Stochastic representation of the quantum quartic oscillator

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$$\left|\int_S \mathrm{d}z\, e^{iaz^2}\right| \le R\int_0^{\pi/4} \mathrm{d}\phi\, e^{-aR^2\sin(2\phi)} = \frac{R}{2}\int_0^{\pi/2} \mathrm{d}\theta\, e^{-aR^2\sin\theta}\,, \tag{62}$$

where last equality follows from introducing $\theta \equiv 2\phi$. Moreover, for $\theta \in (0, \pi/2)$, we have that $2\theta/\pi \le \sin\theta \le \theta$, yielding

$$\left|\int_S \mathrm{d}z\, e^{iaz^2}\right| \le \frac{R}{2}\int_0^{\pi/2} \mathrm{d}\theta\, e^{-aR^2\sin\theta} \le \frac{R}{2}\int_0^{\pi/2} \mathrm{d}\theta\, e^{-2aR^2\theta/\pi} = \frac{\pi}{4aR}\left(1-e^{-aR^2}\right), \tag{63}$$

which vanishes for $R\to\infty$. Finally, we get

$$\int_0^\infty \mathrm{d}x\, e^{iax^2} = e^{i\pi/4}\int_0^\infty \mathrm{d}r\, e^{-ar^2} = e^{i\pi/4}\sqrt{\frac{\pi}{4a}} = \sqrt{\frac{i\pi}{4a}}\,, \tag{64}$$

where we have fixed $\sqrt{i} = e^{i\pi/4}$. This last result can be generalized to

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{i(ax^2+bx)} = \sqrt{\frac{i\pi}{a}}e^{-ib^2/(4a)}\,. \tag{65}$$

A.2 Time-dependent quartic Hamiltonian

In the framework discussed in section 2, it is natural to generalise the expression of Eq. (14) to the case of the time-dependent Hamiltonian

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{x}^2 + \frac{\lambda(t)}{4}\hat{x}^4\,, \tag{66}$$

with a non-negative quartic coupling, $\lambda(t)\ge 0$. The procedure follows the same steps as in the time-independent case, i.e.,

(i) we perform a Trotter-Suzuki splitting of the time-evolution operator, yielding

$$\hat{U}(t) = \lim_{n\to\infty}\left\{\exp\left[-\frac{i\tau_n}{\hbar}\frac{\hat{p}^2}{2m}\right]\exp\left[-\frac{i\tau_n}{\hbar}\left(\frac{m}{2}\omega_n^2\hat{x}^2 + \frac{\lambda_n}{4}\hat{x}^4\right)\right]\right\}^n\,, \tag{67}$$

with $\omega_n^2 \equiv \omega^2(n\tau_n)$ and $\lambda_n \equiv \lambda(n\tau_n)$;

(ii) this is followed by the Hubbard-Stratonovich transformation, performed through an integral of the type

$$\exp\left(-i\frac{\tau_n}{\hbar}\frac{\lambda_n}{4}\hat{x}^4\right) = \sqrt{\frac{\tau_n}{i\hbar\pi}}\int_{-\infty}^{\infty}\mathrm{d}\phi\,\exp\left[\frac{i\tau_n}{\hbar}\left(\phi^2 - \sqrt{\lambda_n}\,\hat{x}^2\,\phi\right)\right]; \tag{68}$$

(iii) Finally, combining all previous steps, we retrieve Eq. (14) with $\Omega^2(t)\equiv\omega^2(t) + 2\sqrt{\lambda(t)}\,\phi(t)/m$ and $S_0[\phi] = \hbar^{-1}\int_0^t \mathrm{d}\tau\,\phi^2(\tau)$.

Once again, this allows one to study the dynamics of a quantum problem by studying a set of classical differential equations.

B Action of $\exp(w\hat{S}^{\pm,z})$ on a Gaussian wave packet

We now investigate the action of the operators appearing in Eq. (12), i.e.,

$$\hat{U}^{\alpha} \equiv \exp(w\hat{S}^{\alpha}), \tag{69}$$

with complex w and $\alpha \in \{+, -, z\}$, on the Gaussian wave packet $|\psi\rangle$, reported in Eq. (16). For a general complex-valued w, the exponential operators are not unitary and do not conserve the normalization of the state. For simplicity, we consider here values of w for which the corresponding $\hat{U}^{\alpha}$ conserves the state normalization. Note that the $\hat{S}^{\alpha}$ operators in Eqs. (9) are, at most, of quadratic order with respect to the operators $\hat{x}$ and $\hat{p}$. In what follows, we assume $\hbar = 1$ and a real-valued w. In order to analyze the action of $\hat{U}^{\alpha}$ in Eq. (69), we have introduced in Eq. (17) the Wigner function $W(x,p)$ for the generic state $|\psi\rangle$.

