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Original citation & hyperlink:
Yang, W-C & Sadeghimanesh, A 2022, 'Matrix decomposition by transforming the unit sphere to an Ellipsoid through Dilation, Rotation and Shearing', *The Electronic Journal of Mathematics and Technology*, vol. 16, no. 3, pp. 176-

ISSN 1933-2823

Publisher: Mathematics and Technology, LLC

Matrix decomposition by transforming the unit sphere to an Ellipsoid through Dilation, Rotation and Shearing

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Abstract

There are various decompositions of matrices in the literature such as lower-upper, singular value and polar decompositions to name a few. In this paper we are concerned with a less standard matrix decomposition for invertible matrices of order 3 with real entries, called TRD decomposition. In this decomposition an invertible matrix is written as product of three matrices corresponding to a shear, a rotation and a dilation map that transform the unit sphere to an ellipsoid. The reason of our interest is the geometric visualization of this decomposition. We also implemented an algorithm to compute this decomposition both in Maple and Matlab.

1 Introduction

There are various matrix decompositions that each of them are designed for a specific computational goal. Probably the most known ones are lower-upper (LU) decomposition [12], which is suitable for solving a system of linear equations, and Polar and Singular Value decompositions [1, 4, 11, 13] which are useful in finding the best rank-k approximation or in Quantum information theory. Here we are interested in a less standard matrix decomposition called TRD decomposition which might not seem to have a specific advantage for a computational problem, but instead has an interesting geometric interpratation. This matrix decomposition is introduced in a blog note by Danny Calegari [6]. Let M be a three times three invertible matrix with real entries. The matrix M can be written as product of three matrices T, R and D, $M = TRD$, where D is a diagonalizable matrix with two equal eigenvalues, R is an orthogonal matrix and finally T is a shear matrix. The product TRD is corresponding to a series of linear transformations that send the unit sphere to the same ellipsoid that M does. The goal of this paper is to provide an algorithm to compute this decomposition.

The structure of this paper is as the following. Section 2 contains some elementary definitions from linear algebra. Section 3 contains a complete discussion on Ellipsoids and their properties needed for presenting the TRD decomposition in Section 4. The main algorithm is given in Section 4. Finally we close the paper with some remarks in Section 5.

1.1 Notations

By a vector $v \in \mathbb{R}^n$ we mean a column vector, i.e. an $n \times 1$ matrix. The ith entry of the vector v is denoted by v_i. Transpose of a matrix M is denoted by M^t. A row vector is represented as transpose of a column vector, i.e. v^t. If M is an $m \times n$ matrix, then the linear map from

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$\mathbb{R}^n$ to $\mathbb{R}^m$, sending a vector v to $M \cdot v$, is also denoted by M. Let A be a subset of $\mathbb{R}^n$ and M an $m \times n$ matrix, The image of A under the linear map M is defined as $\{M(v) \mid v \in A\}$ and denoted by $M(A)$. By $\mathrm{GL_n}(\mathbb{R})$ we mean the general linear group of order n over $\mathbb{R}$ which is the set of invertible $n \times n$ matrices with real entries.

Let x_1, x_2, ..., x_n represent the n coordinate variables in $\mathbb{R}^n$, then we define the vector X to be the column vector $(x_1, x_2, \ldots, x_n)$. In $\mathbb{R}^3$, instead of x_1, x_2 and x_3 we use x, y and z respectively. By $\mathbb{R}[X]$ we mean the set of polynomials in n variables x_is and coefficients from $\mathbb{R}$. For a set $F \subseteq \mathbb{R}[X]$, we denote the set of common solutions of the polynomials in F as a subset of $\mathbb{R}^n$ by $V(F)$. When the set F contains only one polynomial, say f, We simply write $V(f)$ instead of $V(\{f\})$.

We denote the surface of the unit sphere in $\mathbb{R}^n$ by $\mathcal{S}_{n-1}$. That is $\mathcal{S}_{n-1} = V(\sum_{i=1}^{n} x_i^2 - 1)$. In this work all geometric objects are considered centered at origin.

2 Preliminaries

First we recall definition of several important class of matrices.

Definition 1. *A square matrix of order n with real entries, U, is called an orthogonal matrix if $UU^t = U^tU = I_n$ where I_n is the identity matrix of order n.*

An orthogonal matrix is an isometry and geometrically it is corresponding to a rotation, or a reflection or a combination of these two. Therefore it is also called a rotation matrix. A matrix is orthogonal if and only if it sends an orthonormal basis of $\mathbb{R}^n$ to another orthonormal basis [1, Result 7.42].

Definition 2. *A dilation matrix is a diagonalizable matrix with positive eigenvalues.*

The geometric effect of a dilation matrix is scaling a geometric object in the direction of the eigenvectors of this matrix with the scaling factor of the corresponding eigenvalues.

Definition 3. *Let W be a linear subspace of $\mathbb{R}^n$ of dimension m where $1 \leq m \leq n-1$. Pick up a basis for W, say $\{v^1, \ldots, v^m\}$. Extend this basis to a basis for $\mathbb{R}^n$, denote this basis by $B = \{v^1, \ldots, v^m, v^{m+1}, \ldots, v^n\}$. A shear matrix keeping W fixed, or also said parallel to W, is a matrix that its representation in B can be written in the following block form.*

$$\begin{bmatrix} I_m & M \\ 0 & I_{n-m} \end{bmatrix},$$

where M is an $m \times (n-m)$-matrix.

