e^{-\nu x})(s) = \nu^{-s}\Gamma(s)$ , involving the Gamma function  $\Gamma(s)$  [67]. The power series  $e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$  corresponds to the 'singular expansion'  $\Gamma(s) \approx \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$  [82]. The decomposition according to Eq. (E.13) is achieved by writing  $e^{-\nu x} = \cosh \nu x - \sinh \nu x$ . Now, both series in Eq. (E.19) can be evaluated exactly in terms of sine and cosine integrals [67] which are themselves best written with the auxiliary function  $g_{AS}(x)$  defined in Eq. (36)<sup>1</sup>

$$\int_{0}^{\infty} \mathrm{d}x \, e^{-\nu x} \frac{\mathsf{t}_{0}^{2} - x^{2}}{\left(\mathsf{t}_{0}^{2} + x^{2}\right)^{2}} = -\nu \left[\cos(\nu \mathsf{t}_{0}) \operatorname{Ci}(\nu \mathsf{t}_{0}) + \sin(\nu \mathsf{t}_{0}) \operatorname{si}(\nu \mathsf{t}_{0})\right] = \nu g_{\mathrm{AS}}(\nu \mathsf{t}_{0}). \tag{E.20}$$

<sup>1</sup>The  $B_n$  were evaluated using Eq. (06.05.056.0009.01) of Ref. [83].

A straightforward generalisation for any  $n \in \mathbb{N}_0$  is

$$\int_0^\infty \mathrm{d}x \, x^n e^{-\nu x} \frac{\mathsf{t}_0^2 - x^2}{\left(\mathsf{t}_0^2 + x^2\right)^2} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\nu^n} \left(\nu g_{\mathrm{AS}}(\nu \mathsf{t}_0)\right) \tag{E.21}$$

Equations (E.20) and (E.21) are also used in other appendices and in the main text.

Since the exponential function is analytic everywhere, it is worthwhile to benchmark our method as well with a function whose Taylor series has a finite radius of convergence. We choose the function  $f(x) = (1+\sqrt{x})^{-1}$ . Evaluating the series in Eq. (E.19) we find

$$\int_{0}^{\infty} \mathrm{d}x \, \frac{1}{1+\sqrt{x}} \frac{\mathsf{t}_{0}^{2}-x^{2}}{\left(\mathsf{t}_{0}^{2}+x^{2}\right)^{2}} = \frac{\pi}{2\mathsf{t}_{0}} \frac{4\mathsf{t}_{0}^{2}+\sqrt{\mathsf{t}_{0}}\sqrt{2}(\mathsf{t}_{0}+1)\left(\mathsf{t}_{0}^{2}-4\mathsf{t}_{0}+1\right)}{2\left(\mathsf{t}_{0}^{2}+1\right)^{2}} + \frac{1+\mathsf{t}_{0}^{2}+\left(1-\mathsf{t}_{0}^{2}\right)\ln\mathsf{t}_{0}}{\left(\mathsf{t}_{0}^{2}+1\right)^{2}} \tag{E.22}$$

#### Appendix E.2. Asymptotic expansion

The identity (E.20), involving the exponential function, can now be used in order to evaluate the asymptotic behaviour of the quantum noise correlation function in Eq. (E.1). We see that the integrand is invariant under the exchange  $t_1 \leftrightarrow t_2$  apart from the p and q exponential contributions. For a general symmetric function  $F(t_1, t_2) = F(t_2, t_1)$  we may write

$$\int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \, F(t_1, t_2) e^{-pt_1} e^{-qt_2} = \int_0^\infty \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, F(t_1, t_2) \left( e^{-pt_1} e^{-qt_2} + e^{-pt_2} e^{-qt_1} \right). \tag{E.23}$$

Having explicitly symmetrised the above integral, we can now introduce the diagonal coordinates, as in Fig. E1. Using diagonal coordinates has the advantage that we can now isolate the distribution and re-use the formulæ derived in the above examples, in particular in Eq. (E.20), to obtain