$W(x,p)$ provides a phase space description of the state, and allows us to compute expectation values of operators of the type $O(\hat{x},\hat{p})$ as $\int \mathrm{d}x \int \mathrm{d}p\, O(x,p)\, W(x,p)$ [53–55]. The evaluation of the Wigner function in Eq. (18) for the Gaussian wave packet $|\psi\rangle$ in Eq. (16) is obtained by substitution of Eq. (16) into Eq. (17) and integrating with respect to y. First, we consider the action of the operator $\hat{U}^{+} = \exp(iw\hat{S}^{+})$ on $|\psi\rangle$, that we denote as $|\psi_+\rangle \equiv U^{+}|\psi\rangle$. The state $|\psi_+\rangle$ is simply given by

$$|\psi_+\rangle = \int \frac{\mathrm{d}x}{\sqrt[4]{\pi\sigma^2}} \exp\left[-\frac{(x-a)^2}{2\sigma^2} + iw\frac{x^2}{2} + ik(x-a)\right]|x\rangle\,, \tag{70}$$

which follows from the fact that $\hat{S}^{+}$ acts trivially on its eigenstate $|x\rangle$.

By substituting Eq. (70) into (17) we get the Wigner function for $|\psi_+\rangle$

$$W_+(x,p) = \frac{1}{\pi}\exp\left[-\frac{(x-a)^2}{\sigma^2} - \sigma^2(p-k-wx)^2\right], \tag{71}$$

which is equal to W up to position-dependent shift in the momentum. Accordingly, the expectation value of operators of the form $O(\hat{x})$ is unaffected by the $\hat{U}^{+}$ transformation, i.e., $\langle\psi|O(\hat{x})|\psi\rangle = \langle\psi_+|O(\hat{x})|\psi_+\rangle$. On the other hand, the expectation value of a momentum-dependent operator $O(\hat{p})$ can be expressed as

$$\begin{aligned}
\langle\psi_+|O(\hat{p})|\psi_+\rangle &= \int \mathrm{d}x \int \mathrm{d}p\, O(p)\, W_+(x,p) \\
&= \sqrt{\frac{\sigma^2}{\pi(1+\sigma^4 w^2)}} \int \mathrm{d}p\, O(p) \exp\left[-\sigma^2\frac{(p-k-wa)^2}{1+\sigma^4 w^2}\right] \\
&= \sqrt{\frac{\sigma^2}{\pi}} \int \mathrm{d}q\, O\left(\sqrt{1+\sigma^4 w^2}\,(q-k) + wa + k\right) e^{-\sigma^2(q-k)^2} \\
&= \langle\psi|\, O\left(\sqrt{1+\sigma^4 w^2}\,(\hat{p}-k) + wa + k\right)|\psi\rangle\,,
\end{aligned} \tag{72}$$

where the second line comes from direct substitution of Eq. (71) and the final result from the change of variable $p - wa - k = (q-k)\sqrt{1+\sigma^4 w^2}$. Equation (72) tells us that expectation values with respect to the state $|\psi_+\rangle$ of operators depending only on p are equivalent to expectation values with respect to the Gaussian wave packet $|\psi\rangle$ with the rescaled and shifted momentum operator $\sqrt{1+\sigma^4 w^2}\,(\hat{p}-k) + wa + k$. In particular, for the mean and the variance of the momentum operator we can immediately read off from Eq. (72) that

$$\begin{aligned}
\langle p\rangle &\equiv \langle\psi_+|\,\hat{p}\,|\psi_+\rangle = k + wa\,, \\
\langle p^2\rangle_c &\equiv \langle\psi_+|\,\hat{p}^2\,|\psi_+\rangle - \langle\psi_+|\,\hat{p}\,|\psi_+\rangle^2 = \frac{1+\sigma^4 w^2}{2\sigma^2}\,.
\end{aligned} \tag{73}$$

These parameters, together with the unaltered cumulants of the position operator, allow us to fully characterize the state $|\psi_+\rangle$. As an explicit time-dependent example, we consider the evolution of the wave packet under the action of the harmonic oscillator Hamiltonian. Referring to Eqs. (25), we find that

$$\begin{aligned} w(t) &= -m\omega \tan(\omega t)\,, \\ \langle p(t)\rangle &= k - am\omega \tan(\omega t)\,, \\ \langle p^2(t)\rangle_c &= \frac{1+(m\omega\sigma^2)^2\tan^2(\omega t)}{2\sigma^2}\,. \end{aligned} \tag{74}$$

Next, we consider the case of the operator $\hat{U}^z = \exp(iw\{\hat{x},\hat{p}\}/4)$ whose action on the Gaussian wave packet $|\psi\rangle$, which we denote by $|\psi_z\rangle \equiv \hat{U}^z\,|\psi\rangle$, reads

$$|\psi_z\rangle = \int \frac{\mathrm{d}x}{\sqrt[4]{\pi\sigma^2}} \exp\left[\frac{w}{4} - \frac{(x\,e^{w/2}-a)^2}{2\sigma^2} + ik(x\,e^{w/2}-a)\right]|x\rangle\,. \tag{75}$$

This is computed considering the direct action of $\hat{S}^z$ on $|x\rangle$ according the property of the dilation operator $e^{by\frac{d}{dy}}f(y) = f(e^b y)$, where f is any sufficiently smooth function, similarly to what has been done for Eq. (86). The Wigner function $W_z(x,p)$ of the state $|\psi_z\rangle$ can be directly evaluated as

$$W_z(x,p) = \frac{1}{\pi}\exp\left[-\frac{(x\,e^{w/2}-a)^2}{\sigma^2} - \sigma^2(p\,e^{-w/2}-k)^2\right]. \tag{76}$$

Equation (75) shows that the action of $\hat{U}^z$ consists in a uniform rescaling all the x variables by a factor $e^{w/2}$.