Lemma 4. *An inverse of a shear matrix, is a shear matrix keeping the same subspace fixed.*

Proof. Note that a block matrix of the form $\begin{bmatrix} I_m & M \\ 0 & I_{n-m} \end{bmatrix}$ is invertible and its inverse is $\begin{bmatrix} I_m & -M \\ 0 & I_{n-m} \end{bmatrix}$. □

Remember that a real symmetric matrix has real eigenvalues, and even more, it is orthogonally diagonalizable. That is, if M is a square matrix of order n such that $M^t = M$, then there exist an orthogonal matrix U and a diagonal matrix S with eigenvalues of M on its diagonal such that $M = USU^t$. In this paper by positive definite matrix we mean real symmetric matrices with only positive eigenvalues.

3 Ellipsoids

3.1 Definition of an ellipsoid

An ellipsoid is usually defined in one of the following two ways [5, Section 2.2.2].

Definition 5. *Let $M \in \mathrm{GL_n}(\mathbb{R})$, image of $\mathcal{S}_{n-1}$ under M is called a non-degenerate ellipsoid and we denote it by E_M.*

In this paper, by default, when we say an ellipsoid, we mean a non-degenerate ellipsoid.

Definition 6. *Let P be a positive definite matrix of order n. Define f_P to be the following polynomial in $\mathbb{R}[X]$.*

$$X^t P^{-1} X - 1. \tag{1}$$

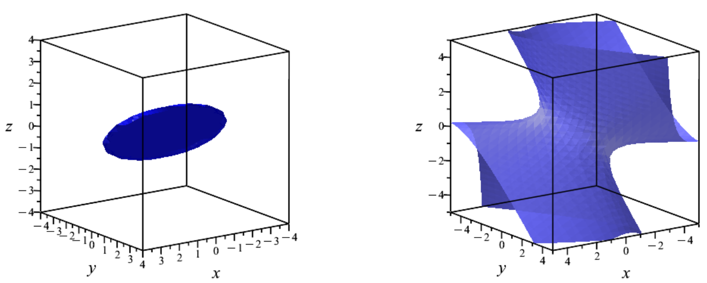
The set $V(f_P)$ is an ellipsoid. We denote this ellipsoid by $\mathcal{E}_P$.

The reader should be careful to not confuse these two definitions, as they do not define the same ellipsoids. More importantly the second definition does not consider every invertible matrix, the matrix used in Definition 6 needs to be positive definite.

Example 7. *Consider the following matrix.*

$$M_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}. \tag{2}$$

The image of the unit sphere under M_1 which is E_{M_1} defined in Definition 5 is depicted in Figure 1a and is indeed an ellipsoid. However, if one forget about the conditions in Definition 6 on the matrix and attempt to plot $V(f_{M_1})$ where f_{M_1} is given as in equation 1, then they will get the geometric shape in Figure 1b which of course is not an ellipsoid. Note that the matrix M_1 here, is not symmetric and thus not positive definite.



(a) (b)

Figure 1: One should not confuse the two definitions of ellipsoids given by a matrix. The ellipsoid in Definition 5, E_M, is defined for any invertible matrix, whereas the ellipsoid in Definition 6, $\mathcal{E}_P$ is defined for positive definite matrices.
(a) The ellipsoid E_{M_1} for the matrix M_1 given in equation 2.
(b) When one ignores the condition on the matrix in Definition 6 and tries to plot $\mathcal{E}_{M_1}$ for M_1 in equation 2, they get a non-ellipsoid surface.

Definition 6 explicitly introduces an equation for the ellipsoid $\mathcal{E}_P$, but what about the defining equation of the ellipsoid E_M of Definition 5?

Lemma 8. *Let $X = (x_1, \ldots, x_n)$ and $F \subseteq \mathbb{R}[X]$. For any $M \in \mathrm{GL_n}(\mathbb{R})$, the image of $V(F)$ under M is defined by the same set of polynomials after substituting $x_i = (M^{-1}X)_i$ for $i = 1, \ldots, n$ which we denote it by $F\mid_{X=M^{-1}X}$. In other words,*

$$M\big(V(F)\big) = V\big(F\mid_{X=M^{-1}X}\big). \tag{3}$$

Before proving this simple lemma, let us use it to answer the question of "how to find the defining equation of E_M of Definition 5".

Example 9. *In Example 7 we saw that Definition 6 can not assign an ellipsoid to an invertible matrix that is not positive definite. That of course shows that the two definitions are not the same, but this alone does not say anything about the case where both definitions are applicable. In another word, do these two definitions assign the same ellipsoid to a positive definite matrix? Consider the following matrix.*

$$M_2 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{4}$$

First we write the defining polynomial of $\mathcal{E}_{M_2}$.

$$\begin{aligned} f_{M_2} &= X^t M_2^{-1} X - 1 \\ &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 1 \\ &= \frac{2}{5}x^2 + \frac{2}{5}xy + \frac{3}{5}y^2 + \frac{1}{2}z^2 - 1. \end{aligned}$$

This ellipsoid, $\mathcal{E}_{M_2} = V(f_{M_2})$, is depicted in Figure 2a.