$$\overline{\overline{F}}(p,q) = \frac{2\hbar}{\pi\gamma} \int_{0}^{\Lambda} dk \, k^{d-1} \int_{0}^{\infty} dt_{1} \int_{0}^{t_{1}} dt_{2} \, e^{-\frac{k^{2}}{\gamma}(t_{1}+t_{2})} \frac{t_{0}^{2} - (t_{1}-t_{2})^{2}}{(t_{0}^{2}+(t_{1}-t_{2})^{2})^{2}} \left[ e^{-pt_{1}}e^{-qt_{2}} + e^{-pt_{2}}e^{-qt_{1}} \right] \\
= \frac{2\hbar}{\pi\gamma} \int_{0}^{\Lambda} dk \, k^{d-1} \int_{0}^{\infty} dv \int_{v}^{\infty} du \, e^{-u\frac{k^{2}}{\gamma}} \frac{t_{0}^{2} - v^{2}}{(t_{0}^{2}+v^{2})^{2}} \left( e^{-\frac{p}{2}(u+v) - \frac{q}{2}(u-v)} + e^{-\frac{p}{2}(u-v) - \frac{q}{2}(u+v)} \right) \\
= \frac{2\hbar}{\pi\gamma} \int_{\mathbf{k},(\Lambda)} \int_{0}^{\infty} dv \int_{v}^{\infty} du \, e^{-u\left(\frac{k^{2}}{\gamma} + \frac{p+q}{2}\right)} \frac{t_{0}^{2} - v^{2}}{(t_{0}^{2}+v^{2})^{2}} \left( e^{\frac{q-p}{2}v} + e^{-\frac{q-p}{2}v} \right) \\
= \frac{2\hbar}{\pi\gamma} \int_{\mathbf{k},(\Lambda)} \frac{1}{k^{2}/\gamma + (p+q)/2} \int_{0}^{\infty} dv \, \frac{t_{0}^{2} - v^{2}}{(t_{0}^{2}+v^{2})^{2}} \left( e^{-v\left(\frac{k^{2}}{\gamma} + p\right)} + e^{-v\left(\frac{k^{2}}{\gamma} + q\right)} \right) \\
= \frac{2\hbar}{\pi\gamma} \int_{\mathbf{k},(\Lambda)} \frac{(k^{2}/\gamma + p) \, g_{\mathrm{AS}}\left(t_{0}\left(k^{2}/\gamma + p\right)\right) + (k^{2}/\gamma + q) \, g_{\mathrm{AS}}\left(t_{0}\left(k^{2}/\gamma + q\right)\right)}{k^{2}/\gamma + (p+q)/2}.$$
(E.24)

We want to extract the leading scaling behaviour of this integral representation, especially for  $t_0 \ll 1$  and for p and q small.<sup>2</sup> Accordingly, we replace the auxiliary function  $g_{AS}$  by its small-argument asymptotics [67]

$$g_{\rm AS}(x) \simeq -(\ln x + C_E) + \frac{\pi}{2}x + O(x^2),$$
 (E.25)

which allows us to identify the leading behaviour of the quantum noise function, up to the order needed in the main text. In general, it turns out  $\overline{\overline{F}}(p,q)$  has a non-universal regular part and an universal irregular part

$$\overline{\overline{F}}(p,q) = \overline{\overline{F}}_{reg}(p,q) + \overline{\overline{F}}_{irr}(p,q).$$
(E.26)

To linear order in p and q, the regular part may be obtained by inserting the expansion (E.25) into Eq. (E.24), with the result

$$\overline{\overline{F}}_{\rm reg}(p,q) \simeq -\frac{4\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^d}{d} \left\{ \ln\left(\frac{t_0\Lambda^2}{\gamma}\right) + C_E - \frac{2}{d} - \frac{\pi}{2} \frac{d}{d+2} \frac{t_0\Lambda^2}{\gamma} + \left[ -\frac{\pi}{4} + \frac{1}{2} \frac{d}{d-2} \frac{\gamma}{t_0\Lambda^2} \Theta(d-2) \right] t_0(p+q) + O(t_0^2) + O(p,q) \right\}.$$
(E.27)

We point out that the term of order zero in p and q exists for all d > 0 and that certain contributions to the first-order term only exist for d > 2. This is expressed above by the Heaviside function  $\Theta$ . Higher-order terms

<sup>2</sup>From Appendix B, we have  $t_0 \sim \gamma^{-1}$ , and the equation of motion (10) is in the over-damped limit  $\gamma$  large.

arise for larger dimensions. In the main text, Eq. (72a) neglects the terms of order  $O(t_0)$ . The presence of the cutoff parameters  $t_0$  and  $\Lambda$  signals that  $\overline{\overline{F}}_{reg}(p,q)$  depends on the details of the cutoff procedures, both temporal and in momentum space (indeed, they only arise through the scaling variable  $t_0 \Lambda^2/\gamma$ ), and they are therefore non-universal.

