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A new method for determining the effective length factor of columns in partially braced frames on elastic supports

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Abstract. The effective buckling length factor is an important parameter in the elastic buckling analysis of steel structures. The present article aims at developing a new method that allows the determination of the buckling factor values for frames. The novelty of the method is that it considers the interaction between the bracing and the elastic supports for asymmetrical frames in particular. The approach consists in isolating a critical column within the frame and evaluating the rotational and translational stiffness of its restraints to obtain the critical buckling load. This can be achieved by introducing, through a dimensionless parameter φ_i, the effects of coupling between the axial loading and bending stiffness of the columns, on the classical stability functions. Subsequently, comparative and parametric studies conducted on several frames are presented for assessing the influence of geometry, loading, bracing, and support conditions of the frame columns on the value of the effective buckling length factor K. The results show that the formulas recommended by different approaches can give rather inaccurate values of K, especially in the case of asymmetric frames. The expressions used refer solely to local stiffness distributions, and not to the overall behavior of the structure.

Keywords: effective length factor; critical load; elastic buckling; stability functions; instability.

1. Introduction

The overall stability of columns is extremely important in the structural design process of buildings. Research has been carried out in this field, but more work is needed in developing a better understanding how the stability of a structure is influenced by the phenomena of buckling, and the need for simpler methods to determine the effective length factor K. This construct allows calculating the elastic critical load using one single formula covering all boundary conditions, expressed by the following equation:

$$N = \frac{\pi^2 EI}{(kL)^2} \qquad (1)$$

Where, k represents the ratio between the effective length L_f and the actual length L of the column:

$$k = L_f/L \qquad (2)$$

For this reason, several researchers have carried out extensive investigation on the stability of frames and on the effective length concept as well. In this context, Bridge and Fraser (1987) proposed an iterative procedure for evaluating the effective length, based on linearized stability functions that consider the presence of axial forces in the restraining elements. In addition, they suggested a procedure to improve the effective buckling length factor alignment chart that consider the positive and negative rotational stiffness ratios in the case of braced frames. The authors indicated that it is quite possible to encounter situations where the value of the effective buckling length factor K can be greater than 1. These situations correspond to negative rotational stiffness values in the case of braced frames, Kalochairetis and Gantes (2012), Corsi *et al.* (2020).

Cheong-Siat-Moy (1999) developed expressions to estimate the buckling length of partially braced frames; he also considered the individual element and the overall behavior of the structural system to achieve an accurate buckling analysis (1986). The following formulation was derived for a column which is laterally restrained at the top, with a lateral restraint S, and restrained rotationally at both ends:

$$Sh^3/EI_c = \beta^4(C-D)/(JC+FD) \tag{3}$$

where,
$\beta = \pi/K$
$C = R_1\beta(1-\cos\beta)+ \beta^2 \mathrm{Sin}\beta$
$D = (R_1+R_2)\beta + R_1R_2\ \mathrm{Sin}\beta$
$F = R_1(\cos\beta -1) - \beta^2$
$J = R_1+R_2+ \beta^2+(R_1R_2 \mathrm{Sin}\ \beta)/\beta$

R_1 and R_2 are factor related to the joint stiffness ratio G, as given in equation (4) below.

However, it is worth noting that the above approach of Cheong-Siat-Moy (1999) does not consider the coupling effect between the rotational and translational stiffness.

Aristizabal-Ochoa (1994a, 1994b and 1996) adopted the effective length concept on the stability of frames. The author derived an analytical relationship for the evaluation of the effective buckling length factor for braced, partially braced and unbraced columns. He also examined the influence of the uniformly distributed axial load on the effective length of braced columns and partially braced columns (1994). It is worth noting that the proposed formulas can be applied to frames with rigid and semi-rigid connections. However, these approaches do not consider the coupling effects between rotational and translational flexibilities.

Gantes and Mageirou (2005) developed an analytical expression similar to that used by Eurocode 3 (2005) for calculating the effective buckling length factor. The results obtained by the approach proposed for three-story sway frames are in perfect agreement with those given by the finite element method. However, the application of design codes such as Eurocode 3 (2005) and Ruiz *et. al.* (1999) leads to significant inaccuracies. In this context, Mageirou and Gantes (2006) developed a simplified approach for the assessment of the buckling load of multi-story frames with semi-rigid connections. In the study, the rotational and translational boundary conditions, including a semi-rigid connection, were taken into consideration. However, the effect of geometric irregularity and that of the loading of vertical elements belonging to the same levels were not considered.

On the other side, Girgin et al. (2006) proposed a practical method to estimate the approximate buckling load values for regular and irregular frames, using fictitious loads to achieve lateral displacements. It is interesting to note that all the approach based on the fictitious load better results compared with the isolated subassembly approach. It is also interesting to cite the work developed by Raftoyiannis (2005) who attempted to assess the effects of connection semi-rigidity and bracing on the elastic critical buckling load of single-story frames. Adman and Saidani (2013) investigated the effect of the boundary conditions at the ends of columns. This was achieved by means of a general stability criterion resulting from the solution of stability equations, while considering the end conditions of the column in terms of the non-dimensional translational and rotational restraint indices, and their coupling effects. Webber *et al.* (2015) suggested two improvements to the effective length method. In the first one, the axial load in neighboring columns was included in the calculation of the effective length. In the second, a modification of the effective length ratio was proposed so that the buckling load of adjacent columns can be taken into consideration. The results obtained by the approach proposed are in good agreement with those given by the finite element method. However, the application of design codes such as Eurocode 3 (2005) leads to inaccuracies. This suggests that the recommendations in the code are limited and may need revisiting.

Li et al. (2016) established a new simplified instability analysis method that considers the vertical interaction effects of all the columns at the different stories. The idealized column model used in the classical G-factor method, as recommended by AISC (2010), was extended to include the columns of all floors and their restraining beams. However, the assumption taken for the rotations at the near and far ends of a beam cannot be fulfilled in all cases.

With the joint stiffness ratio G, is defined by:

$$G = \sum(E_c I_c/L_c)/(E_b I_b/L_b) \tag{4}$$

where E = Young's modulus, I = moment of inertia of the cross-section, L = length of the member, and the summations are over all the columns (beams) intersecting at the joint. The subscripts b and c indicate beam and column, respectively.

Teh and Gilbert (2016) developed an accurate buckling model that allows determining the effective length of a column subjected to intermediate gravity loads for design applications based on a 2D second-order elastic analysis. It is also appropriate to cite the study conducted by Konstantakopoulos *et al.* (2012) for assessing the effects of steel columns with varying cross sections and subjected to axial forces applied concentrically or eccentrically, on the critical buckling load. The governing equation of the problem was solved using the Galerkin method. Several studies were undertaken over the last few decades on analysing the stability of frames and on the effective length concept as well, Zheng *et al.* (2021), Tain *et al.* (2021), Farajian *et al.*

(2021). Rezaiee-Pajand *et al.* (2016) conducted a study in which they considered the effect of varying the cross sections of columns for the determination of the critical load of a simple frame with semi-rigid connections. Ihaddoudène *et al.* (2017) developed a simple mechanical model for determining the elastic buckling load for braced and unbraced multi-story planar frames with semi-rigid connections. Then, based on the stability functions, the stiffness matrix of the proposed model was deduced.