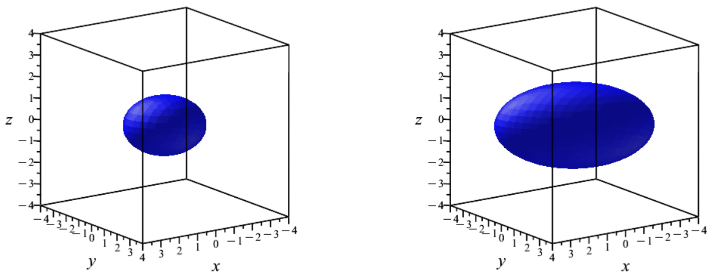
Now we use Lemma 8 to write the defining polynomial of E_{M_2}. Note that the substitution rule given by $X = M_2^{-1}X$ means that every instance of x in a formula should be replaced by $\frac{2}{5}x + \frac{1}{5}y$ and similarly y and z being replaced by $\frac{1}{5}x + \frac{3}{5}y$ and $\frac{1}{2}z$ respectively. Noting that the unit sphere can be written as $V(x^2 + y^2 + z^2 - 1)$, we have the following.

$$\begin{aligned} E_{M_2} &= M_2(\mathcal{S}_2) \\ &= M_2\big(V(x^2 + y^2 + z^2 - 1)\big) \\ &= V\big((x^2 + y^2 + z^2 - 1)\mid_{X=M_2^{-1}X}\big) \\ &= V\big((\frac{2}{5}x + \frac{1}{5}y)^2 + (\frac{1}{5}x + \frac{3}{5}y)^2 + (\frac{1}{2}z)^2 - 1\big) \\ &= V(\frac{1}{5}x^2 + \frac{2}{5}xy + \frac{2}{5}y^2 + \frac{1}{4}z^2 - 1). \end{aligned}$$

Therefore the defining equation of E_{M_2} is $\frac{1}{5}x^2 + \frac{2}{5}xy + \frac{2}{5}y^2 + \frac{1}{4}z^2 - 1 = 0$. This ellipsoid is depicted in Figure 2b.

As one can see from Examples 7 and 9, the two definitions are not the same and when both are applicable they may associate different ellipsoids to the same matrix. In some texts one defines the associated ellipsoid to a matrix M to be $\mathcal{E}_M$ (see [3, Exercise 8.13] as an example). However, we consider E_M as the associated ellipsoid to the matrix M when nothing more is mentioned.

Proof of Lemma 8. Consider the assumptions in the lemma. The proof is simple. One just need to note that if a substitution rule is defined by sending x_i to $(NX)_i$ for a matrix N of



(a) (b)

Figure 2: For a positive definite matrix, M, the two ellipsoids E_M and $\mathcal{E}_M$ are not the same.
(a) The ellipsoid $\mathcal{E}_{M_2}$ for the matrix M_2 given in equation 4.
(b) The ellipsoid E_{M_2} for the matrix M_2 given in equation 4.

order n and f is a function from $\mathbb{R}^n$ to $\mathbb{R}$, then $f\mid_{X=NX}$ is equal to $f\circ N$, where $\circ$ is the function composition operator.

$$\begin{aligned}
u \in M\big(V(F)\big) &\iff \exists v \in V(F) \text{ such that } u = Mv\\
&\iff M^{-1}u \in V(F)\\
&\iff \forall f \in F :\ f(M^{-1}u) = 0\\
&\iff \forall f \in F :\ \big(f \circ M^{-1}\big)(u) = 0\\
&\iff \forall g \in F\mid_{X=M^{-1}X} :\ g(u) = 0\\
&\iff u \in V\big(F\mid_{X=M^{-1}X}\big).
\end{aligned}$$

Note that we used the assumption that M has an inverse. □

The next natural question is if there is any relation between E_M and $\mathcal{E}_M$. The answer is positive. This relation is already known (for example see [5, Section 2.2.2]), but we think it is beneficial for some readers to have a formal proof written somewhere so we bring the following two propositions.

Proposition 10. *Let P be a positive definite matrix of order n. Then $E_P = \mathcal{E}_{P^2}$.*

Proof. We prove this equality by showing that the defining equations of the two ellipsoids in the proposition are equal. Note that since P is symmetric we have $P^t = P$, even more, for every $k \in \mathbb{Z}$ we have $(P^k)^t = P^k$.

$$\begin{aligned}
f_{P^2} &= X^t(P^2)^{-1}X - 1\\
&= X^t(P^{-1}P^{-1})X - 1\\
&= X^t\big((P^{-1})^tP^{-1}\big)X - 1\\
&= \big(X^t(P^{-1})^t\big)(P^{-1}X) - 1\\
&= (P^{-1}X)^t(P^{-1}X) - 1\\
&= (X^tX - 1)\mid_{X=P^{-1}X}
\end{aligned}$$

Define $g = X^tX - 1$, then we proved that $f_{P^2} = g\mid_{X=P^{-1}X}$. Because $V(g) = \mathcal{S}_{n-1}$, by Lemma 8 this shows the following

$$\mathcal{E}_{P^2} = V(f_{P^2}) = V(g\mid_{X=P^{-1}X}) = P\big(V(g)\big) = P(\mathcal{S}_{n-1}) = E_P.$$

□

Proposition 11. *Let $M \in \mathrm{GL_n}(\mathbb{R})$. There exists a positive definite matrix P such that E_M is image of $\mathcal{E}_P$ under a rotation transformation (a rotation, a reflection or a mixture of the two).*