The irregular part is obtained by subtracting  $\overline{F}(0,0)$  from Eq. (E.24), rescaling the integral according to  $x = k/\sqrt{\gamma p}$  and finally taking the limit  $p \to 0$ . To lowest order, this procedure yields

$$\overline{\overline{F}}_{\rm irr}(p,q) \simeq -\frac{4\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} (\gamma p)^{\frac{d}{2}} \mathbb{F}(q/p), \quad \text{as } p,q \to 0, \quad \text{with} \quad \mathbb{F}(1) = \frac{\pi}{d} \frac{1}{\sin\left(\frac{\pi d}{2}\right)} \ . \tag{E.28}$$

Here, we introduced the scaling function

$$\mathbb{F}(z) = \int_0^\infty \mathrm{d}x \, x^{d-1} \frac{\left(x^2 + z\right) \ln\left(1 + z/x^2\right) + \left(x^2 + 1\right) \ln\left(1 + 1/x^2\right)}{\left(x^2 + z\right) + \left(x^2 + 1\right)} \,. \tag{E.29}$$

Remarkably, this can be evaluated explicitly, in terms of hypergeometric and incomplete Beta functions [67], i.e.,

$$\mathbb{F}(z) = \frac{\pi \csc\left(\frac{\pi d}{2}\right)}{4} \left\{ \frac{2}{d} \left[ {}_{2}F_{1}\left(1, -\frac{d}{2}; 1-\frac{d}{2}; \frac{z+1}{2}\right) + \frac{4z^{\frac{d}{2}+1}}{(d+2)(z+1)} \right] - \frac{2}{d-2} {}_{2}F_{1}\left(1, 1-\frac{d}{2}; 2-\frac{d}{2}; \frac{z+1}{2}\right) + 2^{-\frac{d}{2}}(z+1)^{\frac{d}{2}-1} \left(\pi(1-z)\cot\left(\frac{\pi d}{2}\right) + 2zB_{\frac{2z}{z+1}}\left(\frac{d}{2}+1, 0\right) - (z+1)B_{\frac{2z}{z+1}}\left(\frac{d}{2}+2, 0\right) \right) \right\}.$$
(E.30)

This is Eq. (72b) in the main text. Since this scaling function does not contain the parameters  $t_0$  and  $\Lambda$  of the regularisations, it is universal. In addition, the damping parameter  $\gamma$  only enters via the scaling variable  $p\gamma$  and as a trivial scale factor. On the other hand, the form of  $\overline{\overline{F}}_{irr}(p,q)$  should depend on having assumed Ohmic damping.

In the special case  $d = d_u = 2$ , logarithmic corrections to scaling are expected to be present, in analogy to classical dynamics. We do not present a detailed analysis of this case here, but it can be done as outlined above

#### Appendix F. Homogeneity and double Laplace transforms

Well-known Tauberian theorems for the Laplace transform, which go back to Hardy and Littlewood, and Karamata, and Feller, relate the asymptotics of a function f(x) for  $x \to \infty$  with the behaviour of its Laplace transform  $\overline{f}(p)$  as  $p \to 0$ , see ch. XIII.5 in [71]. The non-local structure of the quantum noise correlations requires us to find an extension of these results for functions f(x, y) of two variables and their double Laplace transform, see Eq. (52) and Appendix D. In what follows, f is assumed to be such that  $\overline{\overline{f}}$  exists, see Refs. [68, 69] for sufficient conditions. We are interested in how the asymptotics of f(x, y) for x and y both large is related to the properties of  $\overline{\overline{f}}(p,q)$ . From Ref. [71], the scaling limit  $x, y \to \infty$  with fixed x/y > 1 corresponds to the limit  $p, q \to 0$  with fixed q/p. We are mainly interested in homogeneous functions and look for an explicit transformation formula for the scaling functions, in order to relate the respective asymptotics.