Furthermore, Tian *et al.* (2016, 2017) developed a simplified analytical approach that considers the effects of internal axial loads with the aim of assessing the elastic stability of columns, through a relationship between the internal axial loads and end loads. Using the negative lateral stiffness bonding (Tong and Wang, 2006), the internal axial loads may be viewed as the equivalent end loads so that the critical buckling load can easily be determined using Euler's formula. Tian *et al.* (2020) then investigated the influence of the inclination angle of roofs on the values of the effective buckling length factors while taking into consideration the vertical interaction effects of all columns in different stories. Krzysztof and Alexandre de Macêdo (2021) studied the stability of a very slender thin-walled column with a box section. Stress and strain analysis was carried out using the finite element method, the technical stability theory and Euler's theory. The three methods were used to determine the critical compressive load.

Gunaydin and Aydin (2019) completed a work in which they took into consideration the second order (P-Δ) and (P-δ) effects on the effective length factors in the case of multi-story frames; they also put forward tables of values of the effective length factors based on the developed analytical expressions. In the same context, a decomposition method for frames was developed by Ma and Xu (2020) to assess the stability of partially braced multi-story frames. The proposed method breaks down a frame into several individual stories and assesses the lateral stiffness of each story by means of a stability approach that is based on explicit closed-form solutions. Likewise, Sun *et al.* (2020) proposed modifying the values of the effective length factors based on experimental measurements of supports rotation stiffness of large-size and high-strength steel angle sections (LHS). In addition, a column curve was proposed to predict the LHS member bearing capacity. Simplified procedures for determining approximate values for the effective buckling length factor K of irregular, hence asymmetrical, frames are developed in the literature (Girgin et al. 2006; Slimani et al. 2018; Mahini (2022). However, these approaches only applicable to a specific type of frames and do not consider the coupling effects between of the variation in bracing and the column bases on the factor K, which limits the study. For example, Mahini's method (2022) is only applicable to frames with tapered members.

The above-mentioned studies did not consider the case of asymmetric and partially braced frames with elastic supports and where the coupling effects are taken into account. These aspects are dealt with in the present study, thus suggesting a new method for determining the effective buckling length factor K in different situations. First, the classical stability functions are extended by through a dimensionless parameter φ_i that consider the effects of coupling between the axial loading and bending stiffness of the columns, on the classical stability functions. Then, a parametric study is conducted to investigate the effect of various parameters on the factor K of columns. These parameters are: (i) geometry; (ii) loading; (iii) bracing; and (iv) support conditions of the frame columns. Finally, interaction between bracing and support conditions of the frame columns on the factor K is studied.

2. Analytical formulations

The following analysis refers to a rectangular frame ($1_b\,1_t\ 2_t\ 2_b$) resting on two supports represented by rotational springs (k_{s1b} and k_{s2b}), as shown in Fig. 1. The frame is braced and the bracing is represented by a lateral elastic support consisting of a spring at the top of the column (2_t). The spring has an axial rigidity k_Δ. In addition, the frame is subjected to axial loading from the upper floor. The columns and beam have different section properties.

The frame possesses the following characteristics:
Column (1): length $h_1 = \alpha_1 h$, moment of inertia $I_1 = \gamma_1 I$. Column (2): length $h_2 = \alpha_2 h$, moment of inertia $I_2 = \gamma_2 I$. Beam (1'): length $h'_1 = \alpha'_1 h$, moment of inertia $I'_1 = \gamma'_1 I$. The frame is subjected, at joints 1_t and 2_t, to vertical concentrated loads $\delta_1 P$ and $\delta_2 P$, applied along the centerlines of columns (1) and (2), respectively. The horizontal bar (1') is connected to the columns via rigid connections.

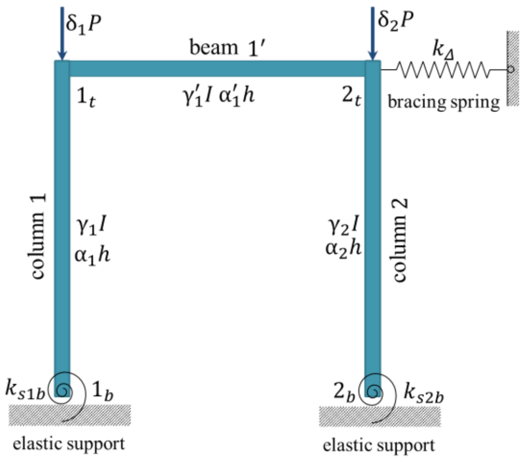

Fig. 1 Single rectangular frame resting on two rotating springs

The following parameters are defined:

$$k_{r0} = \frac{EI}{h}\ ;\ k_{\Delta 0} = \frac{EI}{h^3}\ ;\ \beta_0^2 = \frac{Ph}{k_{r0}}\ ;\ \varphi_i = \alpha_i\sqrt{\frac{\delta_i}{\gamma_i}}\ ;\ k_{ri} = \frac{EI_i}{h_i} = \frac{\gamma_i}{\alpha_i}k_{r0}\ ;$$

$$k_{\Delta i} = \frac{EI_i}{h_i^3} = \frac{\gamma_i}{\alpha_i^3}k_{\Delta 0}$$

Additionally, the following are defined:

$$\frac{\gamma_i}{\alpha_i} = a_i \; ; \quad \frac{\gamma_i}{\alpha_i^3} = b_i \quad \text{where, } k_{ri} = a_i k_{r0} \; ; \quad k_{\Delta_i} = b_i k_{\Delta 0}$$

The slope-deflection equations are used to define the moments (*M*) at the nodes in terms of rotations (θ) and displacements (Δ). For the different elements of the framework, these equations are expressed as follows:

Element 01:

$$M_{1b} = a_1 k_{r0} \left[C_1 \theta_{1b} + S_1 \theta_1 - (C_1 + S_1) \frac{\Delta}{\alpha_1 h} \right] = -k_{S_1 b} \theta_{1b} \tag{5.a}$$

$$M_{1t} = a_1 k_{r0} \left[C_1 \theta_1 + S_1 \theta_{1b} - (C_1 + S_1) \frac{\Delta}{\alpha_1 h} \right] \tag{5.b}$$

With: $\Delta = \Delta_2 - \Delta_1$ and $\Delta_1 = 0$

(C_i, S_i) are the expressions representing the modified stability functions for column (i); they are expressed as:

$$C_i = \frac{\varphi_i \beta_0 \sin(\varphi_i \beta_0) - (\varphi_i \beta_0)^2 \cos(\varphi_i \beta_0)}{2 - 2\cos(\varphi_i \beta_0) - \varphi_i \beta_0 \sin(\varphi_i \beta_0)} \tag{6.a}$$

$$S_i = \frac{(\varphi_i \beta_0)^2 - \varphi_i \beta_0 \sin(\varphi_i \beta_0)}{2 - 2\cos(\varphi_i \beta_0) - \varphi_i \beta_0 \sin(\varphi_i \beta_0)} \tag{6.b}$$

It is worth indicating that the effects of varying the loadings and geometries of the frame columns on the classical stability functions (Livesley and Chandler (1956), Slimani *et al.* (2018)) were introduced using the parameter φ_i.