Proof. Let $M = U_1 S U_2^t$ be the singular value decomposition of M. Therefore U_1 and U_2 are orthogonal matrices and S is a diagonal matrix with positive entries on its diagonal. Define $U_3 = U_1 U_2^t$ and $P_1 = U_2 S U_2^t$, it is easy to verify that U_3 is also orthogonal and P_1 is a positive definite and $M = U_3 P_1$, i.e. this is the polar decomposition of M. Define $P_2 = P_1^2$, clearly P_2 is also positive definite. By Proposition 10 we know that $E_{P_1} = \mathcal{E}_{P_2}$. Thus

$$\begin{aligned} E_M &= M(\mathcal{S}_{n-1}) \\ &= \big(U_3 P_1\big)(\mathcal{S}_{n-1}) \\ &= U_3\big(P_1(\mathcal{S}_{n-1})\big) \\ &= U_3(E_{P_1}) \\ &= U_3(\mathcal{E}_{P_2}). \end{aligned}$$

Remember from Section 2 that a rotation transformation is a linear map defined by an orthogonal matrix. □

3.2 Semi-axes of ellipsoids

Before introducing semi-axes of an ellipsoid, we need the following definition.

Definition 12. *Remember the Euclidean distance function.*

$$\left\{ \begin{array}{lccl} d & : & \mathbb{R}^n \times \mathbb{R}^n & \rightarrow \mathbb{R}_{\geq 0} \\ & & (u, v) & \mapsto \sqrt{\sum_{i=1}^n (u_i - v_i)^2} \end{array} \right. . \tag{5}$$

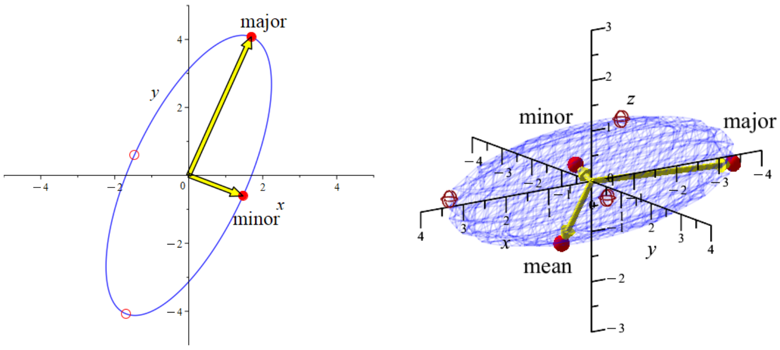
Let $A \subseteq \mathbb{R}^n$, and $c \in \mathbb{R}^n$, define $d_{A,c}$ to be the function $A \to \mathbb{R}_{\geq 0}$, sending $v \in A$ to $d(c, v)$. When $c = (0, \ldots, 0)$, we drop the emphasis on c and simply write d_A. Length of the point (or vector) v is the Euclidean distance of v from origin, $(0, \ldots, 0)$, and is denoted by $|v|$.

Let us start from a familiar case. Consider an ellipse E in $\mathbb{R}^2$. The function d_E has four local extremums, two by two located on same lines passing through the origin, in fact reflection of each other. Such a pair of points are called antipodal. See Figure 3a for an example. We pick up one point from each antipodal pair and call them semi-axes of the ellipse. The semi-axes as vectors are orthogonal and span $\mathbb{R}^2$, so they form an orthogonal basis. At one of the semi-axes d_E attains its maximum value, thus it is called the major semi-axis and at the other one d_E attains its minimum so it is called the minor semi-axis.

In $\mathbb{R}^3$ we have three semi-axes where the Euclidean distance function has a maximum, a saddle point and a minimum called major, mean and minor semi-axes. See Figure 3b. In general for an arbitrary ellipsoid in $\mathbb{R}^n$ we have n semi-axes that we can order them by their length.

A natural question is how to find the coordinates of semi-axes of an ellipsoid given by a matrix M. One way is to use the defining equation of the ellipsoid which now we know how to get its formula thanks to Lemma 8. We first remind the following proposition from algebraic geometry which is not a new result (see [10] for example).

Proposition 13. *Let $X = (x_1, \ldots, x_n)$, $F = \{f_1, \ldots, f_m\} \subseteq \mathbb{R}[X]$, $A = V(F)$ and $c \in \mathbb{R}^n$. The set of critical points of $d_{A,c}$ are the points $v \in A$ such that $v - c$ belong to the normal space of A at v.*



(a) (b)

Figure 3: Semi-axes of ellipsoids.
(a) The ellipse E_M for $M = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$. An ellipsoid in $\mathbb{R}^2$ which is an ellipse has two semi-axes. The function d_{E_M} has four extremums shown in the figure. Two of these points are located on a line through the origin and the other two on a different line through origin. Therefore we have two antipodal pairs. From each of these two pairs, one point is selected. The point where d_{E_M} attains minimum is called the minor semi-axis and the one where d_{E_M} attains its maximum is called the major semi-axis.
(b) The ellipsoid E_{M_1} for M_1 in equation 2. An ellipsoid in $\mathbb{R}^3$ has three semi-axes called major, mean and minor.

Proof. Remember that the tangent space of a manifold A at a point v is the linear space generated by the direction vectors of the tangent lines to A at v, denoted by T_vA, and the normal space of A at v is the orthogonal complement of T_vA, denoted by N_vA. We use the Lagrange multipliers ([9, Chapter 7, Theorem 1.13]) to find the critical points of $d_{A,c}$.