As for $|\psi_z\rangle$, the Wigner function W_z is equivalent to W up to a rescaling of the variables. It follows that the expectation value of an operator $O(\hat{x})$, depending only on x, is given by

$$\begin{aligned} \langle\psi_z|\,O(\hat{x})\,|\psi_z\rangle &= \int \mathrm{d}x \int \mathrm{d}p\, O(x) W_z(x,p) \\ &= \frac{e^{w/2}}{\sqrt{\pi\sigma^2}} \int \mathrm{d}x\, O(x) \exp\left[-\frac{(x\,e^{w/2}-a)^2}{\sigma^2}\right] \\ &= \langle\psi|\,O(\hat{x}\,e^{-w/2})\,|\psi\rangle\,, \end{aligned} \tag{77}$$

and, analogously, for a p—dependent operator $O(\hat{p})$, we get

$$\langle\psi_z|\,O(\hat{p})\,|\psi_z\rangle = \langle\psi|\,O(\hat{p}\,e^{w/2})\,|\psi\rangle\,, \tag{78}$$

which reflects the rescaling action of $\hat{U}^z$. It follows that the first connected moments of $\hat{x}$ and $\hat{p}$ on $|\psi_z\rangle$ are given by

$$\begin{aligned} \langle x\rangle &\equiv \langle\psi_z|\,\hat{x}\,|\psi_z\rangle = a\,e^{-w/2}\,, \\ \langle x^2\rangle_c &\equiv \langle\psi_z|\,\hat{x}^2\,|\psi_z\rangle - \langle\psi_z|\,\hat{x}\,|\psi_z\rangle^2 = \frac{\sigma^2}{2}\,e^{-w}\,, \\ \langle p\rangle &\equiv \langle\psi_z|\,\hat{p}\,|\psi_z\rangle = k\,e^{w/2}\,, \\ \langle p^2\rangle_c &\equiv \langle\psi_z|\,\hat{p}^2\,|\psi_z\rangle - \langle\psi_z|\,\hat{p}\,|\psi_z\rangle^2 = \frac{e^w}{2\sigma^2}\,. \end{aligned} \tag{79}$$

These quantities fully characterize the state $|\psi_z\rangle$. According to Eqs. (25), for a $|\psi\rangle$ evolving under the effect of an harmonic oscillator Hamiltonian, we have

$$
\begin{aligned}
w(t) &= -\log\cos^2(\omega t)\,, \\
\langle x(t)\rangle = a\cos(\omega t), \;\; \langle p\rangle(t) &= \frac{k}{\cos(\omega t)}\,, \\
\langle x^2(t)\rangle_c = \frac{\sigma^2}{2}\cos^2(\omega t), \;\; \langle p^2(t)\rangle_c &= \frac{1}{2\sigma^2\cos^2(\omega t)}\,.
\end{aligned}
\tag{80}
$$

Finally, we consider the action of the operator $\hat{U}^- = \exp\left(iw\hat{p}^2/2\right)$ on the wave packet $|\psi\rangle$; the resulting state $|\psi_-\rangle \equiv \hat{U}^-|\psi\rangle$ is found to be

$$
|\psi_-\rangle = \sqrt{\frac{\sigma^2}{\sigma^2 - iw}} \int \frac{\mathrm{d}x}{(\pi\sigma^2)^{1/4}} \exp\left[-\frac{(x-a-ik\sigma^2)^2}{2(\sigma^2 - iw)} - \frac{\sigma^2 k^2}{2}\right]|x\rangle\,. \tag{81}
$$

The associated Wigner function $W_-(x,p)$ reads

$$
W_-(x,p) = \frac{1}{\pi}\exp\left[-\sigma^2(p-k)^2 - \frac{(x-a+pw)^2}{\sigma^2}\right], \tag{82}
$$

that is equivalent to W up to a $p-$dependent rescaling of the x variable. In this case expectation values of $p-$dependent operators are invariant under the action of $\hat{U}^-$, i.e., $\langle\psi_-|O(\hat{p})|\psi_-\rangle = \langle\psi|O(\hat{p})|\psi\rangle$, while the expectation value of an $x-$dependent operator $O(\hat{x})$ transforms as

$$
\begin{aligned}
\langle\psi_-|O(\hat{x})|\psi_-\rangle &= \int \mathrm{d}x \int \mathrm{d}p\, O(x)\, W_-(x,\,p) \\
&= \sqrt{\frac{\sigma^2}{\pi(\sigma^4+w^2)}} \int \mathrm{d}x\, O(x) \exp\left[-\sigma^2\frac{(x-a+wk)^2}{\sigma^4+w^2}\right] \\
&= \frac{1}{\sqrt{\sigma^2\pi}} \int \mathrm{d}y\, O\left(\frac{y-a}{\sigma^2}\sqrt{\sigma^4+w^2} + a - wk\right) e^{-(y-a)^2/\sigma^2} \\
&= \langle\psi|\, O\left(\frac{\hat{x}-a}{\sigma^2}\sqrt{\sigma^4+w^2} + a - wk\right)|\psi\rangle\,,
\end{aligned}
\tag{83}
$$