Element 02:

$$M_{2b} = a_2 k_{r0} \left[C_2 \theta_{2b} + S_2 \theta_2 - (C_2 + S_2) \frac{\Delta}{\alpha_2 h} \right] = -k_{S_2 b} \theta_{2b} \tag{7.a}$$

$$M_{2t} = a_2 k_{r0} \left[C_2 \theta_2 + S_2 \theta_{2b} - (C_2 + S_2) \frac{\Delta}{\alpha_2 h} \right] \tag{7.b}$$

Element 01´:

$$M_{1'l} = a_1' k_{r0} \left[4\theta_1 + 2\theta_2 \right] \tag{8.a}$$

$$M_{1'R} = a_1' k_{r0} \left[4\theta_2 + 2\theta_1\right] \quad (8.b)$$

With $a_1' = \dfrac{\gamma_1'}{\alpha_1'}$

The relations between the rotations θ_1, θ_{1b}, θ_2 and θ_{2b} should be determined by adding up the moments at the nodes as follows:

The moment equilibrium about node 1_b to calculate rotation θ_{1b} is given by:

$M_{1b} + k_{S_{1b}}\theta_{1b} = 0$

This allows obtaining:

$$\theta_{1b} = -\frac{S_1}{\overline{C_1}}\theta_1 + \frac{(C_1 + S_1)}{\overline{C_1}}\phi_1 \quad (9)$$

With: $S_{r1b} = \dfrac{k_{S_{1b}}}{k_{r0}}$; $\overline{C_1} = C_1 + \dfrac{S_{r1b}}{a_1}$; $\phi_0 = \dfrac{\Delta}{h}$; $\phi_1 = \dfrac{1}{\alpha_1}\phi_0$; $a_1 = \dfrac{\gamma_1}{\alpha_1}$

The moment equilibrium about node 2_b to calculate rotation θ_{2b} is expressed as:

$M_{2b} + k_{S_{2b}}\theta_{2b} = 0$

$$\theta_{2b} = -\frac{S_2}{\overline{C_2}}\theta_2 + \frac{(C_2 + S_2)}{\overline{C_2}}\phi_2 \quad (10)$$

With $S_{r2b} = \dfrac{k_{S_{2b}}}{k_{r0}}$; $\overline{C_2} = C_2 + \dfrac{S_{r2b}}{a_2}$; $\phi_2 = \dfrac{1}{\alpha_2}\phi_0$; $a_2 = \dfrac{\gamma_2}{\alpha_2}$

Based on the results previously obtained, it becomes easy to express the slope-deflection equations for the two columns as follows:

Element 01:

$$M_{1b} = a_1 k_{r0}\left[S_1(1 - \frac{C_1}{\overline{C_1}})\theta_1 - (C_1 + S_1)(1 - \frac{C_1}{\overline{C_1}})\frac{\phi_0}{\alpha_1}\right] \quad (11.a)$$

$$M_{1t} = a_1 k_{r0}\left[(C_1 - \frac{S_1^2}{\overline{C_1}})\theta_1 - (C_1 + S_1)(1 - \frac{S_1}{\overline{C_1}})\frac{\phi_0}{\alpha_1}\right] \quad (11.b)$$

Element 02:

$$M_{2b} = a_2 k_{r0}\left[S_2(1-\frac{C_2}{\overline{C}_2})\theta_2 - (C_2+S_2)(1-\frac{C_2}{\overline{C}_2})\frac{\phi_0}{\alpha_2}\right] \tag{12.a}$$

$$M_{2t} = a_2 k_{r0}\left[(C_2-\frac{S_2^2}{\overline{C}_2})\theta_2 - (C_2+S_2)(1-\frac{S_2}{\overline{C}_2})\frac{\phi_0}{\alpha_2}\right] \tag{12.b}$$

Since there are three unknowns θ_1, θ_2 and ϕ_0, three equilibrium equations are then needed. These are obtained through the summation of the moments at nodes 1_t and 2_t and the shearing forces at the upper ends of the columns.

When node 1_t is in equilibrium:

$$M_{1t} + M_{1'l} - M_1 = 0 \tag{13}$$

When relations (11.b) and (8.a) are taken into account, relation (13) takes the following form:

$$\left[\frac{C_1\overline{C_1}-S_1^2}{\overline{C_1}} + 4\left(\frac{a_1'}{a_1}\right)\right]\theta_1 + 2\left(\frac{a_1'}{a_1}\right)\theta_2 - \left[\frac{(C_1+S_1)(\overline{C_1}-S_1)}{\alpha_1\overline{C_1}}\right]\varphi_0 = \Omega_1 \tag{14}$$

with $\Omega_1 = \dfrac{M_1}{a_1 k_{r0}}$

The equilibrium equation at node 2_t is:

$$M_{2t} + M_{1'R} - M_2 = 0 \tag{15}$$

Similarly, when relations (12.b) and (8.b) are taken into consideration, relation (15) may be expressed in the form:

$$2\left(\frac{a_1'}{a_2}\right)\theta_1 + \left[\frac{C_2\overline{C_2}-S_2^2}{\overline{C_2}} + 4\left(\frac{a_1'}{a_2}\right)\right]\theta_2 - \left[\frac{(C_2+S_2)(\overline{C_2}-S_2)}{\alpha_2\overline{C_2}}\right]\varphi_0 = \Omega_2 \tag{16}$$

with $\Omega_2 = \dfrac{M_2}{a_2 k_{r0}}$

Furthermore, considering equilibrium with respect to the displacement Δ allows writing:

$$H_1 - T_{1t} - T_{2t} - k_\Delta \Delta = 0 \tag{17}$$

The expressions for the shear forces T_{1t} and T_{2t} are given, respectively, as:

$$T_{1t} = -\frac{(M_{1t}+M_{1b})}{\alpha_1 h} - \frac{\delta_1 P\Delta}{\alpha_1 h} \tag{18}$$

$$T_{2t} = -\frac{(M_{2t} + M_{2b})}{\alpha_2 h} - \frac{\delta_2 P \Delta}{\alpha_2 h} \tag{19}$$

