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Specific-heat exponent and modified hyperscaling in the 4D random-field Ising model

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Abstract. We report a high-precision numerical estimation of the critical exponent α of the specific heat of the random-field Ising model in four dimensions. Our result $\alpha = 0.12(1)$ indicates a diverging specific-heat behavior and is consistent with the estimation coming from the modified hyperscaling relation using our estimate of θ via the anomalous dimensions η and $\bar{\eta}$. Our analysis benefited from a high-statistics zero-temperature numerical simulation of the model for two distributions of the random fields, namely a Gaussian and Poissonian distribution, as well as recent advances in finite-size scaling and reweighting methods for disordered systems. An original estimate of the critical slowing down exponent z of the maximum-flow algorithm used is also provided.

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1. Introduction

The random-field Ising model (RFIM) is one of the archetypal disordered systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], extensively studied due to its theoretical interest, as well as its close connection to experiments in hard [16, 17, 15, 18] and soft condensed matter systems [19]. Its beauty is that the mixture of random fields and the standard Ising model creates rich physics and leaves many still unanswered problems. The existence of an ordered ferromagnetic phase for the RFIM, at low temperature and weak disorder, followed from the seminal paper of Imry and Ma [1], when the space dimension is greater than two ($D > 2$) [8, 9, 10, 20, 21]. This has provided us with a general qualitative agreement on the sketch of the phase boundary, separating the ordered ferromagnetic phase from the high-temperature paramagnetic one [22, 23, 24, 25, 26, 27].

Although nowadays the view that the phase transition of the RFIM is of second order, irrespective of the form of the random-field distribution and for all values of the disorder strength [28], there are still some puzzling behavior that remains contradictory. One of the main problems, that we also consider in the current work, refers to the scaling behavior of the specific heat and the corresponding value of the critical exponent α ; the latter having severe implications for the scaling relations [28, 29, 30, 31, 32, 33, 34].

The RFIM Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle xy \rangle} S_x S_y - \sum_x h_x S_x, \quad (1)$$

with the spins $S_x = \pm 1$ occupying the nodes of a hyper-cubic lattice in space dimension D with nearest-neighbor ferromagnetic interactions J and h_x independent random magnetic fields with zero mean and dispersion σ . In the present work we consider the Hamiltonian (1) on a $D = 4$ hyper-cubic lattice with periodic boundary conditions and energy units $J = 1$. Our random fields h_x follow either a Gaussian (\mathcal{P}_G), or a Poissonian (\mathcal{P}_P) distribution

$$\mathcal{P}_G(h, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{h^2}{2\sigma^2}}; \quad \mathcal{P}_P(h, \sigma) = \frac{1}{2|\sigma|} e^{-\frac{|h|}{\sigma}}, \quad (2)$$

where $-\infty < h < \infty$ and σ our single disorder-strength control parameter.

Note that the fact that we consider the model exactly at $T = 0$ is no restriction, because the temperature is an irrelevant perturbation [15, 8, 9, 10, 20]. As it is well-established, in order to describe the critical behavior of the model one needs two correlation functions, namely the connected and disconnected propagators, $C_{xy}^{(\text{con})}$ and $C_{xy}^{(\text{dis})}$. At the critical point and for large r (r : distance between x and y), they decay as

$$C_{xy}^{(\text{con})} \equiv \frac{\partial \overline{\langle S_x \rangle}}{\partial h_y} \sim \frac{1}{r^{D-2+\eta}}; \quad C_{xy}^{(\text{dis})} \equiv \overline{\langle S_x \rangle \langle S_y \rangle} \sim \frac{1}{r^{D-4+\bar{\eta}}}, \quad (3)$$

where the $\langle \dots \rangle$ are thermal mean values as computed for a given realization, a sample, of the random fields $\{h_x\}$. Over-line refers to the average over the samples. The correlation length corresponding to $C^{(\text{con})}$ is denoted by $\xi^{(\text{con})}$ (we use $\xi^{(\text{dis})}$ for $C^{(\text{dis})}$, respectively).