where the second line is found by integrating $W_-(x,\,p)$ in Eq. (82) with respect to p, and the last two lines are obtained by performing the change of variable $x = (y-a)\sigma^{-2}\sqrt{\sigma^4+w^2} + a - wk$. We deduce that, in case of $x-$dependent operators, the expectation value with respect to $|\psi_-\rangle$ is equivalent to the expectation value with respect to $|\psi\rangle$ where the position operator has been rescaled and shifted according to the final line of Eq. (83). In particular, the first two cumulants of the position operator $\hat{x}$ read

$$
\begin{aligned}
\langle x\rangle \equiv \langle\psi_-|\,\hat{x}\,|\psi_-\rangle &= a - wk\,, \\
\langle x^2\rangle_c \equiv \langle\psi_-|\,\hat{x}\,|\psi_-\rangle - \langle\psi_-|\,\hat{x}^2\,|\psi_-\rangle^2 &= \frac{\sigma^4+w^2}{2\sigma^2}\,.
\end{aligned}
\tag{84}
$$

The evolution of the state $|\psi_-\rangle$ under the harmonic oscillator dynamics can be explicitly determined from Eq. (25):

$$
\begin{aligned}
w(t) &= -\frac{\tan(\omega t)}{m\omega}\,, \\
\langle x(t)\rangle &= a + k\frac{\tan(\omega t)}{m\omega}\,, \\
\langle x^2(t)\rangle_c &= \frac{(m\omega\sigma^2)^2 + \tan^2(\omega t)}{2(m\omega\sigma)^2}\,.
\end{aligned}
\tag{85}
$$

As a last remark, we note that by combining the results in Eqs. (74), (80), and (85) into the factorised expression for the time evolution of the harmonic oscillator, Eq. (12), we can construct the time evolution of observable, e.g., the position and momentum moments in Eqs. (26).

C Time evolution of a Gaussian wave packet

Here we report the detailed calculations of the expectation values of the moments of the position and momentum operators on the Gaussian wave packet in Eq. (16). As a preliminary step to the calculation of Eq. (19), we consider the action of the operator $\hat{U}(t)$ on an eigenstate $|x\rangle$ of the position operator, given by

$$
\begin{aligned}
\hat{U}(t)|x\rangle &= \left\langle \int \frac{\mathrm{d}p}{\sqrt{2\pi}}\, e^{\xi^+(t)\hat{x}^2/2}\, e^{i\xi^z(t)\{\hat{x},\hat{p}\}/4}\, e^{\xi^-(t)p^2/2-ipx}\,|p\rangle \right\rangle_\phi \\
&= \left\langle e^{-\xi^z/4} \int \frac{\mathrm{d}p}{\sqrt{2\pi}}\, e^{\xi^+(t)\hat{x}^2/2}\, e^{-(\xi^z(t)/2)p\frac{\partial}{\partial p}}\, e^{\xi^-(t)p^2/2-ipx}\,|p\rangle \right\rangle_\phi \\
&= \left\langle e^{-\xi^z/4} \int \frac{\mathrm{d}p}{\sqrt{2\pi}}\, e^{\xi^+(t)\hat{x}^2/2} \exp\left[\xi^-(t)\frac{p^2}{2}e^{-\xi^z} - ipxe^{-\xi^z/2}\right]|p\rangle \right\rangle_\phi \\
&= \left\langle e^{-\xi^z/4} \int \frac{\mathrm{d}p}{\sqrt{2\pi}} \int \mathrm{d}y\; e^{\xi^+(t)y^2/2}\,|y\rangle\langle y| \exp\left[\xi^-(t)\frac{p^2}{2}e^{-\xi^z} - ipxe^{-\xi^z/2}\right]|p\rangle \right\rangle_\phi \\
&= \left\langle e^{-\xi^z/4} \int \frac{\mathrm{d}y}{\sqrt{2\pi}}\, e^{\xi^+(t)y^2/2} \int \frac{\mathrm{d}p}{\sqrt{2\pi}} \exp\left[\xi^-(t)\frac{p^2}{2}e^{-\xi^z} - ipxe^{-\xi^z/2} + ipy\right]|y\rangle \right\rangle_\phi \\
&= \left\langle \frac{\exp\left[\xi^z/4 + x^2/(2\xi^-)\right]}{\sqrt{-2\pi\xi^-}} \int \mathrm{d}y\; \exp\left[\frac{y^2}{2}\left(\frac{e^{\xi^z}}{\xi^-} + \xi^+(t)\right) - y\frac{xe^{\xi^z/2}}{\xi^-}\right]|y\rangle \right\rangle_\phi ,
\end{aligned}
\tag{86}
$$

where $\langle\ldots\rangle_\phi$ denotes the expectation value with respect to the Gaussian action S_0.