When relations (11.a) and (11.b) are taken into account, relation (18) becomes:

$$T_{1t} = -\left[\frac{a_1 k_{r0}}{\alpha_1 h}(C_1 + S_1)\left(1 - \frac{S_1}{\bar{\bar{C}}_1}\right)\theta_1 - \frac{a_1 k_{r0}}{\alpha_1^2 h}(C_1 + S_1)\left(2 - \frac{(C_1 + S_1)}{\bar{C}_1}\right)\phi_0 + \frac{\delta_1 P \Delta}{\alpha_1 h}\right] \tag{20}$$

Likewise, when relations (12.a) and (12.b) are taken into account, relation (19) becomes:

$$T_{2t} = -\left[\frac{a_2 k_{r0}}{\alpha_2 h}(C_2 + S_2)\left(1 - \frac{S_2}{\bar{\bar{C}}_2}\right)\theta_2 - \frac{a_2 k_{r0}}{\alpha_2^2 h}(C_2 + S_2)\left(2 - \frac{(C_2 + S_2)}{\bar{C}_2}\right)\phi_0 + \frac{\delta_2 P \Delta}{\alpha_2 h}\right] \tag{21}$$

Now, if relations (20) and (21) are used in relation (17), the expression below is obtained:

$$\begin{aligned} &-(C_1 + S_1)\left(1 - \frac{S_1}{\bar{\bar{C}}_1}\right)\frac{a_1}{\alpha_1}\theta_1 - (C_2 + S_2)\left(1 - \frac{S_2}{\bar{\bar{C}}_2}\right)\frac{a_2}{\alpha_2}\theta_2 \\ &+\left[(C_1 + S_1)\left(2 - \frac{(C_1 + S_1)}{\bar{C}_1}\right)\frac{a_1}{\alpha_1^2} + (C_2 + S_2)\left(2 - \frac{(C_2 + S_2)}{\bar{C}_2}\right)\frac{a_2}{\alpha_2^2} + \frac{k_\Delta}{k_{\Delta 0}} - \beta_0^2\left(\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2}\right)\right]\phi_0 = H\frac{h}{k_{r0}} \end{aligned} \tag{22}$$

The quantity β_0 is a dimensionless load parameter associated with the global stability analysis. Finally, using Eqs. (14), (16) and (22), the following system of equations are obtained:

$$\begin{aligned} &\left[\frac{C_1\bar{C}_1 - S_1^2}{\bar{C}_1} + 4\left(\frac{a_1'}{a_1}\right)\right]\theta_1 + 2\left(\frac{a_1'}{a_1}\right)\theta_2 - \left[\frac{(C_1 + S_1)(\bar{C}_1 - S_1)}{\alpha_1 \bar{C}_1}\right]\varphi_0 = \Omega_1 \\ &2\left(\frac{a_1'}{a_2}\right)\theta_1 + \left[\frac{C_2\bar{C}_2 - S_2^2}{\bar{C}_2} + 4\left(\frac{a_1'}{a_2}\right)\right]\theta_2 - \left[\frac{(C_2 + S_2)(\bar{C}_2 - S_2)}{\alpha_2 \bar{C}_2}\right]\varphi_0 = \Omega_2 \\ &-(C_1 + S_1)\left(1 - \frac{S_1}{\bar{\bar{C}}_1}\right)\frac{a_1}{\alpha_1}\theta_1 - (C_2 + S_2)\left(1 - \frac{S_2}{\bar{\bar{C}}_2}\right)\frac{a_2}{\alpha_2}\theta_2 \\ &+\left[(C_1 + S_1)\left(2 - \frac{(C_1 + S_1)}{\bar{C}_1}\right)\frac{a_1}{\alpha_1^2} + (C_2 + S_2)\left(2 - \frac{(C_2 + S_2)}{\bar{C}_2}\right)\frac{a_2}{\alpha_2^2} + \frac{k_\Delta}{k_{\Delta 0}} - \beta_0^2\left(\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2}\right)\right]\phi_0 = H\frac{h}{k_{r0}} \end{aligned} \tag{23}$$

Therefore, the system of Eqs. (23) can be written in the following matrix form:

$$[\psi]X = B \tag{24}$$

where vectors X and B are given by Eqs. (25), and the matrix $[\psi]$ is represented the by expression (26) below.

$$X = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \phi_0 \end{pmatrix} \text{ and } B = \begin{pmatrix} \frac{M_1}{k_{r0}} \\ \frac{M_2}{k_{r0}} \\ \frac{Hh}{k_{r0}} \end{pmatrix} \tag{25}$$

Furthermore, the buckling equation of the system is obtained by putting the determinant of the unknown constants in Eqs. (24) as equal to zero. It is given as:

$$det[\psi] = 0 \tag{26}$$

$$[\psi] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ Sym & & k_{33} \end{bmatrix} \tag{27}$$

with

$$k_{11} = \left(C_1 - \frac{S_1^2}{C_1} \right) a_1 + 4a_1' \tag{28.a}$$

$$k_{12} = 2a_1' \tag{28.b}$$

$$k_{13} = -(C_1 + S_1)\left(1 - \frac{S_1}{C_1} \right) \frac{a_1}{\alpha_1} \tag{28.c}$$

$$k_{22} = \left(C_2 - \frac{S_2^2}{C_2} \right) a_2 + 4a_1' \tag{28.d}$$

$$k_{23} = -(C_2 + S_2)\left(1 - \frac{S_2}{C_2} \right) \frac{a_2}{\alpha_2} \tag{28.e}$$

$$k_{33} = (C_1 + S_1)\left(2 - \frac{(C_1 + S_1)}{\overline{C_1}}\right)\frac{a_1}{\alpha_1^2} + (C_2 + S_2)\left(2 - \frac{(C_2 + S_2)}{\overline{C_2}}\right)\frac{a_2}{\alpha_2^2} + \frac{k_\Delta}{k_{\Delta 0}} - \beta_0^2\left(\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2}\right) \tag{28.f}$$

As for the dimensionless critical load β_{cr}, it is obtained by solving Eqs. (26). Subsequently, the reference elastic critical buckling load of the frame under study may be calculated from:

$$P_{cr} = \beta_{cr}^2 \frac{EI}{h^2} = \frac{\pi^2 EI}{(Kh)^2} \tag{29}$$

The elastic critical buckling loads of columns (1) and (2) are given by the following formulas:

$$P_1 = \delta_1 P_{cr} \tag{30.a}$$

$$P_2 = \delta_2 P_{cr} \tag{30.b}$$

Eqs. (29) may be used to deduce the reference effective buckling length factor K by means of the relation:

$$K = \frac{\pi}{\beta_{cr}} \tag{31}$$

Therefore, the effective buckling length factors of columns (1) and (2) may be expressed as:

$$K_1 = \frac{K}{\varphi_1} \tag{32.a}$$

$$K_2 = \frac{K}{\varphi_2} \tag{32.b}$$

3. Comparison with other numerical and analytical approaches

In what follows is a comparison of the values of the effective buckling length factor K obtained using different approaches. The first approach consists in using the analytical method proposed in the present study, while the second one is based on a numerical analysis using structural analysis software, i.e. Autodesk Robot Structural Analysis (2013). This second approach consists in solving an eigenvalue problem by considering the classical stiffness matrix of an element and associating it with the geometrical stiffness matrix. Looking at the other approaches, it was decided to develop the methods of Le Messurier (1977), Lui (1992), Smyrell (1994) and Raftoyiannis (2005). To do this, the example of a rectangular frame, as shown in Fig. 2 was adopted. Note that that frame (a) corresponds to the situation where the supports are fixed, while the frame (b) represents the situation where the supports are hinged.