**Lemma 1:** The double Laplace transform of a homogeneous function  $f(x,y) = y^{-\alpha}\phi(x/y)$  where  $\alpha < 2$  and  $\phi(0)$  is a finite constant and where  $\phi(u) \simeq \phi_{\infty} u^{-\lambda}$ , asymptotically for  $u \to \infty$ , also admits a scaling form

$$\overline{f}(p,q) = p^{\alpha-2} \Phi(q/p), \tag{F.1}$$

with the scaling function

$$\Phi(u) = \Gamma(2-\alpha) u^{\alpha-1} \int_0^\infty d\xi \, \phi(\xi u) \, (\xi+1)^{\alpha-2}.$$
 (F.2)

In particular, for  $0 < \lambda < 1$  and  $\alpha < 1 + \lambda$ , one has asymptotically for  $u \to \infty$ 

$$\Phi(u) \simeq \Phi_{\infty} u^{\alpha - 1 - \lambda} \quad \text{with} \quad \Phi_{\infty} = \phi_{\infty} \Gamma(1 - \lambda) \Gamma(1 + \lambda - \alpha).$$
(F.3a)

For  $1 < \lambda < 2$ , one has asymptotically for  $u \to \infty$ 

$$\Phi(u) \simeq \phi^{(1)} u^{\alpha - 2} + \Phi_{\infty} u^{\alpha - 1 - \lambda} \quad \text{with} \quad \phi^{(n)} = (-1)^{n - 1} \frac{\Gamma(n + 1 - \alpha)}{(n - 1)!} \int_0^\infty \mathrm{d}u \, u^{n - 1} \phi(u). \tag{F.3b}$$

More generally, for  $n < \lambda < n+1$  with  $n \in \mathbb{N}$ , one has asymptotically

$$\Phi(u) \simeq \phi^{(1)} u^{\alpha-2} + \ldots + \phi^{(n)} u^{\alpha-1-n} + \Phi_{\infty} u^{\alpha-1-\lambda}$$
(F.3c)

**Proof:** The scaling assumption on f(x, y) is equivalent to requiring homogeneity

$$f(\ell x, \ell y) = \ell^{-\alpha} f(x, y),$$

with the index  $\alpha$  and for all positive  $\ell \in \mathbb{R}_+$ . It follows that  $\overline{\overline{f}}(p,q)$  is homogeneous with index  $2 - \alpha$ , i.e.,

$$\overline{\overline{f}}(\ell p, \ell q) = \ell^{-(2-\alpha)} \overline{\overline{f}}(p, q).$$
(F.4)

Choosing  $\ell = 1/p$  in Eq. (F.4) gives the scaling form of the double Laplace transform

$$\overline{\overline{f}}(p,q) = p^{-2+\alpha}\overline{\overline{f}}(1,q/p) = p^{\alpha-2}\Phi(q/p),$$

with the scaling function

$$\begin{split} \Phi(u) &= p^{2-\alpha} \overline{\overline{f}}(p, pu) = u^{\alpha-1} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, y^{-\alpha} \phi\left(\frac{x}{y}u\right) e^{-x-y} \\ &= u^{\alpha-1} \int_0^\infty \mathrm{d}\xi \, \phi(\xi u) \int_0^\infty \mathrm{d}\eta \, \eta^{1-\alpha} e^{-(\xi+1)\eta} \, = \, \Gamma(2-\alpha) \, u^{\alpha-1} \int_0^\infty \mathrm{d}\xi \, \phi(\xi u) \, (\xi+1)^{\alpha-2}, \end{split}$$

as anticipated in Eq. (F.2). We now derive the large-u behaviour of  $\Phi(u)$ . We begin with a heuristic discussion. In general, one expects a decomposition into a regular part and an irregular part

$$\Phi(u) = \Gamma(2-\alpha)u^{\alpha-2} \left[ \Phi_{\rm reg}(u) + \Phi_{\rm irr}(u) \right]$$
$$= \Gamma(2-\alpha)u^{\alpha-2} \left[ \int_0^{\eta} \mathrm{d}\xi \,\phi(\xi) \left(1 + \frac{\xi}{u}\right)^{\alpha-2} + \int_{\eta}^{\infty} \mathrm{d}\xi \,\phi(\xi) \left(1 + \frac{\xi}{u}\right)^{\alpha-2} \right],$$

with a cut  $\eta$ . Expanding formally the regular part in u leads to

$$\Phi_{\rm reg}(u) = \sum_{n \ge 0} \begin{pmatrix} \alpha - 2 \\ n \end{pmatrix} \int_0^{\eta} \mathrm{d}\xi \,\phi(\xi) \left(\frac{\xi}{u}\right)^n$$