In the first line a completeness relation for the momentum basis, $\int dp\,|p\rangle\langle p| = \mathbb{I}$, was inserted between the last exponential operator and the position eigenket, leading to the appearance of the plane wave $\langle p|x\rangle = e^{-ipx}/\sqrt{2\pi}$, where $\hbar = 1$. In the second line, we substituted $\hat{x}\,|p\rangle = i\frac{\partial}{\partial p}\,|p\rangle$ and the consequent action of the dilation operator was written explicitly, i.e., $e^{by\frac{d}{dy}}f(y) = f(e^b y)$. Finally, a further position completeness relation insertion and a Gaussian integration was performed. The convergence of the Gaussian integral is ensured by the fact that the argument of the exponential is purely imaginary. Analogously, the corresponding dual vector evolves according to

$$
\langle x|U^\dagger(t) = \left\langle \frac{\exp\left[\bar{\xi}^z/4 + x^2/(2\bar{\xi}^-)\right]}{\sqrt{-2\pi\bar{\xi}^-}} \int \mathrm{d}z\; \exp\left[\frac{z^2}{2}\left(\frac{e^{\bar{\xi}^z}}{\bar{\xi}^-} + \bar{\xi}^+(t)\right) - z\frac{xe^{\bar{\xi}^z/2}}{\bar{\xi}^-}\right]\langle z| \right\rangle_{\bar{\phi}} , \tag{87}
$$

with $\langle\ldots\rangle_{\bar{\phi}}$ denoting the expectation value with respect to the Gaussian action $S_0[\bar{\phi}]$, and $\bar{\xi} \equiv [\xi(\bar{\phi}(t))]^*$ the complex conjugate of $\xi^{+,-,z}$. Finally, Eq. (19) follows by plugging Eq. (86) into Eq. (16) and integrating the Gaussian integral with respect to x.

Finally, the evolution of the wave packet, reported in Eq. (19), is eventually computed by integrating the expression of $\hat{U}(t)\,|x\rangle$ over the variable x with respect to the Gaussian measure

$$
\exp\left[-(x-a)^2/(2\sigma^2)^2 + i(x-a)k\right](\pi\sigma^2)^{-1/4} . \tag{88}
$$

The convergence is ensured by requiring that $\mathrm{Re}(\gamma) > 0$. In our description we have $\xi^\pm \in i\mathbb{R}$ and real ξ^z, so that it is useful to define $\xi^\pm \equiv i\xi_i^\pm$ with $\xi_i^\pm \in \mathbb{R}$, which, together with Eq. (20), leads to

$$\mathrm{Re}(\gamma) = \frac{\sigma^2 e^{\xi^z}}{\sigma^4 + (\xi_i^-)^2} \geq 0\,. \tag{89}$$

A better understanding of the behavior of $\mathrm{Re}(\gamma)$ can be achieved by considering the following real-valued auxiliary variables:

$$\begin{aligned} X &\equiv \xi_i^- e^{-\xi^z/2}\,, \\ Y &\equiv e^{-\xi^z/2}\,. \end{aligned} \tag{90}$$

These variables evolve according to the harmonic equations with time-dependent frequency given in Eq. (6),

$$\begin{aligned} \ddot{X}(t) + \Omega^2(t)X(t) = 0\,, \\ \ddot{Y}(t) + \Omega^2(t)Y(t) = 0\,, \end{aligned} \tag{91}$$

with initial conditions $X(0) = 0$, $\dot{X}(0) = -m^{-1}$, $Y(0) = 1$ and $\dot{Y}(0) = 0$. These new variables allow one to write $\mathrm{Re}(\gamma) = \sigma^2(\sigma^4 Y^2 + X^2)^{-1}$, making it apparent that $\mathrm{Re}(\gamma(t)) = 0$ if $X(t)$ or $Y(t)$ are infinite. In either case, $\beta(t) = [\sigma^2 Y(t) - iX(t)]^{-1}$ which multiplies Eq. (19), vanishes, i.e., $|\psi(t)\rangle = 0$. Accordingly, the convergence of the Gaussian integral in Eq. (19) is guaranteed by the fact that $\mathrm{Re}(\gamma) \geq 0$ and that whenever $\mathrm{Re}(\gamma) = 0$ the whole $|\psi\rangle$ vanishes.

As a last remark, we point out that the introduction of the variables X and Y in Eq. (90) explains why in the case of the harmonic oscillator, in which the $\xi^{+,-,z}$ are found to be periodically divergent according to the Eqs. (25), there are no divergences in the expectation values of Eqs. (26). In fact, these values depend on a well-behaved combination of the $\xi^{+,-,z}$, satisfying an harmonic equation with constant frequency but different initial conditions.

D Derivations of the Dyson Series

In order to prove the equivalence of the Dyson series for the quantum quartic oscillator and the asymptotic expansion of $\hat{U}(t)$ around the harmonic case according to Eq. (32), we begin by calculating the second variation of the time-evolution operator of the harmonic oscillator, which will be useful to determine the first-order correction according to Eq. (32), namely

$$\hat{U}^{(1)}(t) \equiv i\frac{\lambda}{4}\int_0^t \mathrm{d}s \left.\frac{\delta^2 \hat{U}_S[\phi]}{\delta\phi(s_1)\delta\phi(s_2)}\right|_{\substack{\phi=0 \\ s_1=s_2=s}}\,. \tag{92}$$

This expression involves the first functional derivative of $\hat{U}_S$, given by

$$\frac{\delta \hat{U}_S[\phi]}{\delta\phi(s)} \equiv G^{(1)}(s|t)\hat{U}_S[\phi]\,, \tag{93}$$

where $G^{(1)}$ is explicitly computed by exploiting the $SU(2)$ commutation relations (10) of the S operators and $\hat{U}_S$, leading to

$$G^{(1)}(s) \equiv \left[\frac{\delta\xi^+}{\delta\phi} - \xi^+\frac{\delta\xi^z}{\delta\phi} - (\xi^+)^2 e^{-\xi^z}\frac{\delta\xi^-}{\delta\phi}\right]\hat{S}^+ + \left(\frac{\delta\xi^z}{\delta\phi} + 2\xi^+ e^{-\xi^z}\frac{\delta\xi^-}{\delta\phi}\right)\hat{S}^z + \frac{\delta\xi^-}{\delta\phi}e^{-\xi^z}\hat{S}^-\,, \tag{94}$$

where the parametric dependence on the final time t is understood.