Frames in which the effects of modifying the geometry and boundary conditions were considered are illustrated in the example considered. The values of factor K for columns (1) and

(2) as well as their deviations from their initial values obtained in this comparative study, are listed in Table 1 given below.

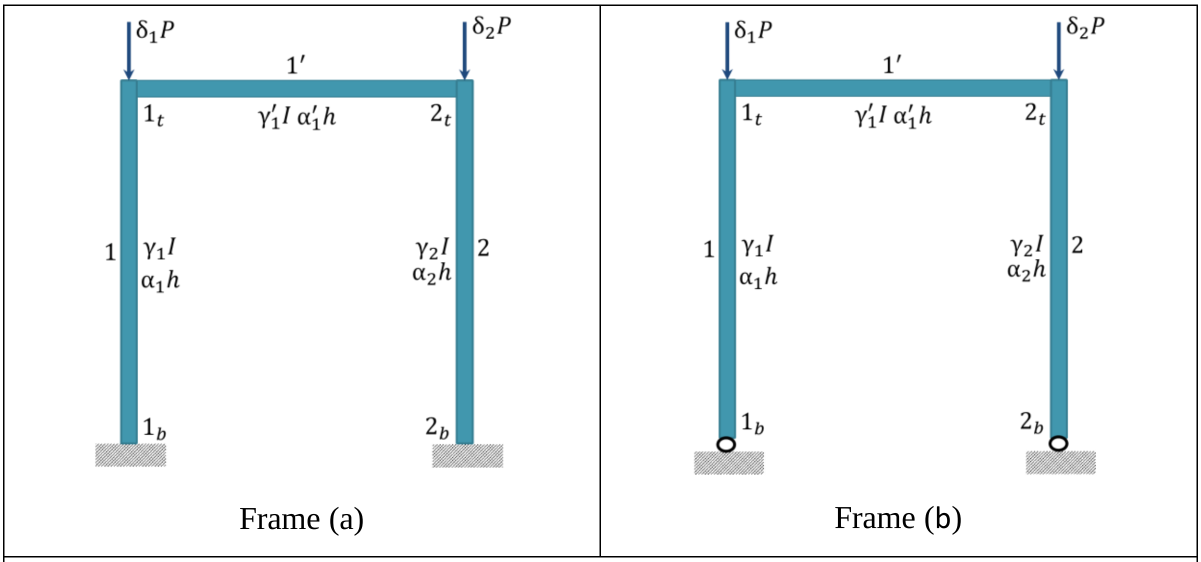

Fig. 2 Frames used in the comparative study

The results obtained by the present study are close to those given by the Robot structural analysis software (2013) (the largest deviation found is less than 2%). These results constitute a positive validation test for the study.

Furthermore, it was observed that, in all cases of symmetrical structures (cases 1 and 4), the results obtained by the approaches of LeMessurier (1977), Smyrell (1994) and Raftoyiannis (2005) are very close to those obtained by the present study, the largest deviation found is less than 1%. This is not the case where the frame is asymmetrical as the current method is more accurate.

For instance, in the case where $\gamma_1' = 0.5$ (case 4) for frame (b), Lui's method (1992) is comparatively less accurate than those used in the other approaches. Moreover, the results of the calculation of factor K, as given by the method of Raftoyiannis (2005), are in perfect agreement with those obtained by the present study, because the two approaches are based on a rigorous analytical formulation. However, unlike the present study, the effects of geometric irregularity and loading were not taken into account in Raftoyiannis' study (2005), in other words it is not applicable to asymmetrical frames.

Additionally, in cases 2 and 3 (asymmetric geometry), the results obtained by LeMessurier (1977) were relatively closer to those reported in the present study (the largest deviation found is less than 8%) than to those proposed by Lui (1992) and Smyrell (1994). These observations are more apparent in case 5 (asymmetric loading) where can we observe that the significant deviations between the present study and the method of LeMessurier (1977) are less than 1%. Cases 1 and 5 show that the method proposed by Smyrell (1994) gives similar results, except that it does not take into account the effect of the variation in loading, as shown by the results reported in Table 1.

Furthermore, it is noted that for case 6 (asymmetric geometry and loading), the results obtained for frame (a) by the present study and those obtained by Lui's method (1992) are close (maximum deviation is less than 2%). However, the results obtained for frame (b) show that LeMessurier's method (1977) is relatively more accurate than that proposed by Lui (1992).

Table 1 Comparison of the values of factor K

	Columns	LeMessurier (1977)	Lui (1992)	Smyrell (1994)	Raftoyiannis (2005)	Robot (2013)	Present study
Case (1) : $\delta_i = 1; \gamma_i = 1; \alpha_i = 1; \gamma_1^{'} = 1; \alpha_1^{'} = 1$							
Frame (a)	1	1.167 (0.95%)	1.170 (1.21%)	1.156 (0.00%)	1.156 (0.00%)	1.155 (0.09%)	**1.156**
	2	1.167 (0.95%)	1.170 (1.21%)	1.156 (0.00%)	1.156 (0.00%)	1.155 (0.09%)	**1.156**
Frame (b)	1	2.340 (0.52%)	2.379 (2.19%)	2.324 (0.17%)	2.328 (0.00%)	2.332 (0.17%)	**2.328**
	2	2.340 (0.52%)	2.379 (2.19%)	2.324 (0.17%)	2.328 (0.00%)	2.332 (0.17%)	**2.328**
Case (2) : $\delta_i = 1; \gamma_1 = 1; \gamma_2 = 0.5; \alpha_i = 1; \gamma_1^{'} = 1; \alpha_1^{'} = 1$							
Frame (a)	1	1.347 (2.05%)	1.490 (12.88%)	1.156 (12.42%)	/	1.325 (0.38%)	**1.320**
	2	0.891 (4.50%)	1.028 (10.18%)	1.082 (15.97%)	/	0.917 (1.71%)	**0.933**
Frame (b)	1	2.702 (1.39%)	2.905 (9.01%)	2.324 (12.80%)	/	2.687 (0.83%)	**2.665**
	2	1.776 (5.73%)	2.003 (6.32%)	2.189 (16.19%)	/	1.857 (1.43%)	**1.884**
Case (3) : $\delta_i = 1; \gamma_i = 1; \alpha_1 = 1; \alpha_2 = 0.5; \gamma_1^{'} = 1; \alpha_1^{'} = 1$							
Frame (a)	1	0.738 (2.89%)	0.723 (4.87%)	1.156 (52.11%)	/	0.755 (0.66%)	**0.760**
	2	1.626 (6.90%)	1.445 (5.00%)	1.276 (16.11%)	/	1.515 (0.39%)	**1.521**
Frame (b)	1	1.480 (7.90%)	1.214 (24.46%)	2.324 (44.62%)	/	1.602 (0.31%)	**1.607**
	2	3.342 (3.98%)	2.427 (24.49%)	2.620 (18.48%)	/	3.220 (0.19%)	**3.214**
Case (4) : $\delta_i = 1; \gamma_i = 1; \alpha_i = 1; \gamma_1^{'} = 0.5; \alpha_1^{'} = 1$							
Frame (a)	1	1.286 (0.55%)	1.303 (1.88%)	1.276 (0.23%)	1.279 (0.00%)	1.285 (0.47%)	**1.279**
	2	1.286 (0.55%)	1.303 (1.88%)	1.276 (0.23%)	1.279 (0.00%)	1.285 (0.47%)	**1.279**
Frame (b)	1	2.642 (0.27%)	2.759 (4.71%)	2.620 (0.57%)	2.635 (0.00%)	2.667 (1.21%)	**2.635**
	2	2.642 (0.27%)	2.759 (4.71%)	2.620 (0.57%)	2.635 (0.00%)	2.667 (1.21%)	**2.635**