The relationship between the anomalous dimensions η and $\bar{\eta}$ has been hotly debated in the literature, especially with respect to the so-called two-exponent scaling scenario $\bar{\eta} = 2\eta$ [11, 12] and its implications on the violation of the hyperscaling exponent θ via

$$\theta = 2 - \bar{\eta} + \eta = 2 - \eta + \Delta_{\eta, \bar{\eta}}, \quad (4)$$

where $\Delta_{\eta, \bar{\eta}} = 2\eta - \bar{\eta}$. In fact, the original prediction $\Delta_{\eta, \bar{\eta}} = 0$ by Schwartz and coworkers [11, 12] has been recently questioned by Tarjus and coworkers [35, 36, 37]. These latter authors suggested that rare events, neglected in Refs. [11, 12], spontaneously break supersymmetry at the intermediate dimension $D_{\text{int}} \approx 5.1$. For $D > D_{\text{int}}$ replica predictions [5] hold: supersymmetry is valid and $\bar{\eta} = \eta$. For $D < D_{\text{int}}$, instead, there are three independent critical exponents. Recent high-precision numerical simulations by the current authors [38] provided clear cut evidence that $\Delta_{\eta, \bar{\eta}} > 0$ in favor of the three-exponent scaling scenario and the spontaneous supersymmetry breaking [35, 36] at some $D_{\text{int}} > 4$. Some of the results of Ref. [38] will be used to corroborate the current analysis.

The rest of the paper is laid out as follows: In the next Section 2 we describe the finite-size scaling methods used and in Section 3 we briefly outline the numerics performed. In Section 4 we present our results for: (i) the critical exponent α of the specific heat, Section 4.1, modified hyperscaling, Section 4.2, and the dynamical aspects of the algorithm used in Section 4.3. Section 5 concludes the article.

2. Finite-size scaling

In the present work we are mostly interested in investigating the controversial issue of the specific heat of the RFIM. The specific heat of the RFIM can be experimentally measured [18] and is of great theoretical importance. Yet, it is well known that it is one of the most intricate thermodynamic quantities to deal with in numerical simulations, even when it comes to pure systems. For the RFIM, Monte Carlo methods at $T > 0$ have been used to estimate the value of its critical exponent α , but were restricted to rather small systems sizes and have also revealed many serious problems, i.e., severe violations of self averaging [39, 40]. A better picture emerged throughout the years from $T = 0$ computations, proposing estimates of $\alpha \approx 0$ [28, 30]. However, even by using the same numerical techniques, but different scaling approaches, some inconsistencies have been recorded in the literature [31, 30, 32, 33]. The origin of these inconsistencies [31, 41] can be traced back to the fact that the specific heat contains, according to simulations, a regular term as well as a singular term with a large exponent of the order of -0.6 . This can be interpreted as a specific-heat exponent with a negative value $\alpha/\nu \simeq -0.6$. Another possibility is a very small value of the specific-heat exponent (which then gives a near constant term) and a corrections-to-scaling term $L^{-\omega}$. Indeed, in three dimensions, ω has a value close to 0.6 [28]. In order to distinguish between these two scenarios, one needs to use a hyperscaling relation in order to get another direct determination of α as a function of the magnetic critical exponents that were measured directly in our

simulations. For the case of the 3D RFIM, it was then shown that $\alpha \simeq 0$ [31, 41]. For the 3D RFIM, there exists also experiments on diluted antiferromagnetic systems, which are expected to be in the same universality class as the random-field magnets [4, 6], and which suggested a logarithmic divergence of the specific heat [18]. For the 4D RFIM that we consider in the present study, we will also compare our direct measurements with the hyperscaling relation to check the validity of our results.

The specific heat can be also estimated using ground-state calculations and applying thermodynamic relations employed by Hartmann and Young [31] and Middleton and Fisher [30]. The method relies on studying the singularities in the bond-energy density E_J [42]. This bond energy density is the first derivative $\partial E/\partial J$ of the ground-state energy with respect to the random-field strength, say σ [30, 31]. The derivative of the sample averaged quantity \overline{E}_J with respect to σ then gives the second derivative with respect to σ of the total energy and thus the sample-averaged specific heat C . The singularities in C can also be studied by computing the singular part of \overline{E}_J , as \overline{E}_J is just the integral of C with respect to σ . The general finite-size scaling form assumed is that the singular part of the specific heat behaves as

$$C_s \sim L^{\alpha/\nu} \tilde{C} [(\sigma - \sigma_c)L^{1/\nu}]. \quad (5)$$

Thus, one may estimate α by studying the behavior of \overline{E}_J at $\sigma = \sigma_c$ [30]. The computation from the behavior of \overline{E}_J is based on integrating the above scaling equation up to σ_c , which gives a dependence of the form

$$\overline{E}_J(L, \sigma_c) = A + BL^{(\alpha-1)/\nu}, \quad (6)$$

with A and B non universal constants.