Similarly, the second order functional derivative $\delta^2\hat{U}_S[\phi]/\delta\phi(s_1)\delta\phi(s_2)$, required to be symmetric under the exchange $s_1 \longleftrightarrow s_2$, can be expressed as

$$\frac{\delta^2\hat{U}_S[\phi]}{\delta\phi(s_1)\delta\phi(s_2)} = \left[G^{(1)}(s_1)G^{(1)}(s_2) + G^{(2)}(s_1,s_2)\right]\hat{U}_S[\phi]\,, \tag{95}$$

where $G^{(2)}(s_1,s_2)$ is found to be

$$\begin{aligned} G^{(2)} \equiv & \left\{\xi^+_{1,2} - \xi^+\xi^z_{1,2} - \frac{1}{2}\left(\xi^z_1\xi^+_2 + \xi^z_2\xi^+_1\right) - \xi^+ e^{-\xi^z}\left[\xi^-_1\xi^+_2 + \xi^-_2\xi^+_1 - \frac{\xi^+}{2}\left(\xi^z_1\xi^-_2 + \xi^z_2\xi^-_1\right) + \xi^+\xi^-_{1,2}\right]\right\}\hat{S}^+ \\ & + e^{-\xi^z}\left[\xi^-_{1,2} - \frac{1}{2}\left(\xi^z_1\xi^-_2 + \xi^z_2\xi^-_1\right)\right]\hat{S}^- \\ & + \left\{\xi^z_{1,2} + e^{-\xi^z}\left[\xi^-_1\xi^+_2 + \xi^-_2\xi^+_1 - \xi^+\left(\xi^z_1\xi^-_2 + \xi^z_2\xi^-_1\right) + 2\xi^+\xi^-_{1,2}\right]\right\}\hat{S}^z\,; \end{aligned} \tag{96}$$

in order to streamline the formulas, the subscripts $\{1,2\}$ above are used to denote the functional differentiation with respect to $\phi(s_1)$ and $\phi(s_2)$, i.e., $\delta\xi(t)/\delta\phi(s_1) \equiv \xi(s_1|t) = \xi_1$. By taking the functional derivative of Eqs. (13) we obtain a system of differential equations for the first functional derivatives $\xi_{1,2}$, namely

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}\xi^+(s|t) + 2\frac{\xi^+}{m}\xi^+(s|t) &= 2\delta(t-s)\,, \\ i\frac{\mathrm{d}}{\mathrm{d}t}\xi^z(s|t) + \frac{2}{m}\xi^+(s|t) &= 0\,, \\ i\frac{\mathrm{d}}{\mathrm{d}t}\xi^-(s|t) - \frac{e^{\xi^z}}{m}\xi^z(s|t) &= 0\,, \end{aligned} \tag{97}$$

with initial conditions $\xi(s|s) = 0$ for $t \leq s$, reflecting the fact that we assume an Itô-like discretization in deriving Eqs. (13) [20]. The solution to these equations reads

$$\begin{aligned} \xi^+(s|t) &= -\theta(t-s)\,2i\,\exp\left\{\frac{2i}{m}\int_s^t \mathrm{d}\tau\,\xi^+(\tau)\right\}, \\ \xi^z(s|t) &= \theta(t-s)\frac{2i}{m}\int_s^t \mathrm{d}\tau\,\xi^+(s|\tau)\,, \\ \xi^-(s|t) &= -\theta(t-s)\frac{i}{m}\int_s^t \mathrm{d}\tau\,\xi^z(s|\tau)\,e^{\xi^z(\tau)}\,. \end{aligned} \tag{98}$$

For $\phi = 0$ they reduce to

$$\begin{aligned} \left.\frac{\delta\xi^+(t)}{\delta\phi(s)}\right|_{\phi=0} &= -i\theta(t-s)\,2\frac{\cos^2(\omega s)}{\cos^2(\omega t)}\,, \\ \left.\frac{\delta\xi^z(t)}{\delta\phi(s)}\right|_{\phi=0} &= \theta(t-s)\frac{4}{m\omega}\cos^2(\omega s)\left[\tan(\omega t) - \tan(\omega s)\right], \\ \left.\frac{\delta\xi^-(t)}{\delta\phi(s)}\right|_{\phi=0} &= -i\theta(t-s)\frac{2}{m^2\omega^2}\cos^2(\omega s)\left[\tan(\omega t) - \tan(\omega s)\right]^2\,. \end{aligned} \tag{99}$$