Define the following new function using the auxiliary variables λ_i, $i = 1, \ldots, m$.

$$\phi = d + \lambda_1 f_1 + \cdots + \lambda_m f_m. \tag{6}$$

Domain of ϕ is $\mathbb{R}^{n+m}$. Its critical points satisfy the following system of equations.

$$\frac{\partial \phi}{\partial x_1} = \cdots = \frac{\partial \phi}{\partial x_n} = \frac{\partial \phi}{\partial \lambda_1} = \cdots = \frac{\partial \phi}{\partial \lambda_m} = 0. \tag{7}$$

Since $d = \sum_{i=1}^n (x_i - c_i)^2$, the equation (7) simplifies to the following.

$$2(x_1 - c_1) + \sum_{i=1}^m \lambda_i \frac{\partial f_i}{\partial x_1} = \cdots = 2(x_n - c_n) + \sum_{i=1}^m \lambda_i \frac{\partial f_i}{\partial x_n} = f_1 = \cdots = f_m = 0. \tag{8}$$

The condition $f_1 = \cdots = f_m = 0$ implies $x \in V(F)$. And the rest of the equation (8) gives us the following.

$$\begin{aligned}(x_1 - c_1, \ldots, x_n - c_n) &= -\frac{1}{2}\Big(\sum_{i=1}^m \lambda_i \frac{\partial f_i}{\partial x_1}, \ldots, \sum_{i=1}^m \lambda_i \frac{\partial f_i}{\partial x_n}\Big) \\ &= \sum_{i=1}^m \Big(-\frac{\lambda_i}{2}\Big)\Big(\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n}\Big) \\ &\in \langle \nabla f_1, \ldots, \nabla f_m \rangle = (T_xA)^{\perp}.\end{aligned}$$

That means $x - c \perp N_xA$. □

Corollary 14. *Let f be the defining polynomial of an ellipsoid, $E \subseteq \mathbb{R}^n$. The semi-axes of E satisfy the system of equations achieved by letting the 2-minors of the following matrix and f equal to 0.*

$$\begin{bmatrix} x_1 & \dots & x_n \\ \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}. \tag{9}$$

Proof. The semi-axes of E are also critical points for d_E. Because $E = V(f)$, by Proposition 13, a point $v \in E$ is a critical point for d_E if it satisfies $v \in \langle \nabla f \rangle$. This is equivalent with the rank of the following matrix being one.

$$\begin{bmatrix} v_1 & \dots & v_n \\ \frac{\partial f}{\partial x_1}|_{X=v} & \dots & \frac{\partial f}{\partial x_n}|_{X=v} \end{bmatrix}. \tag{10}$$

This is the same matrix as in (9) after the substitution $X = v$. Because the rank of a matrix is equal to its determinantal rank, this means that all 2-minors (determinant of the 2 by 2 sub-matrices) must vanish. This together with $f = 0$, gives us a system of $\binom{n}{2} + 1$ polynomial equations in n variables with the degree of the polynomials at most 2. □

Remark 15. *Note that Corollary 14 says that the semi-axes are among the solutions to the introduced system of equations and does not say all of these solutions are semi-axes. Consider the 2-dimensional case. One can check that when the defining polynomial of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, with $a \neq b$, the solution to the system of equations of Corollary 14 gives four points, the two pairs of antipodal points, obviously only two of them should be picked up as the semi-axes which are orthogonal to each other. Now if $a = b$, all the points on the ellipse which now is a circle is a solution to the system! Any two of these infinite choices that are orthogonal to each other can be picked up as the semi-axes.*

In general, consider an ellipsoid in $\mathbb{R}^n$, and let $v^1, \dots, v^n$ be a set of semi-axes for this ellipsoid. If the length of these semi-axes are all different, then the solution set of the system in Corollary 14 is a zero-dimensional set, i.e. a finite set of points, or to be more exact, a set of $2n$ points. Otherwise, its dimension which is equal to the dimension of the irreducible component of this algebraic set with the highest dimension, is equal to the maximum number of semi-axes of the same length.

We will not spend any further on this remark and refer the interested reader to [8, Chapter 9] where they can find several methods to compute dimension of an algebraic set.

Example 16. *Consider the matrix M_1 in equation (2). By Lemma 8, its defining polynomial is*

$$f = \frac{1}{9}x^2 + 3y^2 + \frac{10}{9}z^2 - \frac{2}{3}xy + \frac{2}{9}xz - \frac{8}{9}yz - 1.$$

The matrix in equation (9) becomes

$$\begin{bmatrix} x & y & z \\ \frac{2}{9}x - \frac{2}{3}y - \frac{2}{9}z & 6y - \frac{2}{3}x - \frac{8}{9}z & \frac{20}{9}z + \frac{2}{9}x - \frac{8}{9}z \end{bmatrix}.$$

By Corollary 14, the semi-axes are among the solutions to the following system of equations.

$$\begin{cases} \frac{1}{9}x^2 + 3y^2 + \frac{10}{9}z^2 - \frac{2}{3}xy + \frac{2}{9}xz - \frac{8}{9}yz - 1 & = 0, \\ \frac{52}{9}xy - \frac{8}{3}xz - \frac{2}{3}x^2 + \frac{2}{3}y^2 - \frac{2}{9}yz & = 0, \\ -\frac{8}{3}xy + 2xz + \frac{2}{9}x^2 + \frac{2}{3}yz - \frac{2}{9}z^2 & = 0, \\ -\frac{8}{3}y^2 - \frac{34}{9}yz + \frac{2}{9}xy + \frac{8}{3}z^2 + \frac{2}{3}xz & = 0. \end{cases}$$