Applying standard finite-size scaling methods via equation (6) would require an a priori knowledge of the “exact” value of the critical field strength σ_c . Since what we always have at hand is a numerical estimate this approach would then incorporate a degree of approximation coming from the uncertainty in the value of σ_c . To overcome these scaling problems, we implemented here a variant [29] of the original quotients method, also known as phenomenological renormalization [43, 44, 45]. This method allows for a particularly transparent study of corrections to scaling, that up to now have been considered as the Achilles’ heel in the study of the $D \geq 3$ random-field problem [28]. The general idea is to compare observables computed in pair of lattices $(L, 2L)$. We start imposing scale-invariance by seeking the L -dependent critical point: the value of σ such that $\xi_{2L}/\xi_L = 2$, i.e., the crossing point for ξ_L/L (see also figure 1). Now, for dimensionful quantities O , scaling in the thermodynamic limit as $\xi^{x_O/\nu}$, we consider the quotient $Q_O = O_{2L}/O_L$ at the crossing. Thus, we have:

$$Q_O^{(\text{cross})} = 2^{x_O/\nu} + \mathcal{O}(L^{-\omega}), \quad (7)$$

where x_O/ν , and the scaling-corrections exponent ω are universal.

Since $\alpha - 1$ is negative, equation (6) is dominated by the non-divergent back ground A , forcing us to modify the standard phenomenological renormalization. We get rid of A by considering three lattice sizes in the following sequence: $(L_1, L_2, L_3) = (L, 2L, 4L)$

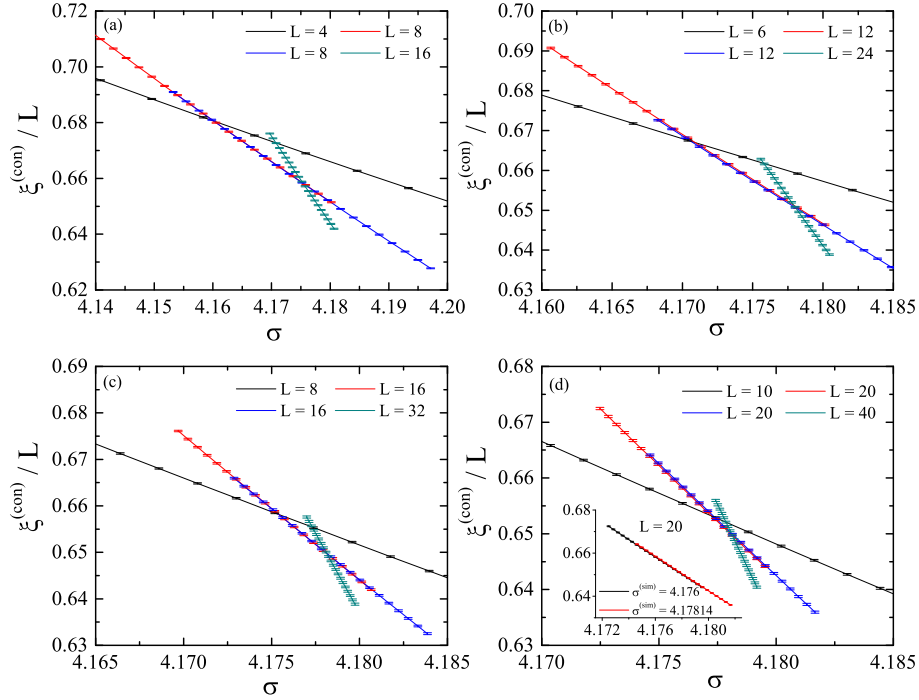


Figure 1. Connected correlation length in units of the system size L versus σ for the case of the Gaussian RFIM and for the sequence of lattice sizes used in the application of the modified quotients method in equations (8) and (9). Data taken from Ref. [38]. The inset of panel (d) shows $\xi^{(\text{con})}/L$ for $L = 20$ and two distinct simulations corresponding to different simulation values, $\sigma^{(\text{sim})}$, and different sets of 10^7 random-field realizations.