As expected, the functional derivative $\xi^+(s|t)$ vanishes for $t < s$, as a consequence of the fact that the differential equation at time t does not depend on the realizations of ϕ at later times, reflecting the causality of the problem. Following the same line of reasoning as before, the second functional derivatives can be computed directly from their differential equations and can be expressed in terms of first functional derivative according to

$$\begin{aligned} \xi^{+}_{1,2} &= \frac{\xi^{z}_{1}\xi^{+}_{2} + \xi^{z}_{2}\xi^{+}_{1}}{2}\,, \\ \xi^{z}_{1,2} &= -e^{-\xi^{z}(t)}\left(\xi^{-}_{1}\xi^{+}_{2} + \xi^{-}_{2}\xi^{+}_{1}\right), \\ \xi^{-}_{1,2} &= \frac{\xi^{z}_{1}\xi^{-}_{2} + \xi^{z}_{2}\xi^{-}_{1}}{2}\,, \end{aligned} \tag{100}$$

and they are non-zero only if $t > \max(s_1, s_2)$. Moreover, by plugging Eqs. (100) into Eq. (96), we get $G^{(2)}(s_1, s_2) = 0$, such that the only contributing term in the functional derivative in Eq. (95) is

$$G^{(1)}(s)|_{\phi=0} = -i\theta(t-s)\,x_0^2\left[\cos(\omega(t-s))\frac{\hat{x}}{x_0} - \sin(\omega(t-s))x_0\,\hat{p}\right]^2, \tag{101}$$

which finally yields

$$\begin{aligned} \left.\frac{\delta^2 \hat{U}_S[\phi]}{\delta\phi(s_1)\delta\phi(s_2)}\right|_{\substack{\phi=0,\\ s_1=s_2=s}} &= -\theta(t-s)\,x_0^4\left[\cos(\omega(t-s))\frac{\hat{x}}{x_0} - \sin(\omega(t-s))x_0\,\hat{p}\right]^4 \hat{U}_0(t) \\ &= -\theta(t-s)\hat{U}_0(t-s)\,\hat{x}^4\,\hat{U}_0(s)\,, \end{aligned} \tag{102}$$

where the time ordering $t > s$ arises naturally from the fact that the equation for ξ^+ depends linearly on ϕ. Collecting the above results of Eqs. (92) and (102), the first-order correction to $\hat{U}(t)$ reads

$$\hat{U}^{(1)} = -i\frac{\lambda}{4}\,\hat{U}_0(t)\int_0^t \mathrm{d}s\,\hat{U}_0^{\dagger}(s)\,\hat{x}^4\,\hat{U}_0(s)\,, \tag{103}$$

which is nothing but the first order term in the Dyson series [56]. This can be seen by noticing that the time-evolution operator in the Schrödinger picture $\hat{U}(t)$ can be written in terms of the interaction time-evolution operator in the interaction picture $\hat{U}_I$ as

$$\hat{U}_I(t) \equiv \hat{U}_0^{\dagger}(t)\,\hat{U}(t)\,\hat{U}_0(0)\,, \tag{104}$$

so that, since $\hat{U}_0(0) = \mathbb{I}$,

$$\hat{U}(t) = \hat{U}_0(t)\,\hat{U}_I(t)\,. \tag{105}$$

The fact that $G^{(2)} = 0$ makes all functional derivatives of order larger than one to depend only on $G^{(1)}$. This allows one to easily generalize the result above to an arbitrary order n, leading to

$$\left.\frac{\delta^{2n}\hat{U}_S[\phi]}{\delta\phi(s_1)\cdots\delta\phi(s_{2n})}\right|_{\substack{\phi=0\\ s_1=s_2=t_1\\ \cdots\\ s_{2n-1}=s_{2n}=t_n}} = [G^{(1)}(t_1)]^2[G^{(1)}(t_2)]^2\cdots[G^{(1)}(t_n)]^2\hat{U}_0(t)\,, \tag{106}$$

with $t > t_n > t_{n-1} > \cdots > 0$. Using the expression for $G^{(n)}$, this readily yields

$$\hat{U}^{(n)}(t) = \left(-i\frac{\lambda}{4}\right)^n \hat{U}_0(t)\int_0^t \mathrm{d}t_n\int_0^{t_n}\mathrm{d}t_{n-1}\cdots\int_0^{t_2}\mathrm{d}t_1\,\hat{x}^4(t_n)\cdots\hat{x}^4(t_1)\,, \tag{107}$$

where $\hat{x}^4(t) = \hat{U}_0^{\dagger}(t)\,\hat{x}^4\,\hat{U}_0(t)$. Equation (107) is precisely the n-th order contribution to the Dyson series in the Schrödinger picture.