There are various methods developed for solving a system of polynomial equations with symbolic exact solutions such as using Gröbner basis [8, Chapters 2 and 3], resultant techniques [7, *Chapter 3], or with numeric solutions such as using numerical homotopy methods [2]. Solving this system we get 6 points shown in Figure 3b. One can use the predefined command* `HilbertDimension` *in the* `PolynomialIdeals` *package of Maple to compute the dimension of the algebraic set which is the solution set of the above system. The result is 0 as expected.*

A more linear algebra flavour approach is to use singular value decomposition. Let $M \in \mathrm{GL_n}(\mathbb{R})$. Denote the singular value decomposition of M as $U_1SU_2^t$, where S is a diagonal matrix with the singular values of M, denoted by σ_is on its diagonal, ordered from the largest to the smallest value, U_1 and U_2 two orthogonal matrices with columns denoted by u^is and v^is respectively. u^is and v^is are called the left and the right singular vectors of M. From [4, Chapter 3] remember that σ_1 is the maximum possible length of Mv for $v \in \mathcal{S}_{n-1}$. The vector Mv which its length is σ_1, is in fact the major semi-axis of the ellipsoid $E_M = M(\mathcal{S}_{n-1})$. Again from [4, Chapter 3], the first right singular vector of M, v^1, is the point v which Mv has length σ_1, and the first left singular vector of M, u^1 is $\frac{1}{\sigma_1}Mv^1$. Therefore the major semi-axis of E_M is equal to Mv^1 or equivalently $\sigma_1 u^1$. Similarly the rest of semi-axes of E_M can be computed as Mv^i or $\sigma_i u^i$.

Proposition 17. *Let $M \in \mathrm{GL_n}(\mathbb{R})$ and let $\sigma_1, \ldots, \sigma_n$ to be the singular values of M ordered from the largest to the smallest and $u^1, \ldots, u^n$ be the corresponding left singular vectors of the σ_is. The semi-axes of E_M ordered by their length are $\sigma_1 u^1, \ldots, \sigma_n u^n$.*

4 TRD decomposition

In this section we restrict ourselves to ellipsoids in $\mathbb{R}^3$. For the rest of the section fix the notation. Let $M \in \mathrm{GL}_3(\mathbb{R})$, $E = E_M$, $M = U_1SU_2^t$ the singular value decomposition with σ_i, u^i and v^is the singular values, singular left vectors and singular right vectors, and A_1, A_2 and A_3 the major, mean and minor semi-axes of E respectively.

A unique ellipse passes through each two choices of the three semi-axes on the surface of the ellipsoid. The minor-mean and the mean-major ellipses are the smallest and the largest possible ellipses on the surface of the ellipsoid. Denote the plane containing the minor-mean ellipse by π_1. By rotating π_1 around the vector A_2 (see Figure 4) about α for $0 \leq \alpha \leq \frac{\pi}{2}$, we get a new plane π_2 that intersects E in a different ellipse with two semi-axes, one being A_2 and the other A_3' a point on the minor-major ellipse of E. For a unique choice of α, A_3' has the same length as A_2. Clearly if length of A_2 and A_3 are the same, then $\alpha = 0$, otherwise $\alpha > 0$. We want to use a shear map parallel to the plane π_2 to transform E to a new ellipsoid. So before going any further, we should know how to find this plane. The plane π_2 is the plane passing through the three points; the origin, A_2 and A_3'. Therefore, we need to find the coordinates of A_3'. This will uniquely determines π_2 as well.

There are different ways to do this. A linear algebra flavour one is to use the singular value decomposition. There exists a vector $v^4 \in \mathcal{S}_2$ such that $A_3' = Mv^4$. Since A_3' belongs to the minor-major ellipse of E, v^4 should be written as $\lambda_1 v^1 + \lambda_2 v^3$ for two real scalar values λ_1 and λ_2. At the same time we want Mv^4 to have the same length as A_2, therefore we have the following system of 2 equations with 2 variables.

$$\begin{cases} |\lambda_1 v^1 + \lambda_2 v^3| & = 1, \\ |\lambda_1 Mv^1 + \lambda_2 Mv^3| & = \sigma_2. \end{cases}$$

Equivalently

$$\begin{cases} |\lambda_1 v^1 + \lambda_2 v^3| & = 1, \\ |\lambda_1 \sigma_1 u^1 + \lambda_2 \sigma_3 u^3| & = \sigma_2. \end{cases}$$

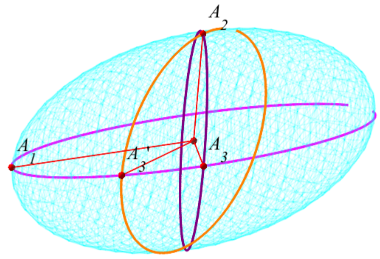


Figure 4: An ellipsoid in $\mathbb{R}^3$. The major, mean and minor semi-axes are the points named A_1, A_2 and A_3 respectively. The minor-mean ellipse is colored in purple, we tilted the plane containing this ellipse around the line connecting the origin to A_2, about α, where $0 \leq \alpha \leq \frac{\pi}{2}$ until it intersects the minor-major ellipse (colores in magenta) in A'_3, a point with the same length as A_2's. The ellipse passing through A_2 and A'_3 is a circle (colored in orange).