(see figure 1 for an instructive illustration of the three-lattice variant of the quotients method based on the crossings of the $\xi^{(\text{con})}/L$). We generalize equation (7) by taking now the quotient of the differences at the crossings of the pairs $(L, 2L)$ and $(2L, 4L)$, respectively

$$\hat{Q}_O = \frac{(O_{4L} - O_{2L})|_{(\xi_{4L}/\xi_{2L})=2}}{(O_{2L} - O_L)|_{(\xi_{2L}/\xi_L)=2}}. \quad (8)$$

Applying this formula to the bond energy we obtain

$$\hat{Q}_{E_J}^{(\text{cross})} = 2^{(\alpha-1)/\nu} + \mathcal{O}(L^{-\omega}). \quad (9)$$

3. Numerical simulations

Our numerical simulations were described in Ref. [38]. We therefore only recall here the crucial details. In order to apply the above formula (9), we simulated systems with linear sizes up to $L = 40$, which provided us with four sets of three-lattice size sequences, as shown in figure 1(a) - (d). For each set (L, σ) and for each field distribution, Gaussian and Poissonian, we simulated 10^7 independent random-field realizations (exceeding previous relevant studies [46, 47] by a factor of 10^3 at least). Suitable generalized

Table 1. Effective critical exponent ratio $(\alpha - 1)/\nu$ using a three lattice-size variant $(L_1, L_2, L_3) = (L, 2L, 4L)$, see equation (9), of the original quotients method.

Crossing Point	(L_1, L_2, L_3)	$(\alpha - 1)/\nu$
$G^{(\text{con})}$	(4, 8, 16)	-0.697(6)
	(6, 12, 24)	-0.848(6)
	(8, 16, 32)	-0.912(8)
	(10, 20, 40)	-0.941(12)
$G^{(\text{dis})}$	(4, 8, 16)	-0.954(4)
	(6, 12, 24)	-0.975(4)
	(8, 16, 32)	-0.990(5)
	(10, 20, 40)	-0.998(4)
$P^{(\text{con})}$	(4, 8, 16)	-1.031(7)
	(6, 12, 24)	-1.014(9)
	(8, 16, 32)	-1.002(11)
	(10, 20, 40)	-0.998(8)
$P^{(\text{dis})}$	(4, 8, 16)	-1.205(4)
	(6, 12, 24)	-1.129(3)
	(8, 16, 32)	-1.084(6)
	(10, 20, 40)	-1.057(12)

fluctuation-dissipation formulas and reweighting extrapolations have been applied and facilitated our analysis, as exemplified in Ref. [29]. A comparative illustration with respect to the errors induced by the reweighting method and the disorder averaging process is shown in the inset of panel (d) of figure 1 for the universal ratio $\xi^{(\text{con})}/L$ of an $L = 20$ Gaussian RFIM and for two sets of simulations, as outlined in the figure. Clearly, this latter accuracy test serves in favor of the proposed scheme. The calculation of the ground states of the RFIM was based on the well-established mapping [48, 49, 50, 51, 28, 30, 31, 32, 33, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64] to the maximum-flow problem [65, 66, 67]. This is a combinatorial optimization problem which can be solved exactly using efficient, i.e., polynomial-time, algorithms. The most efficient network flow algorithm used to solve the RFIM is the push-relabel algorithm of Tarjan and Goldberg [68]. We prepared our own C version of the algorithm, involving a modification that removes the source and sink nodes, reducing memory usage and also clarifying the physical connection [54, 55]. Additionally, the computational efficiency of our algorithm has been increased via the use of periodic global updates [54, 55].