Case (5) : $\delta_1 = 1; \delta_2 = 2; \gamma_i = 1; \alpha_i = 1; \gamma_1^{'} = 1; \alpha_1^{'} = 1$							
Frame (a)	1	1.429 (0.85%)	1.727 (21.88%)	1.156 (18.42%)	/	1.407 (0.71%)	**1.417**
	2	1.010 (0.80%)	1.221 (21.86%)	1.156 (15.37%)	/	1.002 (0.00%)	**1.002**
Frame (b)	1	2.866 (0.46%)	3.261 (14.30%)	2.324 (18.54%)	/	2.845 (0.28%)	**2.853**
	2	2.027 (0.45%)	2.306 (14.27%)	2.324 (15.16%)	/	2.022 (0.20%)	**2.018**
Case (6) : $\delta_1 = 2; \delta_2 = 1; \gamma_1 = 2; \gamma_2 = 1; \alpha_1 = 2; \alpha_2 = 1; \gamma_1^{'} = 2; \alpha_1^{'} = 2$							
Frame (a)	1	0.830 (11.11%)	0.760 (1.74%)	1.156 (54.75%)	/	0.755 (1.07%)	**0.747**
	2	1.660 (11.11%)	1.520 (1.74%)	1.156 (22.62%)	/	1.465 (1.94%)	**1.494**
Frame (b)	1	1.655 (8.88%)	1.270 (16.45%)	2.324 (52.89%)	/	1.535 (0.99%)	**1.520**
	2	3.310 (8.92%)	2.477 (18.49%)	2.324 (23.53%)	/	2.990 (1.61%)	**3.039**

4. Parametric study relating to the overall effective buckling length factor

Parametric studies were conducted on several frames in order to assess the influence of loading, bracing and column support conditions on the value of the overall effective buckling length factor K.

4. 1. Influence of bracing and column loading

A parametric study was performed on a frame with hinged supports (Fig. 2 (a)) and on another one with fixed supports (Fig. 2 (b)); the dimensionless stiffness of the bracing spring (R_Δ) of the frame and the loading of column (2) were simultaneously varied while those of column (1) were kept unchanged. In the present study, the length ratios and inertia moment ratios are taken as equal to 1 ($\alpha_1 = \alpha_2 = \alpha_1^{`} = \gamma_1 = \gamma_2 = \gamma_1^{`} = 1$).

4. 1. 1. Frames with hinged supports ($Sr_i = 0$)

It should be specified that this parametric study was carried out by considering the variation of the values of the dimensionless stiffness of the bracing spring (R_Δ) of the frame. For this, it was considered interesting to vary the stiffness ratio of the spring (R_Δ), from 0 to infinity; these situations correspond to unbraced and braced frames, respectively. Note also that this study was carried out by considering five combinations of load ratios, namely ($\delta_1 = 1$) with ($\delta_2 = 0.25$, 0.5, 1, 2, and 4), as shown in Fig. 3.

The numerical results are represented, in graphical form, in Fig. 4 which shows the variation of the overall effective buckling length factor K as a function of the stiffness ratio of the spring (R_Δ), for a frame with various loading situations ($\delta_2 = 0.25$, 0.5, 1, 2, and 4).