and taking the limit  $\eta \to \infty$ , one only keeps those terms where the corresponding moment  $\phi^{(n)}$  exists, which depends on the value of  $\lambda$ . These are the regular terms in (F.3c). The irregular term is estimated as follows, where for sufficiently large  $\eta$  the asymptotic form of  $\phi(u)$  is used

$$\Phi_{\rm irr}(u) = u \int_{\eta/u}^{\infty} \mathrm{d}\xi \,\phi(\xi u) \,(1+\xi)^{\alpha-2} \simeq u^{1-\lambda} \phi_{\infty} \int_{\eta/u}^{\infty} \mathrm{d}\xi \,\xi^{-\lambda} \,(1+\xi)^{\alpha-2} \stackrel{u \to \infty}{=} u^{1-\lambda} \phi_{\infty} \frac{\Gamma(\lambda+1-\alpha)\Gamma(1-\lambda)}{\Gamma(2-\alpha)}$$

where, in the second step, we consider first the asymptotic limit  $u \to \infty$  and then express the integral via a Beta function [67]. The final result is then independent of the cut  $\eta$  and corresponds to the second part of Eq. (F.3c).

Not all terms in this formal expansion really occur. For example, for  $0 < \lambda < 1$  and also with  $\alpha < 1 + \lambda$ , we consider the regular part as taken from (F.2). The asymptotic approximation  $\phi(u) \sim u^{-\lambda}$  should work as long as  $\xi \gtrsim 1/u$  is sufficiently large. If on the other hand,  $\xi \leq 1/u$  and if  $\phi(0)$  is a finite constant, that part of the integral contributes a term of order  $O(\phi(0)/u)$ , compared to the contribution  $O(u^{-\lambda})$  from the main term. Accordingly, the small- $\xi$  contribution, for  $\lambda < 1$ , will be a sub-dominant correction, see Eq. (F.3a).

We now turn to a more systematic method which does not rely on heuristics. It is convenient to re-write the scaling function as

$$\Phi(u) = \Gamma(2-\alpha) \int_0^\infty \mathrm{d}z \,\phi\left(1/z\right) z^{-2} \left(u + \frac{1}{z}\right)^{\alpha-2}.$$

The required asymptotics for large u, we are interested in, is obtained by first renaming  $\phi(1/z) = f(z)$  and second expressing f(z) through its Mellin transform, see Eq. (E.8) in Appendix E. Exchanging the order of integrations, we first calculate the z-integration and find

$$\Phi(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}s \,\breve{f}(s) u^{s+\alpha-1} \Gamma(s+1) \Gamma(1-s-\alpha) = \sum_{s_0} \operatorname{Res}_{s=s_0} \left[ \breve{f}(s) u^{s+\alpha-1} \Gamma(s+1) \Gamma(1-s-\alpha) \right],$$

with  $c \in (0, 1 - \alpha)$ . As explained in Appendix E, we then shift the contour of integration towards having  $c \to -\infty$  and express the integral as a sum over the set of enclosed poles  $s_0$ . Summing all relevant residues yields an ordered series in u, beginning with the most relevant contributions as  $u \gg 1$ . The integrand has three potentially singular contributions, i.e., for  $s_0 \in \{-1 - n \mid n \in \mathbb{N}_0\}$ , the poles of the Mellin transform itself and for  $s_0 \in \{1 - \alpha + n \mid n \in \mathbb{N}_0\}$ . The last ones do not contribute to the asymptotic behaviour since they are located to the right of the original integration domain. We thus need to identify the poles of the Mellin transform. This is done by specifying the asymptotic behaviour of the function  $\phi(u)$ , e.g.,

$$\phi(u) \simeq u^{-\lambda} \left( A_0 + \frac{A_1}{u} + \frac{A_2}{u^2} + \dots \right) + B_0 + \frac{B_1}{u} + \frac{B_2}{u^2} + \dots \quad \text{for} \quad u \to \infty$$

which translates into

$$f(z) \simeq z^{\lambda} (A_0 + A_1 z + A_2 z^2 + \ldots) + B_0 + B_1 z + B_2 z^2 + \ldots$$
 for  $z \to 0$ .