4. Results

4.1. Specific-heat exponent

As we applied the quotients method at both the crossings of the connected and disconnected correlation length over the system size, i.e., $\xi^{(\text{con})}/L$ and $\xi^{(\text{dis})}/L$, typically

the sets of simulations were doubled for each system size as the crossings between the connected and disconnected cases varied. Note also, that throughout the main manuscript we have used the notation $Z^{(x)}$, where Z denotes the distribution, i.e., G for Gaussian and P for Poissonian, and the superscript x refers to the connected (con) and disconnected (dis) type of the universal ratio $\xi^{(x)}/L$.

Our results for the effective exponent ratio $(\alpha - 1)/\nu$ are given in Table 1 and their extrapolation is shown in figure 2. The solid lines in figure 2 show a joint polynomial fit, second order in $L^{-\omega}$, where ω was set to the estimated value $\omega = 1.30$ [38]. The extrapolated value for the exponent ratio is $(\alpha - 1)/\nu = -1.01(1)$ and is marked by the filled star at $L^{-\omega} = 0$. The quality of the fit is excellent, $\chi^2/\text{DOF} = 6.5/7$, where DOF denoted the degrees of freedom in the fit, and is also given in the plot. Using now our previous estimate $\nu = 0.8718(58)$ for the critical exponent of the correlation length [38], simple algebra (and error propagation) gives the value

$$\alpha = 0.12(1), \tag{10}$$

for the critical exponent of the specific heat.

Middleton [46], using also ground-state simulations and the scaling of the bond energy at the candidate critical field value estimated in his analysis, $\sigma_c = 4.179$, proposed a value of $\alpha = 0.26(5)$. As it will be shown below, this value is not in agreement with modified hyperscaling. Noteworthy, another relevant ground-state numerical work of the 4D RFIM by Hartmann [47] was unable to conclude whether the numerical data are better described by a logarithmic divergence or by an algebraic behavior with a small exponent.

4.2. Modified hyperscaling

A crucial point in the scaling theory of the RFIM is the hyperscaling violation exponent θ [recall equation (4)] and its relation to the critical exponent α of the specific heat via the modified hyperscaling relation

$$(D - \theta)\nu = 2 - \alpha. \tag{11}$$

Numerous works have tried in the past to reconcile the numerical estimates of α with those stemming from the above equation (11) but have mostly failed [17, 31, 56, 40].

We now compare our numerical estimate $\alpha = 0.12(1)$ with the one obtained via equations (4) and (11) and our previous estimates for the anomalous dimension $\eta = 0.1930(13)$ and the two-exponent difference $\Delta_{\eta, \bar{\eta}} = 2\eta - \bar{\eta} = 0.0322(23)$ [38]. In particular, plugging these values to equation (4) we obtain

$$\theta = 1.839(3), \tag{12}$$

which is compatible to the value $\theta = 1.82(7)$ of Middleton [46], but more accurate. On a second step, we can use this value of θ and ν to get a further estimate for the critical exponent of the specific heat:

$$\alpha = 2 - \nu(D - 2 + \eta - \Delta_{\eta, \bar{\eta}}). \tag{13}$$

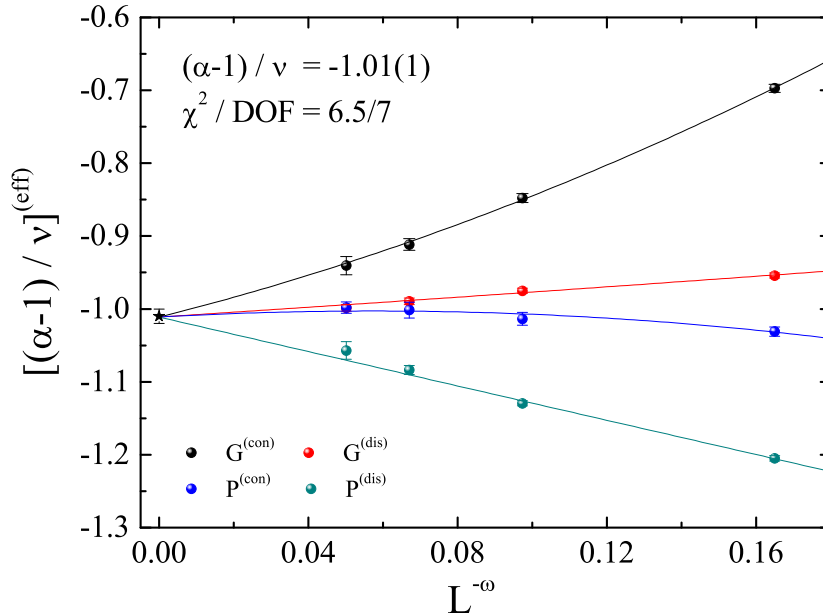


Figure 2. Infinite limit-size extrapolation of the effective exponent ratio $(\alpha - 1)/\nu$.