E Semiclassical limit

In this section we provide details of the computation of the semiclassical approximation for the propagator. The stationary path in Eq. (43) is computed considering the first functional derivative of Eq. (36), namely

$$\begin{aligned}\frac{\delta S_{\mathrm{HO}}(y_f,t|y_i,0)}{\delta\phi(\tau)} &= \frac{\delta}{\delta\phi(\tau)}\frac{m}{2}\int_0^t \mathrm{d}s\left[\dot{y}^2(s)-\Omega^2(s)y^2(s)\right] \\ &= -y^2(\tau)+m\int_0^t \mathrm{d}s\left[\dot{y}(s)\dot{y}_1(\tau|s)-\Omega^2(s)y(s)y_1(\tau|s)\right] \\ &= -y^2(\tau)+m\left[\dot{y}(t)y_1(\tau|t)-\dot{y}(0)y_1(\tau|0)\right] \\ &\qquad -m\int_0^t \mathrm{d}s\, y_1(\tau|s)\left[\ddot{y}(s)+\Omega^2(s)y(s)\right] \\ &= -y^2(\tau),\end{aligned} \tag{108}$$

where $y_1(\tau|s)\equiv\delta y(t)/\delta\phi(\tau)$, $y_f=\sqrt{\lambda}\,x_f$, $y_i=\sqrt{\lambda}\,x_i$, and the t dependence is understood. In the third line of Eq. (108) we integrated by parts the right-hand side and finally we exploited Eq. (38) and the explicit expression of $y_1(\tau|s)$

$$\begin{aligned}y_1(\tau|s) =& -\left[\frac{\xi_1^-(\tau|t)}{\xi^-(t)}+\frac{\xi_1^z(\tau|s)}{2}\right]y(\tau|t) \\ &+\frac{e^{-\xi^z(s)/2}}{\xi^-(t)}\left[y_f\,e^{-\xi^z(t)/2}\left(\frac{\xi^-(\tau)}{2}\xi_1^z(\tau|t)+\xi_1^-(\tau|s)\right)+y_i\left(\xi_1^-(\tau|t)-\xi_1^-(\tau|s)\right)\right],\end{aligned} \tag{109}$$

where $\xi_1(t_1|t_2)$ is null if $t_1\geq t_2$, so that $y_1(\tau|t)=y_1(\tau|0)=0$. Note that by taking the functional derivative of Eq. (38) computed along the stationary solution $\bar{\phi}$ in Eq. (43) one has

$$\ddot{\bar{y}}_1(t_2|t_1)+\left[\omega^2+\frac{\bar{y}^2(t_1)}{m}\right]\bar{y}_1(t_2|t_1)+\frac{2}{m}\bar{y}(t_1)\delta(t_1-t_2)=0\,, \tag{110}$$

with boundary conditions $\bar{y}_1(t_2|0)=\bar{y}_1(t_2|0)=0$. The path integral in the second line of Eq. (45) can be expressed in terms of the determinant of the operator $H(t_1,t_2)$, defined in Eq. (46) as

$$H(t_1,t_2)\equiv\delta(t_1-t_2)+\frac{1}{2}\frac{\delta^2 S_{\mathrm{HO}}(y_f,t|y_i,0)}{\delta\phi(t_1)\delta\phi(t_2)}\bigg|_{\bar{\phi}_k}=\delta(t_1-t_2)-\bar{y}(t_1)\bar{y}_1(t_2|t_1)\,, \tag{111}$$

where last equality follows from direct functional derivation of Eq. (108). The determinant of H can be evaluated by relying on the fact that this operator can be recast as the product of two operators whose determinant can be computed exactly. We start by defining the operators

$$\begin{aligned}O_1(t_1,t_2)&\equiv\delta(t_1-t_2)\frac{1}{\bar{y}(t_1)}\left[\frac{d^2}{dt_1^2}+\omega^2+\frac{\bar{y}^2(t_1)}{m}\right], \\ O_2(t_1,t_2)&\equiv\delta(t_1-t_2)\frac{1}{\bar{y}(t_1)}\left[\frac{d^2}{dt_1^2}+\omega^2+3\frac{\bar{y}^2(t_1)}{m}\right]=O_1(t_1,t_2)+2\,\delta(t_1-t_2)\frac{\bar{y}(t_1)}{m}\,.\end{aligned} \tag{112}$$

It follows that, given Eqs. (112), one can recast $H(t_1,t_2)$ in Eq. (111) as

$$H(t_1,t_2)=\int_0^t \mathrm{d}\tau\, O_2(t_1,\tau)O_1^{-1}(\tau,t_2)=\delta(t_1-t_2)+2\frac{\bar{y}(t_1)}{m}O_1^{-1}(t_1,t_2)\,, \tag{113}$$

where the inverse operator satisfies the following relation

$$\int_0^t d\tau\, O_1(t_1, \tau) O_1^{-1}(\tau, t_2) = \delta(t_1 - t_2). \tag{114}$$

By plugging Eq. (110) in the inverse operator definition in Eq. (114) we identify $O_1^{-1}(t_1, t_2)$ as

$$O_1^{-1}(t_1, t_2) = -\frac{m}{2} \bar{y}_1(t_2|t_1). \tag{115}$$

Hence, we have proved Eq. (111) to be true. It then follows that the path integral evaluates to

$$\begin{aligned} &\int \mathcal{D}\varphi \exp\left\{-\int_0^t dt_1 \int_0^t dt_2\, \varphi(t_1) H(t_1, t_2) \varphi(t_2)\right\} \\ &= \left[\det\left(O_2 O_1^{-1}\right)\right]^{-1/2} = \sqrt{\frac{\det O_1}{\det O_2}} = \sqrt{\frac{f(t)}{F(t)}}, \end{aligned} \tag{116}$$

where in the last relation we exploited the fact that $f(t)$ and $F(t)$ are respectively proportional to the determinant of O_1 and O_2 with the same proportionality constant, according to Eq. (39) and the fact that the determinant of a product of operators is given by the product of the determinants of the individual operators.

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