A second approach is as follows. The defining equation of the plane containing the minor-major ellipse can be calculated by first letting $v = A_1 \times A_3$ to be the cross-product of these two vectors. This vector is the normal vector of the minor-major plane. A plane with the normal vector v and containing a point u is a solution set of the equation $v_1(x-u_1)+v_2(y-u_2)+v_3(z-u_3) = 0$. We can use either A_1 or A_3 as u. Let g be the linear polynomial in this equation. The point A'_3 that we are looking for satisfies the following system of equations.

$$f = g = x^2 + y^2 + z^2 - |A_2|^2 = 0.$$

Example 18. *Consider the matrix M_1 in (2). To find the coordinates of the tilted minor using the first approach, we have to solve the following system of equations (numbers are rounded, for more exact values see the computation files).*

$$\begin{cases} |\lambda_1(-0.2351, -0.5831, -0.7777) + \lambda_2(-0.6949, -0.4586, 0.5539)| & = 1 \\ |3.7955\lambda_1(-0.2351, -0.5831, -0.7777) + 0.5183\lambda_2(-0.6949, -0.4586, 0.5539)| & = 1.5251 \end{cases}$$

That can simplify to the following.

$$\begin{cases} \lambda_1^2 + \lambda_2^2 & = 1 \\ 14.405\lambda_1^2 + 0.2686\lambda_2^2 & = 2.3259 \end{cases}$$

By solving this system of equations numerically and substituting the solution into $\lambda_1 A_1 + \lambda_2 A_3$ we get $(-1.4704, 0.2015, -0.3511)$. Now using the second approach. The defining polynomial of E_{M_1} is the following.

$$f = 3y^2 - \frac{8}{3}yz + \frac{10}{9}z^2 + \frac{1}{9}x^2 - \frac{2}{3}xy + \frac{2}{9}xz - 1.$$

For the equation of the plane containing A_1 and A_3 we can use A_2 instead of calculating the cross product of $A_1 \times A_3$ as A_2 is also orthogonal to both of them. It gives us $g = 0.2305x - 0.6706y - 1.3502z$. So the alternative system of equations is the following.

$$f = g = x^2 + y^2 + z^2 - 2.3259 = 0.$$

This also gives us the same solution $(-1.4704, 0.2015, -0.3511)$.

A shear map parallel to π_2 maps E to a new ellipsoid E' where two of its semi-axes have the same length, they are A'_3 and A_2, but its third semi-axis is on the line normal to π_2 and is the image of the furthest point of E from π_2 which is not necessarily A_1. Denote this shear map by T_E, the furthest point of E from π_2 by A'_1, and the image of A'_1 under T_E by A''_1. If we find the coordinates of A_1' and A_1'', then we can find T_E by solving the linear system of equations generated by the following three conditions.

$$T_E(A_2) = A_2,\ T_E(A_3') = A_3',\ T_E(A_1') = A_1''.$$

Note that A_2, A_3' and A_1'' are semi-axes of an ellipsoid, E', therefore they form a basis of $\mathbb{R}^3$. In addition to that, A_1' is outside the plane containing A_2 and A_3', therefore the set $\{A_2, A_3', A_1'\}$ is also a basis for $\mathbb{R}^3$. This means that the linear system to find entries of T_E has a unique solution.

Theorem 19. *Let $M \in \mathrm{GL}_3(\mathbb{R})$, there are a shear matrix T, an orthogonal matrix R, and a diagonalizable matrix D with two equal eigenvalues, such that $M = TRD$.*

Proof. Let $M \in \mathrm{GL}_3(\mathbb{R})$ and T_E to be the shear map that transforms E_M to the rotational ellipsoid (an ellipsoid with two semi-axes of equal length) introduced before this theorem. Let $T_E M = UP$ be the polar decomposition of $T_E M$ as in proof of Proposition 11. The matrix U is an orthogonal matrix and the matrix P is a positive definite matrix. From Section 2 remember that real symmetric matrices are orthogonally diagonalizable, therefore their singular values are the same as their eigenvalues. Since the singular values of P and $T_E M$ are the same, and the singular values of $T_E M$ are length of semi-axis of a rotational ellipsoid, P has two equal eigenvalues. Finally, by Lemma 4 the matrix T_E is invertible and its inverse is also a shear map. Let $T = T_E^{-1}$, $R = U$ and $D = P$, we have $M = TRD$. This finishes the proof. □

An algorithm to compute the TRD decomposition of Theorem 19 is given below, Algorithm 1. We implemented this algorithm both in Maple and Matlab.

In Maple we used `LinearAlgebra` package for basic linear algebra computations such as transpose, inverse, cross product, rank etc., for the singular value decomposition we used the command `svd` in `MTM` package. To solve the equations we used the numeric solver command `fsolve`. For line 8 of Algorithm 1 we used `Maximize` command from `Optimization` package. The result together with a few more procedures such as finding the defining polynomial of ellipsoids are wrapped into a new Maple package named `Ellipsoids` accessible online for free from `https://doi.org/10.5281/zenodo.7021479`.

As for the Matlab implementation, we used `vpasolve` for numerically solving the equations. For line number 8 of the Algorithm 1 we used Lagrange multipliers technique and `vpasolve`. All the equivalent versions of the functions implemented in the Maple package `Ellipsoids` can be found in the Matlab script file `Ellipsoids` accessible online for free from the same Zenodo repository.