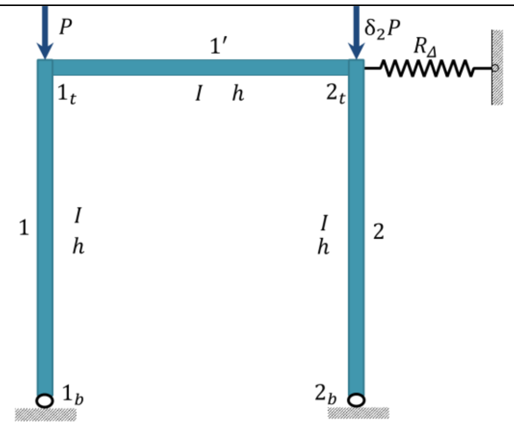

Fig. 3 Geometries and loadings of the frame under study

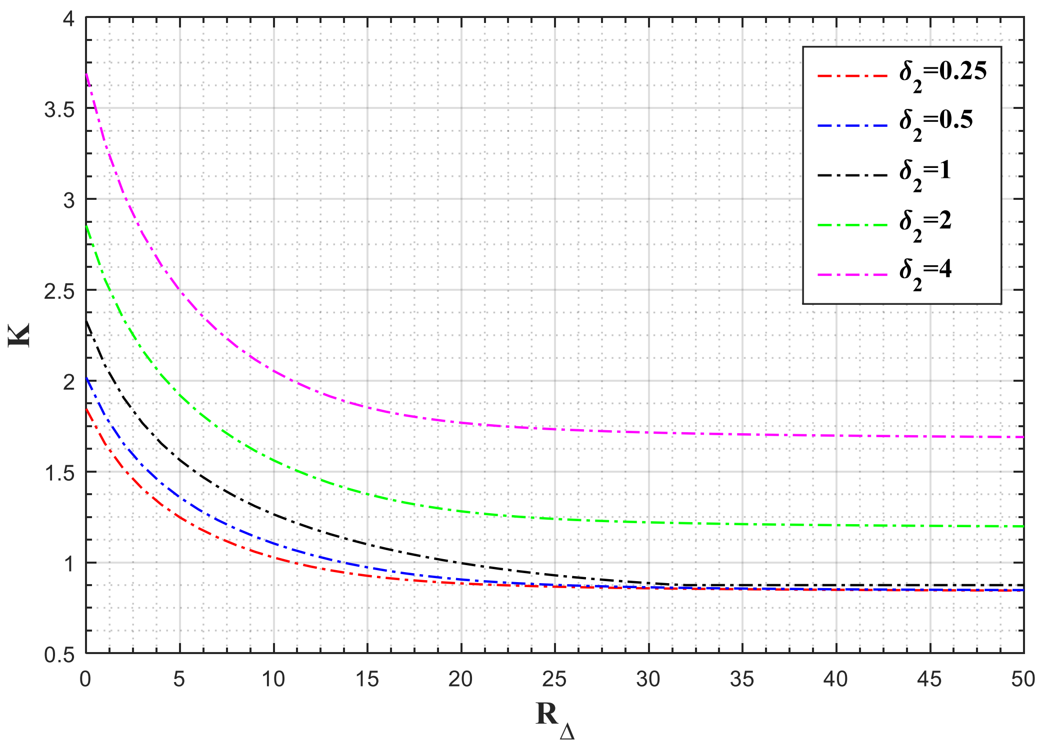

Fig. 4 Variation of K as a function of R_Δ, with $Sr_i = 0$.

4.1.2 Frame with fixed supports ($Sr_i = \infty$)

The parametric study carried out in the previous section, was repeated, but this time by varying the values of the bracing spring ratio (R_Δ) for a frame with fixed supports (Fig. 5).

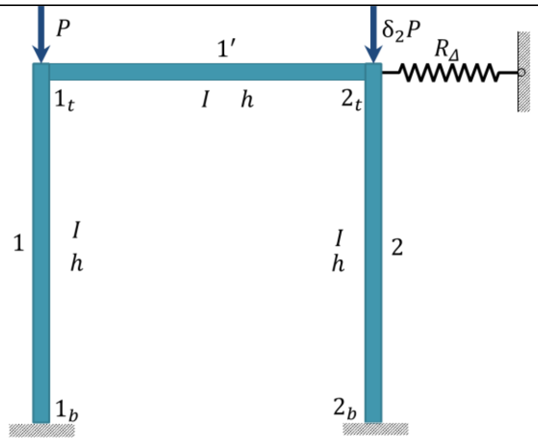

Fig. 5 Geometries and loadings of the frame under study

The numerical results obtained are represented, in graphical form, in Fig. 6 which shows the evolution of the overall effective buckling length factor K as a function of the spring stiffness ratio (R_Δ), for a frame under various loading situations (δ_2 = 0.25, 0.5, 1, 2, and 4).

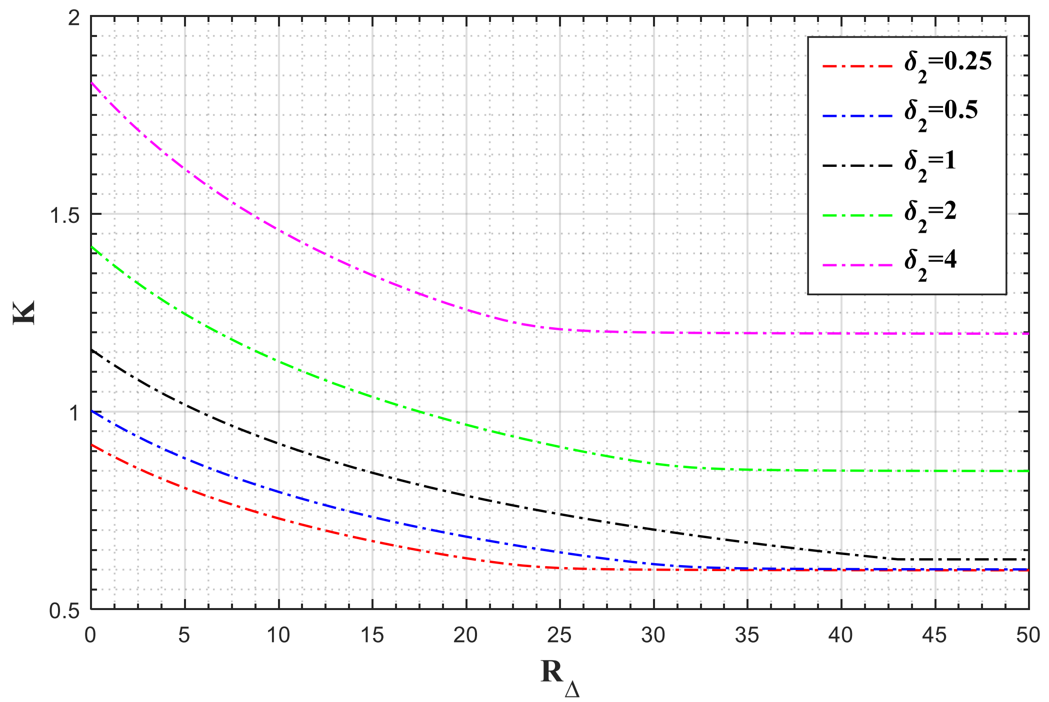

Fig. 6 Variation of K as a function of R_{Δ}, with $Sr_i = \infty$

4. 2. *Influence of bracing and column bases*

A parametric study was performed on a frame resting on two supports represented by rotational springs with dimensionless stiffness values (Sr_{1b} *and* Sr_{2b}). For this, the value of the stiffness ratio of the bracing spring (R_{Δ}) of the frame was varied, while changing the stiffness ratios of the rotational springs (Sr_{1b} *and* Sr_{2b}) in both columns. For this, it was decided to vary the stiffness ratio of the spring (R_{Δ}), from 0 to ∞, which corresponds to situations of unbraced and braced frames, respectively. In addition, the present study was carried out by considering seven combinations of the stiffness ratios of the rotational springs, namely $Sr_{1b} = Sr_{2b}$ = 0, 0.5, 1, 2, 4, 10, ∞). At this stage of the study, the length, inertia and loading ratios are taken equal to 1 ($\alpha_1 = \alpha_2 = \grave{\alpha}_1 = \gamma_1 = \gamma_2 = \grave{\gamma}_1 = 1$), as can be seen in Fig. 7.

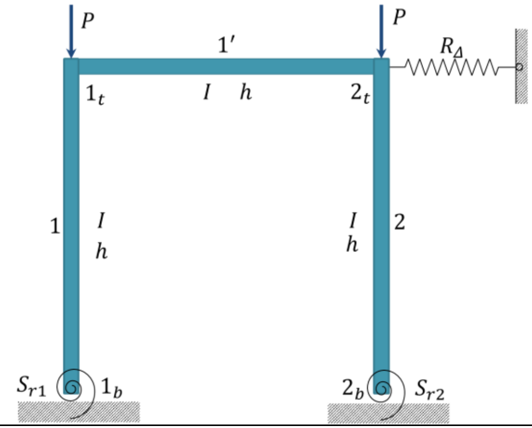

Fig. 7 Geometries and loadings of the frame under study.