The error propagation for the above equation is rather simple (because the final error is basically dominated by the uncertainty in ν). We finally obtain $\alpha = 0.12(1)$, in excellent agreement to the numerical estimate based on the finite-size scaling analysis described above.

4.3. Critical dynamic slowing down

Finally, we present some computational aspects of the implemented push-relabel algorithm and its performance on the study of the RFIM. Although its generic implementation has a polynomial time bound, its actual performance depends on the order in which operations are performed and which heuristics are used to maintain auxiliary fields for the algorithm. Even within this polynomial time bound, there is a power-law critical slowing down of the push-relabel algorithm at the zero-temperature transition [48, 54]. A direct way to measure the dynamics of the algorithm is to examine the dependence of the running time, measured by the number of push-relabel operations, on system size L [54, 55]. Such an analysis has already been performed for the 3D version of the model [29] and a FIFO queue implementation. We present here results for the 4D version of the model using our scaling approach within the quotients method and numerical data for both Gaussian and Poissonian random-field distributions for system sizes up to $L = 60$. In figure 3 we plot the effective exponent values of z at the various crossing points considered, as indicated. The solid line is a joint, second-order in $L^{-\omega}$, polynomial fit for system sizes $L \geq 14$ with $\chi^2/\text{DOF} = 12/15$. The obtained estimate for

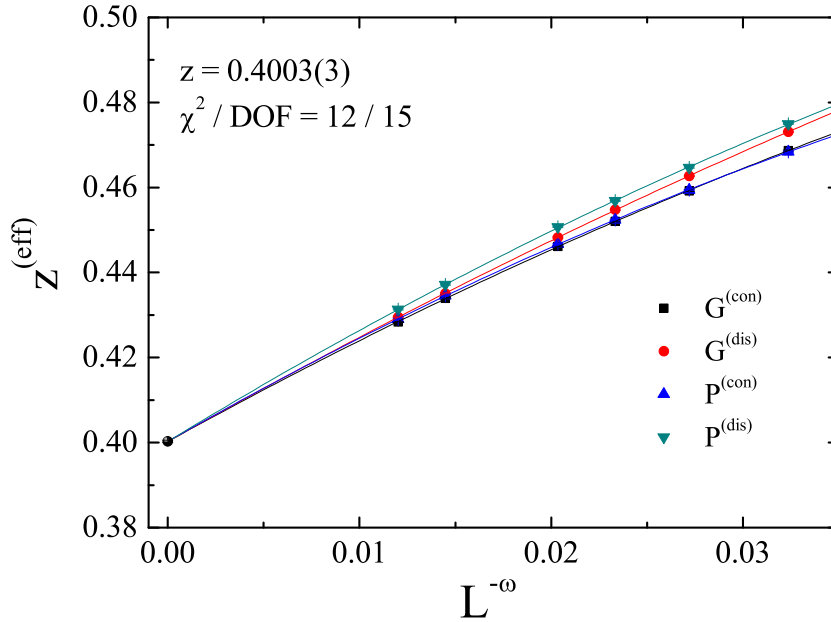


Figure 3. Infinite limit-size extrapolation of the effective exponent z of the push-relabel algorithm.

the dynamic critical exponent z is 0.4003(3), as marked by the filled circle at $L^{-\omega} = 0$.

5. Conclusions

Using extensive numerical simulations at zero temperature and efficient finite-size scaling methods we presented a high-precision numerical estimate of the critical exponent α of the specific heat of the random-field Ising model in four dimensions. Our result is fully consistent with the estimation coming from the modified hyperscaling relation, giving us full credit on the numerical and scaling approach implemented. Finally, we provided an original estimate of the critical slowing down exponent of the maximum-flow algorithm used.

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