We also use the decomposition  $f(z) = f_A(z) + f_B(z)$  if we want to consider these two series separately. The poles of the Mellin transform are located at  $s_0 \in \{-\lambda - n \mid n \in \mathbb{N}_0\} \cup \{-n \mid n \in \mathbb{N}_0\}$  [82]. We assume  $\lambda \notin \mathbb{N}$  such that the first series has only simple poles. Evaluation of the residues leads to the following asymptotic series for the scaling function

$$\Phi(u) \simeq \sum_{n=0}^{\infty} \left\{ A_n \Gamma(1+\lambda-\alpha+n) \Gamma(1-\lambda-n) u^{-\lambda+\alpha-n-1} + \frac{\Gamma(2-\alpha)}{1-\alpha} B_0 - \sum_{n=1}^{\infty} B_n (-1)^n n \frac{\Gamma(1+n-\alpha)}{\Gamma(n+1)} \psi(n) u^{\alpha-n-1} \right\}$$

$$+ \sum_{m \ge 1} \check{f}_A(-m) (-1)^{m-1} \frac{\Gamma(m+1-\alpha)}{(m-1)!} u^{\alpha-m-1},$$
(F.5)

where  $\psi(n)$  is the digamma function [67] and the terms in the last line have to be included as long as  $f_A(-m)$  exists.

In the special case, in which  $A_0 = \phi_{\infty}$  and  $A_n = B_n = B_0 = 0$  for all  $n \ge 1$ , we recover Eq. (F.3a) for  $0 < \lambda < 1$ . For  $1 < \lambda < 2$ , we formally have  $\check{f}_A(-1) = \int_0^\infty dz \, z^{-2} f(z) = \int_0^\infty du \, \phi(u)$  and we obtain what we anticipated in Eq. (F.3b). Similarly, for different ranges of  $\lambda$ , the terms contained in Eq. (F.3c) are read off. This completes the proof.

It follows that a derived asymptotic behaviour  $\Phi(u) \sim u^{-\vartheta}$  must be interpreted carefully in order to identify the exponent  $\lambda$  in  $\phi(u) \sim u^{-\lambda}$  correctly. If effectively  $\vartheta > n - \alpha$  is found, the expansion must be carried up to terms  $O(u^{\alpha - n - 1})$ .

**Corollary:** Consider a function f(x, y) of two variables and such that its double Laplace transform  $\overline{\overline{f}}(p,q)$  exists. Assume that  $f(x, y) = y^{-\alpha} \left( \ln \frac{1}{y} \right)^{-\beta} \phi(x/y)$  admits a logarithmic scaling form, with  $\alpha < 2$  and  $\phi(0)$  being a finite constant. Then the double Laplace transform admits the scaling form

$$\overline{\overline{f}}(p,q) = p^{\alpha-2} \left( \ln p \right)^{-\beta} \Phi(q/p), \quad \text{with} \quad \Phi(u) = \Gamma(2-\alpha) u^{\alpha-1} \int_0^\infty \mathrm{d}\xi \, \phi(\xi u) \, (\xi+1)^{\alpha-2}. \tag{F.6}$$

A different kind of scaling arises if there is a further auxiliary variable, labeled k here. We can formulate the following elementary result.

**Lemma 2:** Consider a function f(x, y; k) of two variables x, y and such that its double Laplace transform  $\overline{\overline{f}}(p,q;k)$  with respect to these variables exists. Assume that f admits the scaling form  $f(x,y;k) = k^{\alpha z} \phi(k^z x, k^z y)$ . Then the double Laplace transforms admits the scaling form

$$\overline{\overline{f}}(p,q;k) = k^{(\alpha-2)z} \Phi(pk^{-z},qk^{-z}) \quad \text{with} \quad \Phi(u,v) = \overline{\overline{\phi}}(u,v).$$
(F.7)

**Proof:** The scaling assumption on f is equivalent to the homogeneity property

$$f(\ell x, \ell y; \ell^{-1/z}k) = \ell^{-\alpha} f(x, y; k).$$

Laplace-transforming this with respect to x and y leads to the transformed homogeneity property

$$\overline{\overline{f}}\left(\frac{p}{\ell},\frac{q}{\ell};\ell^{-1/z}k\right) = \ell^{2-\alpha}\overline{\overline{f}}(p,q;k)$$

and setting  $\ell = k^z$  gives the scaling form in Eq. (F.7). The scaling functions are identified as  $\phi(x, y) := f(x, y; 1)$ and  $\Phi(p, q) := \overline{\overline{f}}(p, q; 1)$ . The relationship between these scaling functions, as stated in Eq. (F.7), readily follows from the definitions.

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