Furthermore, the numerical results are represented, in graphical form, in Fig. 8 which shows the variation of the overall effective buckling length factor K as a function of the stiffness ratio of the spring (R_Δ), for a frame with various situations of the rotational springs (Sr_{1b} = Sr_{2b} = 0, 0.5, 1, 2, 4, 10, ∞).

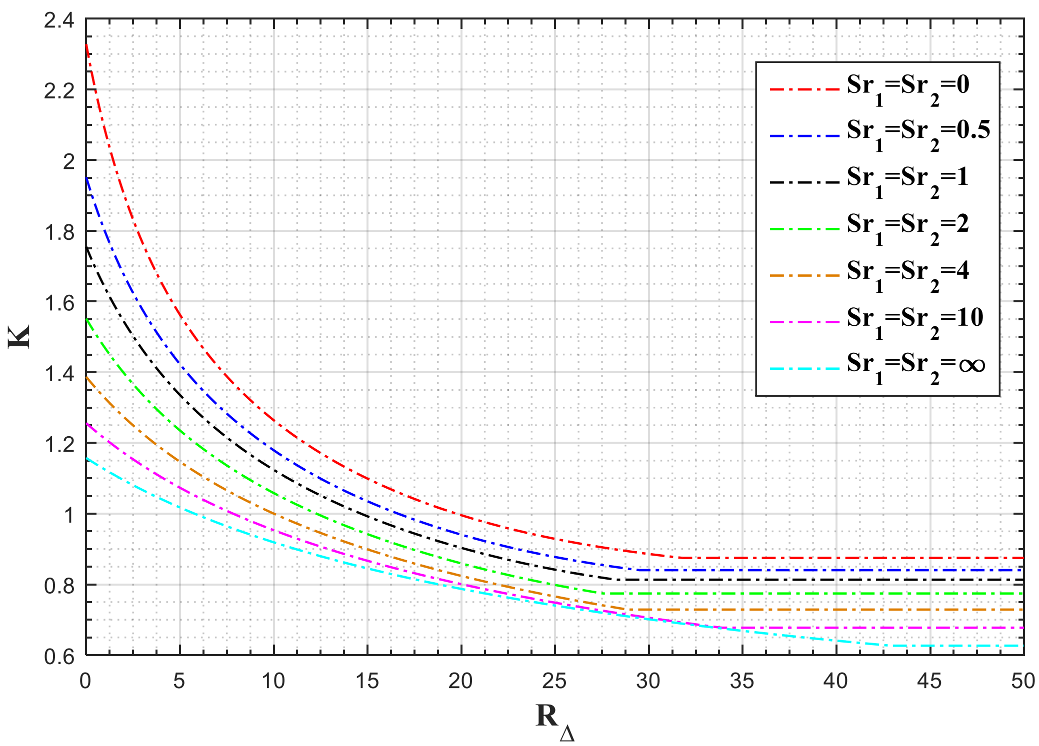

Fig. 8 Variation of K as a function of R_Δ with Sr_i varying from 0 to ∞

5. Discussion of the results

Regarding the influence of the bracing and column loading, for frames with hinged support, it was observed that the stiffness ratio of the bracing spring (R_Δ) increases, the overall effective buckling length factor K decreases, in a monotonous manner, going from a value close to 3.70 for a load ratio equal to 4 ($\delta_2 = 4$) until attaining asymptotic values related to the load ratios of column (2), as illustrated in Fig. 4 above. Afterwards, one may observe that, in this case, the factor K is highly sensitive to the load ratio (δ_2) when this load ratio is higher than 1 for a braced frame ($R_\Delta = \infty$) In addition, when the load ratio (δ_2) increases from 0.25 to 4, the factor K also increases by approximately 100% for a spring stiffness ratio $R_\Delta = 0$ (unbraced frame). For braced frame ($R_\Delta = \infty$), when the ratio of the loading is greater than 1, we observe that the factor K become greater than 1 as the rotational stiffness of the column end is negative (Bridge and Fraser 1987; Corsi et al. 2020). It can therefore be said that the variation of the bracing and the loading have almost the same effects on the values of the factor K. These results contradict the tendency to think that the coefficient K has a higher sensitivity to the bracing compared to the loading.

For frames with fixed supports, as shown in Fig. 6 above, when the stiffness ratio of the bracing spring (R_Δ) increases, the overall effective buckling length factor K decreases, in a monotonous manner, by a value close to 1.83 for a load ratio equal to 4 ($\delta_2 = 4$) until attaining asymptotic values related to the load ratios of column (2). Moreover, it may be noted that in this case, the factor K is highly sensitive to the load ratio (δ_2) when this load ratio is greater than 0.5 in the vicinity of $R_\Delta = \infty$ (braced frame). In addition, when the load ratio (δ_2) increases from 0.25 to 4, the factor K increases by approximately 100% for a spring stiffness ratio $R_\Delta = 0$ (unbraced frame). Finally, it was observed that the values of the factor K for the frame with hinged supports are higher by approximately 100% and 40% for the values $R_\Delta = 0$ and $R_\Delta = \infty$, respectively, in comparison with those obtained in the case of a frame with fixed supports.

Regarding the influence of the bracings and column bases, as shown in Fig. 8, the bracing spring stiffness ratio (R_Δ) increases, the overall effective buckling length factor K decreases from values close to 2.32 (frame with hinged supports) and 1.16 (frame with fixed supports) until attaining asymptotic values related to the stiffness ratios of the rotational springs. Moreover, it is easy to see that, in this case, the factor K is highly sensitive to the stiffness ratios of the rotational springs in the vicinity of $R_\Delta = 0$ (unbraced frame), which is different from the case where the values of K are obtained in the vicinity of $R_\Delta = \infty$ (braced frame). These observations indicate that in accordance with the results obtained in the literature, in the case of an unbraced frame, the variation of the column bases have a large effect on the values of the factor K, in comparison with the values of factor K obtained in the case of a braced frame.

6. Conclusion

The present study aimed at developing an analytical model intended to determine the effective buckling lengths in irregular structures.

- The slope-deflection method was used in the present study along with the accurate classical stability functions for the buckling analysis of the structures studied in this article.

- Using the analytical development for the determination of the exact values of the effective buckling length factor allowed the carrying out the parametric studies proposed in this article.
- The method developed by this research covers both symmetrical and asymmetrical frames, where geometric irregularities and loading are considered.
- The current method gives better accuracy for asymmetrical frames than those methods developed by previous researchers. For symmetrical frames, the results are similar.

Finally, the results obtained from the current parametric studies carried out on several frames, allowed determination of the sensitivity of the factor K to the interaction between the variations of the different parameters presented in the study. Namely:

- The interaction effect between of the variation in bracing and loading of columns on the factor K in frames with hinged supports.
- The interaction effect between of the variation in bracing and loading of columns on the factor K in frames with fixed supports.
- The interaction effect between of the variation in bracing and the column bases on the factor K.

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