Example 20. *Recall the matrix M_1 from (2). The TRD decomposition of this matrix is the following.*

$$T = \begin{bmatrix} 1.5331746196 & 2.1705784446 & -0.9869790414 \\ -0.0730706176 & 0.7025261487 & 0.1352636931 \\ 0.1273275947 & 0.5183565054 & 0.7642992318 \end{bmatrix},$$

$$R = \begin{bmatrix} -0.3248257249 & -0.1398570971 & 0.9353759890 \\ 0.2027109239 & 0.9557267312 & 0.2132948583 \\ -0.9237946362 & 0.2588945879 & -0.2820940669 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.4703492210 & -0.0810228156 & -0.0576003935 \\ -0.0810228156 & 1.4051711889 & -0.0852531757 \\ -0.0576003935 & -0.0852531757 & 1.4644835940 \end{bmatrix}.$$

Note that $TRD = M$, D is diagonalizable with two equal eigenvalues, R is an orthogonal matrix and T is a shear matrix keeping the plane containing the mean semi-axis of E_{M_1} and the titled minor.

Input : $M \in \mathrm{GL}_3(\mathbb{R})$.
Output: Three real matrices of order 3, T, R, D where T is a shear matrix, R is an orthogonal matrix, and D is a diagonalizable matrix with two equal eigenvalues.

compute the singular decomposition of M, $M = U_1 S_1 U_2^t$. Denote the singular values of M by σ_i ordered from the largest to the smallest, and the left and the right singular vectors by u^i and v^i accordingly, $i = 1, 2, 3$;
$A_i = \sigma_i u^i$, $i = 1, 2, 3$;
solve $|\lambda_1 v^1 + \lambda_2 v^3| - 1 = |\lambda_1 A_1 + \lambda_2 A_3| - \sigma_2 = 0$ to find λ_1 and λ_2;
$A_3' = \lambda_1 A_1 + \lambda_2 A_3$;
$v = A_2 \times A_3'$ (cross product);
$g = v(X - A_2)^t$ where $X = (x, y, z)$;
$f = f\ |_{X=M^{-1}X}$;
$A_1' = \underset{f(u)=0}{\mathrm{argmax}}\big(g(u)\big)$;
$A_1'' = \frac{A_1 v^t}{v v^t} v$;
solve $T_E A_2 - A_2 = T_E A_3' - A_3' = T_E A_1' - A_1'' = 0$ to find T_E;
compute the singular decomposition of $T_E M$, $T_E M = U_3 S_2 U_4^t$;
$T = T_E^{-1}$;
$R = U_3 U_4^t$;
$D = U_3 S_2 U_4^t$;

Algorithm 1: An algorithm to decompose an invertible matrix of order 3 to product of three matrices, a shear, a rotation and a dilation.

5 Conclusion

In this paper we presented a computational algorithm, Algorithm 1, to compute the TRD decomposition introduced in [6] for invertible matrices of order 3. The algorithm is implemented in both Maple and Matlab (see `https://doi.org/10.5281/zenodo.7021479`).

Note that all steps of algorithm 1 can be done for $M \in \mathrm{GL_n}(\mathbb{R})$ with $n > 3$ as well, with one difference. In higher dimension, the ellipsoid has more than 3 semi-axes and instead of a unique choice of three semi-axes ordered by length, we have $\binom{n}{3}$ choices. Let $(A_{i_1}, A_{i_2}, A_{i_3})$ be one such choice where $1 \le i_1 \precneqq i_2 \precneqq i_3 \le n$. Instead of the major, mean, minor semi-axes of 3d ellipsoid in Algorithm 1, one should use these three semi-axes which of course there is a 3d ellipsoid passing through them on the surface of the main ellipsoid (compare with the case of ellipse passing through each two semi-axes of a 3d ellipsoid on its surface). Therefore the TRD decomposition is not unique. The shear matrix, T_E, in this case keeps the hyperplane (linear space of codimension 1) that contains A_{i_2}, the tilted A'_{i_3} and all other non-chosen $n-3$ semi-axes. So the image of E_M under this shear map has $n-2$ semi-axes the same as the original one.

One may hope for a possibility of repeating the shearing step of the algorithm several times to get an ellipsoid with more than two semi-axes of the same length and then doing the polar decomposition to get the following statement. However, it should be noted that we do not have a prior control on the relation between the length of A''_{i_1} and the length of A'_{i_3} and other A_js ($j \notin \{i_1, i_2, i_3\}$). This makes it difficult to judge the possibility of choosing the next three semi-axes appropriately.

Question: Let $M \in \mathrm{GL_n}(\mathbb{R})$, $n \ge 3$, and $k \in \{2, 3, \ldots, n-1\}$. Is it possible to find $k-1$ shear maps T_1, ..., T_{k-1}, an orthogonal matrix R, and a diagonalizable matrix D with k equal eigenvalues, such that $M = T_1 T_2 \cdots T_{k-1} R D$?

Data Access Statement. The code and data described in this paper is openly available from this URL: `https://doi.org/10.5281/zenodo.7021479`.

acknowledgement

The second author acknowledge the support of EPSRC Grant EP/T015748/1, “Pushing Back the Doubly-Exponential Wall of Cylindrical Algebraic Decomposition”.

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